B. Ticknor

A

COURSE

OF

MATHEMATICS;

FOR THE

USE OF ACADEMIES

AS WELL AS

PRIVATE TUITION.

IN TWO VOLUMES.

BY

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THE FIFTH AMERICAN, FROM THE NINTH LONDON EDITION,

WITH MANY CORRECTIONS AND IMPROVEMENTS.

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WITH THE ADDITIONS

OF

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THE WHOLE

CORRECTED AND IMPROVED.

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PREFACE.

The present American edition is in part a reprint of the Ninth English edition by Dr. Olinthus Gregory, with most of the improvements introduced into former American editions by Dr. Adrain, together with such modifications of the English editions as appeared calculated to increase the general usefulness of the work. At the same time two or three Chapters, devoted to subjects of no great value at present to the American student, have been omitted, to leave room for matter of more interest and importance.

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OF

VOL. I.

A COURSE OF MATHEMATICS, &c.

GENERAL PRINCIPLES.

1. **Quantity**, or **Magnitude**, is any thing that will admit of increase or decrease; or that is capable of any sort of calculation or mensuration; such as numbers, lines, space, time, motion, weight, &c.

2. **Mathematics** is the science which treats of all kinds of quantity whatever, that can be numbered or measured.—That part which treats of numbering is called *Arithmetic*; and that which concerns measuring, or figured extension, is called *Geometry*.—Not only these two, but *Algebra* and *Fluxions*, which are conversant about multitude, magnitude, form, and motion, being the foundation of all the other parts, are called *Pure* or *Abstract Mathematics*; because they investigate and demonstrate the properties of abstract numbers and magnitudes of all sorts. And when these two parts are applied to particular or practical subjects, they constitute the branches or parts called *Mixed Mathematics*.—Mathematics is also distinguished into *Speculative* and *Practical*: viz. *Speculative*, when it is concerned in discovering properties and relations; and *Practical*, when applied to practice and real use concerning physical objects.

The peculiar topics of investigation in the four principal departments of pure mathematics may be indicated by four words: viz. *arithmetic* by *number*, *geometry* by *form*, *algebra* by *generality*, *fluxions* by *motion*.

3. In mathematics are several general terms or principles; such as, Definitions, Axioms, Propositions, Theorems, Problems, Lemmas, Corollaries, Scholia, &c.

4. *A Definition* is the explication of any term or word in a science; showing the sense and meaning in which the term is employed.—Every Definition ought to be clear, and expressed in words that are common and perfectly well understood.

5. *A Proposition* is something proposed to be demonstrated, or something required to be done; and is accordingly either a Theorem or a Problem.

6. *A Theorem* is a demonstrative Proposition; in which some property is asserted, and the truth of it required to be proved. Thus, when it is said that, The sum of the three angles of a plane triangle is equal to two right angles, that is a Theorem, the truth of which is demonstrated by Geometry. —A set or collection of such Theorems constitutes a *Theory*.

7. *A Problem* is a proposition or a question requiring something to be done; either to investigate some truth or property, or to perform some operation. As, to find out the quantity or sum of all the three angles of any triangle, or to draw one line perpendicular to another.—*A Limited Problem* is that which has but one answer or solution. *An Unlimited Problem* is that which has innumerable answers. And a *Determinate Problem* is that which has a certain number of answers.

8. *Solution* of a Problem, is the resolution or answer given to it. *A Numerical* or *Numeral Solution*, is the answer given in numbers. *A Geometrical Solution*, is the answer given by the principles of Geometry. And a *Mechanical Solution*, is one which is gained by trials.

9. *A Lemma* is a preparatory proposition, laid down in order to shorten the demonstration of the main proposition which follows it.

10. *A Corollary*, or *Consectary*, is a consequence drawn immediately from some proposition or other premises.

11. *A Scholium* is a remark or observation made upon some foregoing proposition or premises.

12. *An Axiom*, or *Maxim*, is a self-evident proposition; requiring no formal demonstration to prove its truth; but received and assented to as soon as mentioned. Such as, The whole of any thing is greater than a part of it; or, The whole is equal to all its parts taken together; or, Two quantities that are each of them equal to a third quantity, are equal to each other.

13. *A Postulate*, or *Petition*, is something required to be done, which is so easy and evident that no person will hesitate to allow it.

14. *An Hypothesis* is a supposition assumed to be true, in order to argue from, or to found upon it the reasoning and demonstration of some proposition.

15. *Demonstration* is the collecting the several arguments and proofs, and laying them together in proper order to show the truth of the proposition under consideration.

16. *A Direct, Positive*, or *Affirmative Demonstration*, is that which concludes with the direct and certain proof of the proposition in hand.

17. *An Indirect*, or *Negative Demonstration*, is that which shows a proposition to be true, by proving that some absurdity would necessarily follow if the proposition advanced were false. This is also sometimes called *Reductio ad Absurdum*; because it shows the absurdity and falsehood of all suppositions contrary to that contained in the proposition.

18. *Method* is the art of disposing a train of arguments in a proper order, to investigate either the truth or falsity of a proposition, or to demonstrate it to others when it has been found out.—This is either Analytical or Synthetical.

19. *Analysis* or the *Analytic Method*, is the art or mode of finding out the truth of a proposition, by first supposing the thing to be done, and then reasoning back, step by step, till we arrive at some known truth. This is also called the *Method of Invention*, or *Resolution*; and is that which is commonly used in Algebra.

20. *Synthesis*, or the *Synthetic Method*, is the searching out truth, by first laying down some simple and easy principles, and then pursuing the consequences flowing from them till we arrive at the conclusion.—This is also called the *Method of Composition*; and is the reverse of the Analytic method, as this proceeds from known principles to an unknown conclusion; while the other goes in a retrograde order, from the thing sought, considered as if it were true, to some known principle or fact. Therefore, when any truth has been found out by the Analytic method, it may be demonstrated by a process in the contrary order, by Synthesis: and in the solution of geometrical propositions, it is very instructive to carry through both the *analysis* and the *synthesis*.

ARITHMETIC.

Arithmetic is the art or science of numbering; being that branch of Mathematics which treats of the nature and properties of numbers.—When it treats of whole numbers, it is called *Vulgar*, or *Common Arithmetic*; but when of broken numbers, or parts of numbers, it is called *Fractions*.

Unity, or an *Unit*, is that by which every thing is called one; being the beginning of number; as, one man, one ball, one gun.

Number is either simply one, or a compound of several units; as, one man, three men, ten men.

An *Integer*, or *Whole Number*, is some certain precise quantity of units; as, one, three, ten.—These are so called as distinguished from *Fractions*, which are broken numbers, or parts of numbers; as, one-half, two-thirds, or three-fourths.

A *Prime Number* is one which has no other divisor than unity; as 2, 3, 5, 7, 17, 19, &c. A *Composite Number* is one which is the product of two or more numbers; as, 4, 6, 8, 9, 28, &c.

NOTATION AND NUMERATION.

These rules teach how to denote or express any proposed number, either by words or characters: or to read and write down any sum or number.

The Numbers in Arithmetic are expressed by the following ten digits, or Arabic numeral figures, which were introduced into Europe by the Moors, about eight or nine hundred years since; viz. 1 one, 2 two, 3 three, 4 four, 5 five, 6 six, 7 seven, 8 eight, 9 nine, 0 cipher, or nothing. These characters or figures were formerly all called by the general name of *Ciphers*; whence it came to pass that the art of Arithmetic was then often called *Ciphering*. The first nine are called *Significant Figures*, as distinguished from the cipher, which is of itself quite insignificant.

Besides this value of those figures, they have also another, which depends on the place they stand in when joined together; as in the following table:

&c.	Hundreds of Millions	Tens of Millions	Millions	Hundreds of Thousands	Tens of Thousands	Thousands	Hundreds	Tens	Units
&c.	9	8	7	6	5	4	3	2	1
		9	8	7	6	5	4	3	2
			9	8	7	6	5	4	3
				9	8	7	6	5	4
					9	8	7	6	5
						9	8	7	6
							9	8	7
								9	8
									9

Here, any figure in the first place, reckoning from right to left, denotes only its own simple value; but that in the second place, denotes ten times its simple value; and that in the third place, a hundred times its simple value; and so on: the value of any figure, in each successive place, being always ten times its former value.

Thus, in the number 1796, the 6 in the first place denotes only six units, or simply six; 9 in the second place signifies nine tens, or ninety; 7 in the third place, seven hundred; and the 1 in the fourth place, one thousand: so that the whole number is read thus, one thousand seven hundred and ninety-six.

As to the cipher, 0, though it signify nothing of itself, yet being joined on the right-hand side to other figures, it increases their value in the same ten-fold proportion: thus, 5 signifies only five; but 50 denotes 5 tens, or fifty; and 500 is five hundred; and so on.

For the more easily reading of large numbers, they are divided into periods and half-periods, each half-period consisting of three figures; the name of the first period being units; of the seceud, millions; of the third, millions of millions, or bi-millions, contracted to billions; of the fourth, millions of millions of millions, or tri-millions, contracted to trillions, and so on. Also the first part of any period is so many units of it, and the latter part so many thousands.

The following Table contains a summary of the whole doctrine.

Periods.	Quadril. ; Trillions ; Billions ; Millions ; Units.
Half-per.	th. un. th. un. th. un. th. un. th. un.
Figures.	123,456 ; 789,098 ; 765,432 ; 101,234 ; 567,890.

Numeration is the reading of any number in words that is proposed or set down in figures ; which will be easily done by help of the following rule, deduced from the foregoing tables and observations—viz.

Divide the figures in the proposed number, as in the summary above, into periods and half-periods ; then begin at the left-hand side, and read the figures with the names set to them in the two foregoing tables.

EXAMPLES.

Express in words the following numbers ; viz.

34	15080	13405670
96	72003	47050023
380	109026	309025600
704	483500	4723507689
6134	2500639	274856390000
9028	7523000	6578600307024

Notation is the setting down in figures any number proposed in words ; which is done by setting down the figures instead of the words or names belonging to them in the summary above ; supplying the vacant places with ciphers where any words do not occur.

EXAMPLES.

Set down in figures the following numbers :

Fifty-seven.
Two hundred eighty-six.
Nine thousand two hundred and ten.
Twenty-seven thousand five hundred and ninety-four.
Six hundred and forty thousand, four hundred and eighty-one,
Three millions, two hundred sixty thousand, one hundred and six.

Four hundred and eight millions, two hundred and fifty-five thousand, one hundred and ninety-two.

Twenty-seven thousand and eight millions, ninety-six thousand two hundred and four.

Two hundred thousand and five hundred and fifty millions, one hundred and ten thousand, and sixteen.

Twenty-one billions, eight hundred and ten millions, sixty-four thousand, one hundred and fifty.

OF THE ROMAN NOTATION.

The Romans, like several other nations, expressed their numbers by certain letters of the alphabet. The Romans used only seven numeral letters, being the seven following capitals: viz. I for *one*; V for *five*; X for *ten*; L for *fifty*; C for an *hundred*; D for *five hundred*; M for a *thousand*; The other numbers they expressed by various repetitions and combinations of these, after the following manner:

1 = I		
2 = II	As often as any character is repeated, so many times is its value repeated.
3 = III	
4 = IIII or IV	A less character before a greater diminishes its value.
5 = V	
6 = VI	A less character after a greater increases its value.
7 = VII	
8 = VIII	
9 = IX	
10 = X	
50 = L	
100 = C	
500 = D or IↃ	For every Ↄ annexed, this becomes 10 times as many.
1000 = M or CIↃ	For every C and Ↄ, placed one at each end, it becomes 10 times as much.
2000 = MM	
5000 = $\overline{\text{V}}$ or IↃↃ	A bar over any number increases it 1000 fold.
6000 = $\overline{\text{VI}}$	
10000 = $\overline{\text{X}}$ or CCIↃↃ	
50000 = $\overline{\text{L}}$ or IↃↃↃ	
60000 = $\overline{\text{LX}}$	
100000 = $\overline{\text{C}}$ or CCCIↃↃↃ	
1000000 = $\overline{\text{M}}$ or CCCCIↃↃↃↃ	
2000000 = $\overline{\text{MM}}$	
&c. &c.	

EXPLANATION OF CERTAIN CHARACTERS.

There are various characters or marks used in Arithmetic, and Algebra, to denote several of the operations and propositions; the chief of which are as follow:

$+$ signifies *plus*, or addition.
$-$ - - *minus*, or subtraction.
$\times$ or - multiplication.
$\div$ - - division.
$: \; :: \; :$ - proportion.
$=$ - - equality.
$\sqrt{}$ - - square root.
$\sqrt[3]{}$ - - cube root, &c.
$\backsim$ - - diff. between two numbers when it is not known which is the greater.

Thus,
$5 + 3$, denotes that 3 is to be added to 5.
$6 - 2$, denotes that 2 is to be taken from 6.
7×3, or $7 . 3$, denotes that 7 is to be multiplied by 3.
$8 \div 4$, denotes that 8 is to be divided by 4.
$2 : 3 :: 4 : 6$, shows that 2 is to 3 as 4 is to 6.
$6 + 4 = 10$, shows that the sum of 6 and 4 is equal to 10.
$\sqrt{3}$, or $3^{\frac{1}{2}}$, denotes the square root of the number 3.
$\sqrt[3]{5}$, or $5^{\frac{1}{3}}$, denotes the cube root of the number 5.
7^2, denotes that the number 7 is to be squared.
8^3, denotes that the number 8 is to be cubed.
&c.

OF ADDITION.

Addition is the collecting or putting of several numbers together, in order to find their *sum*, or the total amount of the whole. This is done as follows:

Set or place the numbers under each other, so that each figure may stand exactly under the figures of the same value, that is, units under units, tens under tens, hundreds under hundreds, &c. and draw a line under the lowest number, to separate the given numbers from their sum, when it is found. —Then add up the figures in the column or row of units, and find how many tens are contained in that sum.—Set down exactly below, what remains more than those tens, or if nothing remains, a cipher, and carry as many ones to the next row as there are tens.—Next add up the second row, together with the number carried, in the same manner as the first. And thus proceed till the whole is finished, setting down the total amount of the last row.

TO PROVE ADDITION.

First Method.—Begin at the top, and add together all the rows of numbers downwards, in the same manner as they were before added upwards; then if the two sums agree, it may be presumed the work is right.—This method of proof is only doing the same work twice over, a little varied.

Second Method.—Draw a line below the uppermost number, and suppose it cut off.—Then add all the rest of the numbers together in the usual way, and set their sum under the number to be proved.—Lastly, add this last found number and the uppermost line together; then if their sum be the same as that found by the first addition, it may be presumed the work is right.—This method of proof is founded on the plain axiom, that "The whole is equal to all its parts taken together."

Third Method.—Add the figures in the uppermost line together, and find how many nines are contained in their sum.—Reject those nines, and set down the remainder towards the right hand directly even with the figures in the line, as in the annexed example.—Do the same with each of the proposed lines of numbers, setting all these excesses of nines in a column on the right-hand, as here 5, 5, 6. Then, if the excess of 9's in this sum, found as before, be equal to the excess of 9's in the total sum 18304, the work is probably right.—Thus, the sum of the right-hand column, 5, 5, 6, is 16, the excess of which above 9 is 7. Also the sum of the figures in

EXAMPLE I.

	Excess of nines.
3497	5
6512	5
8295	6
18304	7

the sum total 18304, is 16, the excess of which above 9 is also 7, the same as the former*.

OTHER EXAMPLES.

2.	3.	4.
12345	12345	12345
67890	67890	876
98765	9876	9087
43210	543	56
12345	21	234
67890	9	1012
302445	90684	23610
290100	78339	11265
302445	90684	23610

Ex. 5. Add 3426; 9024; 5106; 8890; 1204, together. Ans. 27650.

6. Add 509267; 235809; 72920; 8392; 420; 21; and 9, together. Ans. 826838.

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* This method of proof depends on a property of the number 9, which, except the number 3, belongs to no other digit whatever; namely, that "any number divided by 9, will leave the same remainder as the sum of its figures are digits divided by 9:" which may be demonstrated in this manner.

Demonstration. Let there be any number proposed, as 4658. This, separated into its several parts, becomes, $4000+600+50+8$. But $4000=4\times1000=4\times(999+1)=(4\times999)+4$. In like manner $600=(6\times99)+6$; and $50=(5\times9)+5$. Therefore the given number $4658=(4\times999)+4+(6\times99)+6+(5\times9)+5+8=(4\times999)+(6\times99)+(5\times9)+4+6+5+8$; and $4658\div9=(4\times999+6\times99+5\times9+4+6+5+8)\div9$. But $(4\times999)+(6\times99)+(5\times9)$ is evidently divisible by 9, without a remainder; therefore if the given number 4658 be divided by 9, it will leave the same remainder as $4+6+5+8$ divided by 9. And the same, it is evident, will hold for any other number whatever.

In like manner, the same property may be shown to belong to the number 3; but the preference is usually given to the number 9, on account of its being more convenient in practice.

Now, from the demonstration above given, the reason of the rule itself is evident: for the excess of 9's in two or more numbers being taken separately, and the excess of 9's taken also out of the sum of the former excesses, it is plain that this last excess must be equal to the excess of 9's contained in the total sum of all these numbers; all the parts taken together being equal to the whole.—This rule was first given by Dr. Wallis in his Arithmetic, published in the year 1657.

7. Add 2; 19; 817; 4298; 50916; 730205; 9180634, together. Ans. 9966891.

8. How many days are in the twelve calendar months? Ans. 365.

9. How many days are there from the 15th day of April to the 24th day of November, both days included? Ans. 224.

10. An army consisting of 52714 infantry*, or foot, 5110 horse, 6250 dragoons, 3927 light-horse, 928 artillery, or gunners, 1410 pioneers, 250 sappers, and 406 miners: what is the whole number of men? Ans. 70995.

OF SUBTRACTION.

SUBTRACTION teaches to find how much one number exceeds another, called their *difference*, or the *remainder*, by taking the less from the greater. The method of doing which is as follows:

Place the less number under the greater, in the same manner as in Addition, that is, units under units, tens under tens, and so on; and draw a line below them.—Begin at the right hand, and take each figure in the lower line, or number, from the figure above it, setting down the remainder below it.—But if the figure in the lower line be greater than that above it, first borrow, or add, 10 to the upper one, and then take the lower figure from that sum, setting down the remainder, and carrying 1, for what was borrowed, to the next lower figure, with which proceed as before; and so on till the whole is finished.

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* The whole body of foot soldiers is denoted by the word *Infantry*; and all those that charge on horseback by the word *Cavalry*.—Some authors conjecture that the term infantry is derived from a certain Infanta of Spain, who, finding that the army commanded by the king her father had been defeated by the Moors, assembled a body of the people together on foot, with which she engaged and totally routed the enemy. In honour of this event, and to distinguish the foot soldiers, who were not before held in much estimation, they received the name of Infantry, from her own title of Infanta.

TO PROVE SUBTRACTION.

Add the remainder to the less number, or that which is just above it; and if the sum be equal to the greater or uppermost number, the work is right*.

EXAMPLES.

	1		2		3
From	5386427	From	5386427	From	1234567
Take	2164315	Take	4258792	Take	702973
Rem.	3222112	Rem.	1127635	Rem.	531594
Proof.	5386427	Proof.	5386427	Proof.	1234567

4. From 5331806 take 5073918. Ans. 257888.
5. From 7020974 take 2766809. Ans. 4254165.
6. From 8503402 take 574271. Ans. 7929131.
7. Sir Isaac Newton was born in the year 1642, and he died in 1727: how old was he at the time of his decease? Ans. 85 years.
8. Homer was born 2560 years ago, and Christ 1827 years ago: then how long before Christ was the birth of Homer? Ans. 733 years.
9. Noah's flood happened about the year of the world 1656, and the birth of Christ about the year 4000: then how long was the flood before Christ? Ans. 2344 years.
10. The Arabian or Indian method of notation was first known in England about the year 1150: then how long is it since to this present year 1827? Ans. 677 years.
11. Gunpowder was invented in the year 1330: how long was that before the invention of printing, which was in 1441? Ans. 111 years.
12. The mariner's compass was invented in Europe in the year 1302: how long was that before the discovery of America by Columbus, which happened in 1492? Ans. 190 years.

* The reason of this method of proof is evident; for if the difference of two numbers be added to the less, it must manifestly make up a sum equal to the greater.

OF MULTIPLICATION.

MULTIPLICATION is a compendious method of Addition, teaching how to find the amount of any given number when repeated a certain number of times; as, 4 times 6, which is 24.

The number to be multiplied, or repeated, is called the *Multiplicand*.—The number you multiply by, or the number of repetitions, is the *Multiplier*.—And the number found, being the total amount, is called the *Product*.—Also, both the multiplier and multiplicand are, in general, named the *Terms* or *Factors*.

Before proceeding to any operations in this rule, it is necessary to learn off very perfectly the following Table, of all the products of the first 12 numbers, commonly called the Multiplication Table, or sometimes Pythagoras's Table, from its inventor.

MULTIPLICATION TABLE.

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	120	132	144

To multiply any Given Number by a Single Figure, or by any Number not exceeding 12.

* Set the multiplier under the units' figure or right-hand place, of the multiplicand, and draw a line below it.—Then, beginning at the right-hand, multiply every figure in this by the multiplier.—Count how many tens there are in the product of every single figure, and set down the remainder directly under the figure that is multiplied; and if nothing remains, set down a cipher.—Carry as many units or ones as there are tens counted, to the product of the next figures; and proceed in the same manner till the whole is finished.

EXAMPLE.

Multiply 9876543210 the Multiplicand.
By - - - - - 2 the Multiplier.

19753086420

To multiply by a Number consisting of Several Figures.

† Set the multiplier below the multiplicand, placing them as in Addition, namely, units under units, tens under tens, &c. drawing a line below it.—Multiply the whole of the multiplicand by each figure of the multiplier, as in the last article;

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* The reason of this rule is the same as for the process in Addition, in which 1 is carried for every 10, to the next place, gradually as the several products are produced one after another, instead of setting them all down below each other, as in the annexed example.

5678		
4		
32	=	8 × 4
280	=	70 × 4
2400	=	600 × 4
20000	=	5000 × 4
22712	=	5678 × 4

† After having found the product of the multiplicand by the first figure of the multiplier, as in the former case, the multiplier is supposed to be divided into parts, and the product is found for the second figure in the same manner: but as this figure stands in the place of tens, the product must be ten times its simple value; and therefore the first figure of this product must be set in the place of tens; or, which is the same thing, directly under the figure multiplying by. And proceeding

setting down a line of products for each figure in the multiplier, so as that the first figure of each line may stand straight under the figure multiplying by. Add all the lines of products together, in the order in which they stand, and their sum will be the answer or whole product required.

TO PROVE MULTIPLICATION.

There are three different ways of proving multiplication, which are as below:

First Method. — Make the multiplicand and multiplier change places, and multiply the latter by the former in the same manner as before. Then if the product found in this way be the same as the former, the number is right.

Second Method.—* Cast all the 9's out of the sum of the figures in each of the two factors, as in Addition, and set down the remainders. Multiply these two remainders together, and cast the 9's out of the product, as also out of the whole product or answer of the question, reserving the remainders of these last two, which remainders must be equal when the work is right.—*Note*, It is common to set the four remainders within the four angular spaces of a cross, as in the example below.

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in this manner separately with all the figures of the multiplier, it is evident that we shall multiply all the parts of the multiplicand by all the parts of the multiplier, or the whole of the multiplicand by the whole of the multiplier: therefore these several products being added together, will be equal to the whole required product; as in the example annexed.

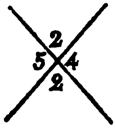
1234567		the multiplicand.
4567		
8641969	=	7 times the mult.
7407402	=	60 times ditto.
6172835	=	500 times ditto.
4938268	=	4000 times ditto.
5638267489	=	4567 times ditto.

* This method of proof is derived from the peculiar property of the number 9, mentioned in the proof of Addition, and the reason for the one includes that of the other. Another more ample demonstration of this rule may, however, be as follows:—Let P and Q denote the number of 9's in the factors to be multiplied, and a and b what remain; then $9P + a$ and $9Q + b$ will be the numbers themselves, and their product is $(9P \times 9Q) + (9P \times b) + (9Q \times a) + (a \times b)$; but the first three of these products are each a precise number of 9's, because their factors are so, either one or both: these therefore being cast away, there remains only $a \times b$; and if the 9's also be cast out of this, the excess is the excess of 9's in the total product: but a and b are the excesses in the factors themselves, and $a \times b$ is their product; therefore the rule is true. This mode of proof, however, is not an ample check against the errors that might arise from a transposition of figures.

Third Method.—Multiplication is also very naturally proved by Division; for the product divided by either of the factors, will evidently give the other. But this cannot be practised till the rule of division is learned.

EXAMPLES.

Mult. 3542		or Mult. 6196
by 6196	Proof.	by 3542
21252		12392
31878		24784
3542		30980
21252		18588
21946232		21946232 Proof.



OTHER EXAMPLES.

Multiply 123456789	by 3.	Ans. 370370367.
Multiply 123456789	by 4.	Ans. 493827156.
Multiply 123456789	by 5.	Ans. 617283945.
Multiply 123456789	by 6.	Ans. 740740734.
Multiply 123456789	by 7.	Ans. 864197523.
Multiply 123456789	by 8.	Ans. 987654312.
Multiply 123456789	by 9.	Ans. 1111111101.
Multiply 123456789	by 11.	Ans. 1358024679.
Multiply 123456789	by 12.	Ans. 1481481468.
Multiply 302914603	by 16.	Ans. 4846633648.
Multiply 273580961	by 23.	Ans. 6292362103.
Multiply 402097316	by 195.	Ans. 78408976620.
Multiply 82164973	by 3027.	Ans. 248713373271.
Multiply 7564900	by 579.	Ans. 4380077100.
Multiply 8496427	by 874359.	Ans. 7428927415293.
Multiply 2760325	by 37072.	Ans. 102330768400.

CONTRACTIONS IN MULTIPLICATION.

I. *When there are Ciphers in the Factors.*

If the ciphers be at the right-hand of the numbers; multiply the other figures only, and annex as many ciphers to the right-hand of the whole product, as are in both the factors.—When the ciphers are in the middle parts of the multiplier; neglect them as before, only taking care to place the first figure of every line of products exactly under the figure you are multiplying with.

EXAMPLES.

1.		2.
Mult. 9001635		Mult. 390720400
by - 70100		by - 406000
9001635		23443224
63011445		15628816
631014613500	Products	158632482400000

3. Multiply 81503600 by 7030. Ans. 572970308000.
4. Multiply 9030100 by 2100. Ans. 18963210000.
5. Multiply 8057069 by 70050. Ans. 564397683450.

II. *When the Multiplier is the Product of two or more Numbers in the Table; then*

* Multiply by each of those parts separately, instead of the whole number at once.

EXAMPLES.

1. Multiply 51307298 by 56, or 7 times 8.

51307298
7
359151086
8
2873208688

2. Multiply 31704592 by 36. Ans. 1141365312.
3. Multiply 29753804 by 72. Ans. 2142273888.
4. Multiply 7128368 by 96. Ans. 684323328.
5. Multiply 160430800 by 108. Ans. 17326526400.
6. Multiply 61835720 by 1320. Ans. 81623150400.

* The reason of this rule is obvious enough; for any number multiplied by the component parts of another, must give the same product as if it were multiplied by that number at once. Thus, in the 1st example, 7 times the product of 8 by the given number, makes 56 times the same number, as plainly as 7 times 8 make 56.

7. There was an army composed of 104* battalions, each consisting of 500 men; what was the number of men contained in the whole? Ans. 52000.

8. A convoy of ammunition †bread, consisting of 250 waggons, and each waggon containing 320 loaves, having been intercepted and taken by the enemy, what is the number of loaves lost? Ans. 80000.

OF DIVISION.

Division is a kind of compendious method of Subtraction, teaching to find how often one number is contained in another, or may be taken out of it: which is the same thing.

The number to be divided is called the *Dividend*.—The number to divide by, is the *Divisor*.—And the number of times the dividend contains the divisor, is called the *Quotient*.—Sometimes there is a *Remainder* left, after the division is finished.

The usual manner of placing the terms, is, the dividend in the middle, having the divisor on the left hand, and the quotient on the right, each separated by a curve line; as, to divide 12 by 4, the quotient is 3,

	Dividend	
Divisor 4)	12	(3 Quotient;

showing that the number 4 is 3 times contained in 12, or may be 3 times subtracted out of it, as in the margin.

```
12
 4 subtr.
--
 8
 4 subtr.
--
 4
 4 subtr.
--
 0
--
```

‡ *Rule.*—Having placed the divisor before the dividend, as above directed, find how often the divisor is contained in as many figures of the dividend as are just necessary, and place the number on the right in the quotient. Mul-

---

* A battalion is a body of foot, consisting of 500, or 600, or 700 men, more or less.

† The ammunition bread, is that which is provided for, and distributed to, the soldiers; the usual allowance being a loaf of 6 pounds to every soldier, once in 4 days.

‡ In this way the dividend is resolved into parts, and by trial is found how often the divisor is contained in each of those parts, one after another, arranging the several figures of the quotient one after another, into one number.

tiply the divisor by this number, and set the product under the figures of the dividend before-mentioned.—Subtract this product from that part of the dividend under which it stands, and bring down the next figure of the dividend, or more if necessary, to join on the right of the remainder.—Divide this number, so increased, in the same manner as before; and so on, till all the figures are brought down and used.

Note. If it be necessary to bring down more figures than one to any remainder, in order to make it as large as the divisor, or larger, a cipher must be set in the quotient for every figure so brought down more than one.

TO PROVE DIVISION.

* Multiply the quotient by the divisor; to this product add the remainder, if there be any; then the sum will be equal to the dividend, when the work is right.

When there is no remainder to a division, the quotient is the whole and perfect answer to the question. But when there is a remainder, it goes so much towards another time, as it approaches to the divisor: so, if the remainder be half the divisor, it will go the half of a time more; if the 4th part of the divisor, it will go one-fourth of a time more; and so on. Therefore, to complete the quotient, set the remainder at the end of it, above a small line, and the divisor below it, thus forming a fractional part of the whole quotient.

* This method of proof is plain enough: for since the quotient is the number of times the dividend contains the divisor, the quotient multiplied by the divisor must evidently be equal to the dividend.

There are several other methods sometimes used for proving Division, some of the most useful of which are as follow:

Second Method.—Subtract the remainder from the dividend, and divide what is left by the quotient; so shall the new quotient from this last division be equal to the former divisor, when the work is right.

Third Method.—Add together the remainder and all the products of the several quotient figures by the divisor, according to the order in which they stand in the work; and the sum will be equal to the dividend, when the work is right.

EXAMPLES.

1	Quot.	2	Quot.
3) 1234567	(411522	37) 12345678	(333666
12	mult. 3	111	37
3	1234566	124	2335662
3	add 1	111	1000998
			rem. 36
4	1234567	135	
3		111	12345678
	Proof.		
15		246	Proof.
15		222	
6		247	
6		222	
7		258	
6		222	
Rem. 1		Rem. 36	

3. Divide 73146085 by 4. Ans. $18286521\frac{1}{4}$.
4. Divide 53179860027 by 7. Ans. $7597122289\frac{4}{7}$.
5. Divide 570196382 by 12. Ans. $47516365\frac{2}{12}$.
6. Divide 74638105 by 37. Ans. $2017246\frac{3}{37}$.
7. Divide 137896254 by 97. Ans. $1421610\frac{84}{97}$.
8. Divide 35821649 by 764. Ans. $46886\frac{745}{764}$.
9. Divide 72091365 by 5201. Ans. $13861\frac{304}{5201}$.
10. Divide 4637064283 by 57606. Ans. $80496\frac{11707}{57606}$.

11. Suppose 471 men are formed into ranks of 3 deep, what is the number in each rank? Ans. 157.

12. A party, at the distance of 378 miles from the head quarters, receive orders to join their corps in 18 days: what number of miles must they march each day to obey their orders? Ans. 21.

13. The annual revenue of a nobleman being 37960*l.*; how much per day is that equivalent to, there being 365 days in the year? Ans. 104*l.*

CONTRACTIONS IN DIVISION.

There are certain contractions in Division, by which the operation in particular cases may be performed in a shorter manner: as follows:

I. *Division by any Small Number*, not greater than 12, may be expeditiously performed, by multiplying and subtracting mentally, omitting to set down the work except only the quotient immediately below the dividend.

EXAMPLES.

3) 56103961	4) 52619675	5) 1379192
Quot. $18701320\frac{1}{3}$		
6) 38672940	7) 81396627	8) 23718920
9) 43981962	11) 57614230	12) 27980373

II. * *When Ciphers are annexed to the Divisor;* cut off those ciphers from it, and cut off the same number of figures from the right-hand of the dividend; then divide with the remaining figures, as usual. And if there be any thing remaining after this division, place the figures cut off from the dividend to the right of it, and the whole will be the true remainder; otherwise, the figures cut off only will be the remainder.

EXAMPLES.

1. Divide 3704196 by 20.

```
2,0) 370419,6
     --------
Quot. 185209 16/20
```

2. Divide 31086901 by 7100.

```
71,00) 310869,01 (4378 3101/7100.
       284
       ---
        268
        213
        ---
         556
         497
         ---
          599
          568
          ---
           31
```

* This method serves to avoid a needless repetition of ciphers, which would happen in the common way. And the truth of the principle on

3. Divide 7380964 by 23000. Ans. $320\frac{20964}{23000}$.
4. Divide 2304109 by 5800. Ans. $397\frac{1509}{5800}$.

III. *When the Divisor is the exact Product of two or more of the small Numbers not greater than* 12: * Divide by each of those numbers separately, instead of the whole divisor at once.

Note. There are commonly several remainders in working by this rule, one to each division; and to find the true or whole remainder, the same as if the division had been performed all at once, proceed as follows: Multiply the last remainder by the preceding divisor, or last but one, and to the product add the preceding remainder; multiply this sum by the next preceding divisor, and to the product add the next preceding remainder; and so on till you have gone backward through all the divisors and remainders to the first. As in the example following:

EXAMPLES.

1. Divide 31046835 by 56 or 7 times 8.

```
7) 31046835                            6 the last rem.
   --------                      mult. 7 preced. divisor.
8)  4435262—1 first rem.               —
   --------                            42
     554407—6 second rem.        add    1 to the 1st rem.
   --------                            —
Ans. 554407 43/56                      43 whole rem.
   --------                            —
```

2. Divide 7014596 by 72. Ans. $97424\frac{68}{72}$.
3. Divide 5130652 by 132. Ans. $38868\frac{76}{132}$.
4. Divide 83016572 by 240. Ans. $345902\frac{92}{240}$.

---

which it is founded, is evident; for, cutting off the same number of ciphers, or figures, from each, is the same as dividing each of them by 10, or 100, or 1000, &c. according to the number of ciphers cut off; and it is evident, that as often as the whole divisor is contained in the whole dividend, so often must any part of the former be contained in a like part of the latter.

* This follows from the second contraction in Multiplication, being only the converse of it; for the half of the third part of any thing, is evidently the same as the sixth part of the whole; and so of any other numbers.—The reason of the method of finding the whole remainder from the several particular ones, will best appear from the nature of Vulgar Fractions. Thus, in the first example above, the first remainder being 1, when the divisor is 7, makes $\frac{1}{7}$; this must be added to the second remainder, 6, making $6\frac{1}{7}$ to the divisor 8, or to be divided by 8. But $6\frac{1}{7} = \frac{6\times7+1}{7} = \frac{43}{7}$; and this divided by 8 gives $\frac{43}{7\times8} = \frac{43}{56}$.

IV. *Common Division may be performed more concisely,* by omitting the several products, and setting down only the remainders; namely, multiply the divisor by the quotient figures as before, and, without setting down the product, subtract each figure of it from the dividend, as it is produced; always remembering to carry as many to the next figure as were borrowed before.

EXAMPLES.

1. Divide 3104679 by 833.

$$
\begin{array}{l}
833)\ 3104679\ (3727\tfrac{88}{833}. \\
\quad\quad\ \ 6056 \\
\quad\quad\quad 2257 \\
\quad\quad\quad\ \ 5919 \\
\quad\quad\quad\quad\ 88
\end{array}
$$

2. Divide 79165238 by 238. Ans. $332627\frac{12}{238}$.
3. Divide 29137062 by 5317. Ans. $5479\frac{5219}{5317}$.
4. Divide 62015735 by 7803. Ans. $7947\frac{5294}{7803}$.

OF REDUCTION.

REDUCTION is the changing of numbers from one name or denomination to another, without altering their value.—This is chiefly concerned in reducing money, weights, and measures.

When the numbers are to be reduced from a higher name to a lower, it is called *Reduction descending;* but when, contrarywise, from a lower name to a higher, it is *Reduction ascending*.

Before we proceed to the rules and questions of Reduction, it will be proper to set down the usual tables of money, weights, and measures, which are as follow :

OF MONEY, WEIGHTS, AND MEASURES.

TABLES OF MONEY.

2 Farthings	= 1 Halfpenny	½	qrs		d		
4 Farthings	= 1 Penny	d	4 =		1	s	
12 Pence	= 1 Shilling	s	48 =		12 =	1	£
20 Shillings	= 1 Pound	£	960 =		240 =	20 =	1

PENCE TABLE.			
d		s	d
20	is	1	8
30	—	2	6
40	—	3	4
50	—	4	2
60	—	5	0
70	—	5	10
80	—	6	8
90	—	7	6
100	—	8	4
110	—	9	2
120	—	10	0

SHILLINGS TABLE.		
s		d
1	is	12
2	—	24
3	—	36
4	—	48
5	—	60
6	—	72
7	—	84
8	—	96
9	—	108
10	—	120
11	—	132

Note.—£ denotes pounds, *s* shillings, and *d* denotes pence.
¼ denotes 1 farthing, or one quarter of any thing.
½ denotes a halfpenny, or the half of any thing.
¾ denotes 3 farthings, or three quarters of any thing.

The full weight and value of the English gold and silver coin, both old and new, are as here below.

GOLD.	Value. £	s	d	Weight. dwt	gr
Guinea	1	1	0	5	9½
Half do.	0	10	6	2	16¾
Third do.	0	7	0	1	19⅙
Double Sov.	2	0	0	10	$6\frac{4}{11}$
Sovereign	1	0	0	5	$3\frac{3}{11}$
Half do.	0	10	0	2	$13\frac{7}{11}$

SILVER.	Value. s	d	Old Wt. dwt	gr	New Wt. dwt	gr.
A Crown	5	0	19	8½	18	$4\frac{4}{11}$
Half-crown	2	6	9	16¼	9	$2\frac{8}{11}$
Shilling	1	0	3	21	3	$15\frac{3}{11}$
Sixpence	0	6	1	22½	1	$19\frac{7}{11}$

The usual value of gold is nearly 4*l* an ounce, or 2*d* a grain; and that of silver is nearly 5*s* an ounce. Also the value of any quantity of gold, was to the value of the same weight of standard silver, as $15\frac{8}{124}$ to 1, in the old coin; but in the new coin they are as $14\frac{7}{44}$ to 1.

Pure gold, free from mixture with other metals, usually called fine gold, is of so pure a nature, that it will endure the fire without wasting, though it be kept continually melted. But silver, not having the purity of gold, will not endure the fire like it: yet fine silver will waste but very little by being in the fire any moderate time; whereas copper, tin, lead, &c. will not only waste, but may be calcined, or burnt to a powder.

Both gold and silver, in their purity, are so soft and flexible (like new lead, &c.) that they are not so useful, either in coin or otherwise (except to beat into leaf gold or silver), as when they are alloyed, or mixed and hardened with copper or brass. And though most nations differ, more or less, in the quantity of such alloy, as well as in the same place at different times, yet in England the standard for gold and silver coin has been for a long time as follows—viz. That 22 parts of fine gold, and 2 parts of copper, being melted together, shall be esteemed the true standard for gold coin: And that 11 ounces and 2 pennyweights of fine silver, and 18 pennyweights of copper, being melted together, be esteemed the true standard for silver coin, called Sterling silver.

In the old coin the pound of sterling gold was coined into 44½ guineas, of 21 shillings each, of which the pound of sterling silver was divided into 62. The new coin is also of the same quality or degree of

TROY WEIGHT*.

Grains	- -		marked *gr*
24 Grains make	1	Pennyweight	*dwt*
20 Pennyweights	1	Ounce	*oz*
12 Ounces	1	Pound	*lb*

gr	*dwt*		
24=	1	*oz*	
480=	20=	1	*lb*
5760=	240=	12=	1

By this weight are weighed Gold, Silver, and Jewels.

APOTHECARIES' WEIGHT.

Grains	- -		marked *gr*
20 Grains make	1	Scruple	*sc* or ℈
3 Scruples	1	Dram	*dr* or ʒ
8 Drams	1	Ounce	*oz* or ℥
12 Ounces	1	Pound	*lb* or ℔

gr	*sc*			
20 =	1	*dr*		
60 =	3 =	1	*oz*	
480 =	24 =	8 =	1	*lb*
5760 =	288 =	96 =	12 =	1

This is the same as Troy weight, only having some different divisions. Apothecaries make use of this weight in compounding their medicines; but they buy and sell their Drugs by Avoirdupois weight.

AVOIRDUPOIS WEIGHT.

Drams	- -		- -	marked *dr*
16 Drams	make	1	Ounce - -	*oz*
16 Ounces	- -	1	Pound - -	*lb*
28 Pounds	-	1	Quarter - -	*qr*
4 Quarters	-	1	Hundred weight	*cwt*
20 Hundred Weight		1	Ton - -	*ton*

---

fineness with that of the old sterling gold and silver above described, but divided into pieces of other names or values; viz. the pound of the silver into 66 shillings, of course each shilling is the 66th part of a pound; and 20 pounds of the gold into $934\frac{1}{2}$ pieces called sovereigns, or the pound weight into $46\frac{29}{40}$ sovereigns, each equal to 20 of the new shillings. So that the weight of the sovereign is $46\frac{29}{40}$ths of a pound, which is equal to $5\frac{85}{623}$ pennyweights, or equal to 5 dwt. $3\frac{3}{11}$ gr. very nearly, as stated in the preceding table. And multiples and parts of the sovereign and shilling in their several proportions.

* The original of all weights used in England, was a grain or corn of wheat, gathered out of the middle of the ear, and, being well dried, 32 of them were to make one pennyweight, 20 pennyweights one

dr	oz	lb	qr	cwt	ton
16 =	1				
256 =	16 =	1			
7168 =	448 =	28 =	1		
28672 =	1792 =	112 =	4 =	1	
573440 =	35840 =	2240 =	80 =	20 =	1

By this weight are weighed all things of a coarse or drossy nature, as Corn, Bread, Butter, Cheese, Flesh, Grocery Wares, and some Liquids; also all metals, except Silver and Gold.

		oz	dwt	gr	
Note, that 1*lb* Avoirdupois	=	14	11	$15\frac{1}{2}$	Troy
1*oz* - -	=	0	18	$5\frac{1}{2}$	
1*dr* - -	=	0	1	$3\frac{1}{2}$	

LONG MEASURE.

2 Barley-corns make	1 Inch	- -	*In*
12 Inches - -	1 Foot	- -	*Ft*
3 Feet - -	1 Yard	- -	*Yd*
6 Feet - -	1 Fathom	-	*Fth*
5 Yards and a half	1 Pole or Rod	-	*Pl*
40 Poles - -	1 Furlong	-	*Fur*
8 Furlongs -	1 Mile	- -	*Mile*
3 Miles - -	1 League	-	*Lea*
$69\frac{1}{12}$ Miles nearly	1 Degree	- -	*Deg* or °.

In	Ft	Yd	Pl	Fur	Mile
12 =	1				
36 =	3 =	1			
198 =	$16\frac{1}{2}$ =	$5\frac{1}{2}$ =	1		
7920 =	660 =	220 =	40 =	1	
63360 =	5280 =	1760 =	320 =	8 =	1

CLOTH MEASURE.

2 Inches and a quarter make	1 Nail	- -	*Nl*
4 Nails - - -	1 Quarter of a Yard		*Qr*
3 Quarters - -	1 Ell Flemish	-	*E F*
4 Quarters - -	1 Yard	- -	*Yd*
5 Quarters - - -	1 Ell English	-	*E E*
4 Quarters $1\frac{1}{2}$ Inch -	1 Ell Scotch	-	*E S*

ounce, and 12 ounces one pound. But in later times it was thought sufficient to divide the same pennyweight into 24 equal parts, still called grains, being the least weight now in common use; and from thence the rest are computed, as in the Tables above.

SQUARE MEASURE.

144	Square Inches make	1 Sq Foot	-	*Ft*
9	Square Feet -	1 Sq Yard	-	*Yd*
30¼	Square Yards -	1 Sq Pole	-	*Pole*
40	Square Poles -	1 Rood	-	*Rd*
4	Roods - -	1 Acre	-	*Acr*

Sq Inc		*Sq Ft*								
144	=	1		*Sq Yd*						
1296	=	9	=	1		*Sq Pl*				
39204	=	272¼	=	30¼	=	1		*Rd*		
1568160	=	10890	=	1210	=	40	=	1		*Acr*
6272640	=	43560	=	4840	=	160	=	4	=	1

By this measure, Land, and Husbandmen and Gardeners' work are measured; also Artificers' work, such as Board, Glass, Pavements, Plastering, Wainscoting, Tiling, Flooring, and every dimension of length and breadth only.

When three dimensions are concerned, namely, length, breadth, and depth or thickness, it is called cubic or solid measure, which is used to measure Timber, Stone, &c.

The cubic or solid Foot, which is 12 inches in length and breadth and thickness, contains 1728 cubic or solid inches, and 27 solid feet make one solid yard.

DRY, OR CORN MEASURE.

2 Pints make	1 Quart	- -	*Qt*
2 Quarts -	1 Pottle	- -	*Pot*
2 Pottles -	1 Gallon	- -	*Gal*
2 Gallons -	1 Peck	- -	*Pec*
4 Pecks -	1 Bushel	- -	*Bu*
8 Bushels	1 Quarter	- -	*Qr*
5 Quarters	1 Wey, Load, or Ton		*Wey*
2 Weys - -	1 Last	- -	*Last*

Pts		*Gal*										
8	=	1		*Pec*								
16	=	2	=	1		*Bu*						
64	=	8	=	4	=	1		*Qr*				
512	=	64	=	32	=	8	=	1		*Wey*		
2560	=	320	=	160	=	40	=	5	=	1		*Last*
5120	=	640	=	320	=	80	=	10	=	2	=	1

By this are measured all dry wares, as, Corn, Seeds, Roots, Fruits, Salt, Coals, Sand, Oysters, &c.

The standard Gallon dry-measure contained $268\frac{4}{5}$ cubic or solid inches, and the corn or Winchester bushel $2150\frac{2}{5}$ cubic inches; for the dimensions of the Winchester bushel, by the old Statute, were 8 inches deep, and $18\frac{1}{2}$ inches wide or in diameter. But the Coal bushel was to be $19\frac{1}{2}$ inches in diameter; and 36 bushels, heaped up, made a London chaldron of coals, the weight of which was 3136 lb Avoirdupois, or 1 ton 8 cwt nearly. See, however, page 29.

ALE AND BEER MEASURE.

2 Pints make	-	1 Quart	-	*Qt*
4 Quarts	-	1 Gallon	-	*Gal*
36 Gallons	-	1 Barrel	-	*Bar*
1 Barrel and a half		1 Hogshead	-	*Hhd*
2 Barrels	-	1 Puncheon	-	*Pun*
2 Hogsheads	-	1 Butt	-	*Butt*
2 Butts	-	1 Tun	-	*Tun*

Pts	*Qt*				
2 =	1	*Gal*			
8 =	4 =	1	*Bar*		
288 =	144 =	36 =	1	*Hhd*	
432 =	216 =	54 =	$1\frac{1}{2}$ =	1	*Butt*
864 =	432 =	108 =	3 =	2 =	1

Note. The Ale Gallon contained 282 cubic or solid inches, by which also milk was measured.

WINE MEASURE.

2 Pints make	-	1 Quart	-	*Qt*
4 Quarts	-	1 Gallon	-	*Gal*
42 Gallons	-	1 Tierce	-	*Tier*
63 Gallons or $1\frac{1}{2}$ Tierces	-	1 Hogshead	-	*Hhd*
2 Tierces	-	1 Puncheon	-	*Pun*
2 Hogsheads	-	1 Pipe or Butt		*Pi*
2 Pipes or 4 Hhds	-	1 Tun	-	*Tun*

Pts	*Qt*						
2 =	1	*Gal*					
8 =	4 =	1	*Tier*				
336 =	168 =	42 =	1	*Hhd*			
504 =	252 =	63 =	$1\frac{1}{2}$ =	1	*Pun*		
672 =	336 =	84 =	2 =	$1\frac{1}{3}$ =	1	*Pi*	
1008 =	504 =	126 =	3 =	2 =	$1\frac{1}{2}$ =	1	*Tun*
2016 =	1008 =	252 =	6 =	4 =	3 =	2 =	1

Note, by this are measured all Wines, Spirits, Strong-waters, Cyder, Mead, Perry, Vinegar, Oil, Honey, &c.

The old Wine Gallon contained 231 cubic or solid inches. And it is remarkable that these Wine and Ale Gallons have the same proportion to each other, as the Troy and Avoirdupois Pounds have; that is, as one Pound Troy is to one Pound Avoirdupois, so is one Wine Gallon to one Ale Gallon.

OF TIME.

60 Seconds or 60″ make	- 1 Minute	-	*M* or ′
60 Minutes	- 1 Hour	-	*Hr*
24 Hours	- 1 Day	-	*Day*
7 Days	- 1 Week	-	*Wk*
4 Weeks	- 1 Month	-	*Mo*
13 Months 1 Day 6 Hours, or 365 Days 6 Hours	1 Julian Year	-	*Yr*

Sec	*Min*	*Hr*	*Day*	*Wk*	*Mo*	
60 =	1					
3600 =	60 =	1				
86400 =	1440 =	24 =	1			
604800 =	10080 =	168 =	7 =	1		
2419200 =	40320 =	672 =	28 =	4 =	1	
31557600 =	525960 =	8766 =	365¼		=	1 *Year*

	Wk	*Da*	*Hr*	*Mo*	*Da*	*Hr*	
Or	52	1	6 =	13	1	6 =	1 *Julian Year*

	Da	*Hr*	*M*	*Sec*	
But	365	5	48	45½	= *Solar Year*

IMPERIAL MEASURES.

By the late Act of Parliament for Uniformity of Weights and Measures, which commenced its operation on the 1st of January, 1826, the chief part of the weights and measures are allowed to remain as they were; the Act simply prescribing scientific modes of determining them, in case they should be lost.

The pound *troy* contains 5760 grains.
The pound *avoirdupois* contains 7000 grains.
The *imperial gallon* contains 277·274 cubic inches.
The *corn bushel*, eight times the above.

Hence, with respect to Ale, Wine, and Corn, it will be expedient to possess a

TABLE OF FACTORS,

For converting old measures into new, and the contrary.

	By decimals.			By vulgar fractions nearly.		
	Corn Measure.	Wine Measure.	Ale Measure.	Corn Measure.	Wine Measure.	Ale Measure.
To convert old measures to new.	·96943	·83311	1·01704	$\frac{31}{32}$	$\frac{5}{6}$	$\frac{60}{59}$
To convert new measures to old.	1·03153	1·20032	·98324	$\frac{32}{31}$	$\frac{6}{5}$	$\frac{59}{60}$

N. B. For reducing the *prices*, these numbers must all be reversed.

RULES FOR REDUCTION.

I. *When the Numbers are to be reduced from a Higher Denomination to a Lower:*

Multiply the number in the highest denomination by as many of the next lower as make an integer, or 1, in that higher; to this product add the number, if any, which was in this lower denomination before, and set down the amount.

Reduce this amount in like manner, by multiplying it by as many of the next lower as make an integer of this, taking in the odd parts of this lower, as before. And so proceed through all the denominations to the lowest; so shall the number last found be the value of all the numbers which were in the higher denominations, taken together*.

---

* The reason of this rule is very evident; for pounds are brought into shillings by multiplying them by 20; shillings into pence, by multiplying them by 12; and pence into farthings, by multiplying by 4; and the reverse of this rule by division.—And the same, it is evident, will be true in the reduction of numbers consisting of any denominations whatever.

EXAMPLE.

1. In 1234*l* 15*s* 7*d*, how many farthings?

```
      l    s   d
   1234   15   7
     20
 ------
  24695 Shillings
     12
 ------
 296347 Pence
      4
 -------
Answer 1185388 Farthings.
 -------
```

II. *When the Numbers are to be reduced from a Lower Denomination to a Higher:*

Divide the given number by as many of that denomination as make 1 of the next higher, and set down what remains, as well as the quotient.

Divide the quotient by as many of this denomination as make 1 of the next higher; setting down the new quotient, and remainder, as before.

Proceed in the same manner through all the denominations to the highest; and the quotient last found, together with the several remainders, if any, will be of the same value as the first number proposed.

EXAMPLES.

2. Reduce 1185388 farthings into pounds, shillings, and pence.

```
     4) 1185388
        -------
    12)  296347d
        -------
   2,0)  2469,5s—7d
        -------
Answer  1234l 15s 7d
        -------
```

3. Reduce 24*l* to farthings. Ans. 23040.

4. Reduce 337587 farthings to pounds, &c.

Ans. 351*l* 13*s* 0¾.

5. How many farthings are in 36 guineas? Ans. 36288.
6. In 36288 farthings how many guineas? Ans. 36.
7. In 59 lb 13 dwts 5 gr. how many grains? Ans. 340157.
8. In 8012131 grains how many pounds, &c.
Ans. 1390 lb 11 oz 18 dwt 19 gr.
9. In 35 ton 17 cwt 1 qr 23 lb 7 oz 13 dr how many drams?
Ans. 20571005.

10. How many barley-corns will reach round the earth, supposing it, according to the best calculations, to be 25000 miles? Ans. 4752000000.

11. How many seconds are in a solar year, or 365 days 5 hrs 48 min. 45½ sec? Ans. 31556925½.

12. In a lunar month, or 29 ds 12 hrs 44 min 3 sec, how many seconds? Ans. 2551443.

COMPOUND ADDITION.

Compound Addition shows how to add or collect several numbers of different denominations into one sum.

Rule.—Place the numbers so, that those of the same denomination may stand directly under each other, and draw a line below them. Add up the figures in the lowest denomination, and find, by Reduction, how many units, or ones, of the next higher denomination are contained in their sum. —Set down the remainder below its proper column, and carry those units or ones to the next denomination, which add up in the same manner as before.—Proceed thus through all the denominations, to the highest, whose sum, together with the several remainders, will give the answer sought.

The method of proof is the same as in Simple Addition.

EXAMPLES OF MONEY.

1.			2.			3.			4.		
l	*s*	*d*	*l*	*s*	*d*	*l*	*s*	*d*	*l*	*s*	*d*
7	13	3	14	7	5	15	17	10	53	14	8
3	5	$10\frac{1}{2}$	8	19	$2\frac{1}{4}$	3	14	6	5	10	$2\frac{3}{4}$
6	18	7	7	8	$1\frac{1}{2}$	23	6	$2\frac{3}{4}$	93	11	6
0	2	$5\frac{3}{4}$	21	2	9	14	9	$4\frac{1}{2}$	7	5	0
4	0	3	7	16	$8\frac{1}{2}$	15	6	4	13	2	5
17	15	$4\frac{1}{2}$	0	4	3	6	12	$9\frac{3}{4}$	0	18	7
39	15	$9\frac{3}{4}$									
32	2	$6\frac{3}{4}$									
39	15	$9\frac{3}{4}$									

5.			6.			7.			8.		
l	*s*	*d*	*l*	*s*	*d*	*l*	*s*	*d*	*l*	*s*	*d*
14	0	$7\frac{1}{4}$	37	15	8	61	3	$2\frac{1}{2}$	472	15	3
8	15	3	14	12	$9\frac{3}{4}$	7	16	8	9	2	$2\frac{1}{2}$
62	4	7	17	14	9	29	13	$10\frac{3}{4}$	27	12	$6\frac{1}{4}$
4	17	8	23	10	$9\frac{1}{4}$	12	16	2	370	16	$2\frac{1}{2}$
23	0	$4\frac{3}{4}$	8	6	0	0	7	$5\frac{1}{4}$	13	7	4
6	6	7	14	0	$5\frac{1}{2}$	24	13	0	6	10	$5\frac{1}{4}$
91	0	$10\frac{1}{4}$	54	2	$7\frac{1}{2}$	5	0	$10\frac{3}{4}$	30	0	$11\frac{3}{4}$

EXAM. 9. A nobleman, going out of town, is informed by his steward, that his butcher's bill comes to 197*l* 13*s* $7\frac{1}{2}$*d*; his baker's to 59*l* 5*s* $2\frac{3}{4}$*d*; his brewer's to 85*l*; his wine-merchant's to 103*l* 13*s*; to his corn chandler is due 75*l* 3*d*; to his tallow-chandler and cheesemonger, 27*l* 15*s* $11\frac{1}{4}$*d*; and to his tailor 55*l* 3*s* $5\frac{3}{4}$*d*; also for rent, servants' wages, and other charges, 127*l* 3*s*: Now, supposing he would take 100*l* with him, to defray his charges on the road, for what sum must he send to his banker? Ans. 830*l* 14*s* $6\frac{1}{4}$*d*.

10. The strength of a regiment of foot, of 10 companies, and the amount of their subsistence*, for a month of 30 days, according to the annexed Table, are required?

Numb.	Rank.	Subsistence for a Month.
1	Colonel	£27 0 0
1	Lieutenant Colonel	19 10 0
1	Major	17 5 0
7	Captains	78 15 0
11	Lieutenants	57 15 0
9	Ensigns	40 10 0
1	Chaplain	7 10 0
1	Adjutant	4 10 0
1	Quarter-Master	5 5 0
1	Surgeon	4 10 0
1	Surgeon's Mate	4 10 0
30	Serjeants	45 0 0
30	Corporals	30 0 0
20	Drummers	20 0 0
2	Fifers	2 0 0
390	Private Men	292 10 0
507	Total.	£656 10 0

* Subsistence Money, is the money paid to the soldiers weekly; which is short of their full pay, because their clothes, accoutrements, &c. are to be accounted for. It is likewise the money advanced to officers till their accounts are made up, which is commonly once a year, when they are paid their arrears. The following table shows the full pay and subsistence of each rank on the English establishment.

COMMISSIONED OFFICERS' REGIMENTAL PAY. 1824.

	Life Guards.		Horse Guards.		Foot Guards.		Dr. Gds. & Drs.	R. Wag. Train.	Foot.	R. Staff. Corps.	Royal Artillery.		Royal Eng.	Royal Marines	Militia and Fencib.
	Subsistence per Diem nett.	Gross Pay and Allowance per Diem as borne on the Establishment.	Subsistence per Diem nett.	Gross Pay and Allowance per Diem as borne on the Establishment.	Subsistence per Diem nett.	Gross Pay and Allowance per Diem as borne on the Establishment.					Marching and Invalid Battalion.	Horse Brigade.			
Colonel Commandant	...	...	...	...	...	...	...	...	...	...	2 14 9½	3 0 0	2 14 9½	2 13 0	...
Colonel	1 7 0	1 16 0	1 11 0	2 1 0	1 10 0	1 19 0	1 12 10	...	1 2 6	1 16 10	1 6 3	1 12 0	1 6 3	1 2 6	1 2 6
Lieutenant Colonel	1 3 3	1 11 0	1 2 6	1 9 6	1 1 6	1 8 6	1 3 0	1 12 10	0 17 0	1 3 0	0 18 1	1 7 1	{ 16 1 / 18 1	} 17 0	0 15 11
Major	0 19 6	1 6 0	1 1 6	1 7 0	0 18 6	1 4 6	0 19 3	0*13 0	0 16 0	0 19 3	0 16 11	1 2 11	...	0 16 0	0 14 1
Captain	0 12 0	0 16 0	0 16 6	1 1 6	0 12 6	0 16 6	0 14 7	0*12 6	0 10 6	0 14 7	0 11 1	0 16 1	0 11 1	0 10 6	0 10 6
Ditto with Brevet of Major, or superior rank	...	...	...	...	...	...	...	0*14 6	0 12 6	...	0 13 1	0 18 1	0 13 1	0 12 6	...
Lieutenant	0 8 3	0 11 0	0 11 6	0 15 0	0 6 0	0 7 10	0 9 0	0 *8 6	0 6 6	0 9 0	0 6 10	0 9 10	0 6 10	0 6 6	0 §6 0
Ditto above 7 years standing	...	...	...	...	...	...	...	0 *9 6	0 7 6	...	0 7 10	0 10 10	0 7 10	0 7 6	...
Cornet, Ensign, and 2d Lieut.	0 7 3	0 8 6	0 11 0	0 11 0	0 4 6	0 5 10	0 8 0	0 *7 3	0 5 3	0 8 0	0 5 7	...	0 5 7	0 5 3	0 §5 3
Paymaster	...	...	...	...	...	...	0 15 0	0 15 0	0 15 0	0 15 0	...	...	...	...	0‡15 0
Adjutant	0 13 0	0 13 0	0 10 0	0 10 0	0 10 0	0 10 0	0 10 0	0 10 0	0 8 6	0 10 0	0 8 †6	0 10 6	0 6 0	0 8 6	0 8 6
Quarter Master	0 4 9	0 6 0	0 6 6	0 8 6	0 6 6	0 6 6	0 *8 0	...	0 6 6	0 6 6	0 7 10	0 10 10	0 6 0	0 6 6	0 8 6
Surgeon Major	...	...	...	...	0 16 8	1 0 0	...	...	...	...	...	...	...	...	...
Battalion Surgeon	...	...	...	...	0 10 0	0 12 0	...	...	...	...	...	...	...	...	...
Surgeon	0 9 0	0 12 0	0 9 0	0 12 0	...	...	0 11 4	0 11 4	0 11 4	0 11 4	0 11 4	...	...	...	0 11 4
Assistant Surgeon	0 8 6	0 8 6	0 8 6	0 8 6	0 7 6	0 7 6	0 8 6	...	0 7 6	0 7 6	0 7 6	...	...	...	...
Surgeon's Mate	...	...	...	...	...	...	...	...	...	...	...	...	...	...	0 §5 0
Veterinary Surgeon	0 8 0	0 8 0	0 8 0	0 8 0	...	...	0 8 0	...	...	...	...	...	...	...	...

The difference between the Subsistence and Gross Pay of the Officers of these Regiments, after deducting Poundage, Hospital, and Agency, is paid as "*Arrears*."

* These rates include 2*s.* a day for a horse. † If 2d Captain 12*s.* 8*d.*
‡ Including pay as a Subaltern.
§ If holding another Commission, Lieutenants of Militia receive only 4*s.* 8*d.* Ensigns 3*s.* 8*d.*, and Surgeons' Mates 3*s.* 6*d.*

MEM.—Regimental Surgeons of the Line, those of the Royal Artillery, and Veterinary Surgeons, after certain periods of Service, receive the following Rates of Pay, viz.

Surgeons of the Line and Royal Artillery.	*s.*	*d.*	
After 7 years service	14	1	per diem.
— 20 ditto	18	10	ditto.

Veterinary Surgeons.	*s.*	
After 3 years service	10	per diem.
— 10 ditto	12	ditto
— 20 ditto		

EXAMPLES OF WEIGHTS, MEASURES, &c.

TROY WEIGHT.

1.

lb	oz	dwt.
17	3	15
7	9	4
0	10	7
9	5	0
176	2	17
23	11	12

2.

oz	dwt	gr
37	9	3
9	5	3
8	12	12
17	7	8
5	9	0
3	0	19

APOTHECARIES' WEIGHT.

3.

lb	oz	dr	sc
3	5	7	2
13	7	3	0
19	10	6	2
0	9	1	2
36	3	5	0
5	8	6	1

4.

oz	dr	sc	gr
3	5	1	17
7	3	2	5
16	7	0	12
7	3	2	9
4	1	2	18
36	4	1	14

AVOIRDUPOIS WEIGHT.

5.

lb	oz	dr
17	10	13
5	14	8
12	9	18
27	1	6
0	4	0
6	14	10

6.

cwt	qr	lb
15	2	15
6	3	24
9	1	14
9	1	17
10	2	6
3	0	3

LONG MEASURE.

7.

mls	fur	pls
29	3	14
19	6	29
7	0	24
9	1	37
7	0	3
4	5	9

8.

yds	feet	inc
127	1	5
12	2	9
10	0	10
54	1	11
5	2	7
23	0	5

CTOTH MEASURE.

9.

yds	qr	nls
26	3	1
13	1	2
9	1	2
217	0	3
9	1	0
55	3	1

10

el en	qrs	nls
270	1	0
57	4	3
18	1	2
6	3	2
10	1	0
4	4	1

LAND MEASURE.

11.

ac	ro	p
225	3	37
16	1	25
7	2	18
4	2	9
42	1	19
7	0	6

12.

ac	ro	p
19	0	16
270	3	29
6	3	13
23	0	34
7	2	16
75	0	23

WINE MEASURE.

13.

t	hds	gal
13	3	15
8	1	37
14	1	20
25	0	12
3	1	9
72	3	21

14.

hds	gal	pts
15	61	5
17	14	13
29	23	7
3	15	1
16	8	0
4	96	6

ALE AND BEER MEASURE.

15.

hds	gal	pts
17	37	3
9	10	15
3	6	2
5	14	0
12	9	6
8	42	4

16.

hds	gal	pts
29	43	5
12	19	7
14	16	6
6	8	1
57	13	4
5	6	0

COMPOUND SUBTRACTION.

COMPOUND SUBTRACTION shows how to find the difference between any two numbers of different denominations. To perform which, observe the following Rule.

* PLACE the less number below the greater, so that the parts of the same denomination may stand directly under each other; and draw a line below them.—Begin at the right-hand, and subtract each number or part in the lower line, from the one just above it, and set the remainder straight below it.—But if any number in the lower line be greater than that above it, add as many to the upper number as make 1 of the next higher denomination; then take the lower number from the upper one thus increased, and set down the remainder. Carry the unit borrowed to the next number in the lower line; after which subtract this number from the one above it, as before; and so proceed till the whole is finished. Then the several remainders, taken together, will be the whole difference sought.

The method of proof is the same as in Simple Subtraction.

EXAMPLES OF MONEY.

	1.			2.			3.			4.		
	l	*s*	*d*	*l*	*s*	*d*	*l*	*s*	*d*	*l*	*s*	*d*
From	79	17	8¾	103	3	2½	81	10	11	254	12	0
Take	35	12	4¼	71	12	5¾	29	13	3¼	37	9	4¾
Rem.	44	5	4½	31	10	8¾						
Proof	79	17	8¾	103	3	2½						

5. What is the difference between 73*l* 5¼*d* and 19*l* 13*s* 10*d*?
Ans. 53*l* 6*s* 7¼*d*.

* The reason of this Rule will easily appear from what has been said in Simple Subtraction; for the borrowing depends on the same principle, and is only different as the numbers to be subtracted are of different denominations.

Ex. 6. A lends to B 100*l*, how much is B in debt after A has taken goods of him to the amount of 73*l* 12*s* 4$\frac{3}{4}$*d*?

Ans. 26*l* 7*s* 7$\frac{1}{4}$*d*.

7. Suppose that my rent for half a year is 20*l* 12*s*, and that I have laid out for the land-tax 14*s* 6*d*, and for several repairs 1*l* 3*s* 3$\frac{1}{4}$*d*, what have I to pay of my half-year's rent?

Ans. 18*l* 14*s* 2$\frac{3}{4}$*d*.

8. A trader, failing, owes to A 35*l* 7*s* 6*d*, to B 91*l* 13*s* $\frac{1}{2}$*d*, to C 53*l* 7$\frac{1}{4}$*d*, to D 87*l* 5*s*, and to E 111*l* 3*s* 5$\frac{3}{4}$*d*. When this happened, he had by him in cash 23*l* 7*s* 5*d*, in wares 53*l* 11*s* 10$\frac{1}{4}$*d*, in household furniture 63*l* 17*s* 7$\frac{3}{4}$*d*, and in recoverable book-debts 25*l* 7*s* 5*d*. What will his creditors lose by him, supposing these things delivered to them?

Ans. 212*l* 5*s* 3$\frac{1}{2}$*d*.

EXAMPLES OF WEIGHTS, MEASURES, &c.

TROY WEIGHT.

	1.				2.			
	lb	oz	dwt	gr	lb	oz	dwt	gr
From	9	2	12	10	7	10	4	17
Take	5	4	6	17	3	7	16	12
Rem.								
Proof								

APOTHECARIES' WEIGHT.

	3.				
	lb	oz	dr	scr	gr
From	73	4	7	0	14
Take	29	5	3	4	19
Rem.					
Proof					

AVOIRDUPOIS WEIGHT.

	4.			5.		
	c	qrs	lb	lb	oz	dr
From	5	0	17	71	5	9
Take	2	3	10	17	9	18
Rem.						
Proof						

LONG MEASURE.

	6.			7.		
	m	fu	pl	yd	ft	in
From	14	3	17	96	0	4
Take	7	6	11	72	2	9
Rem.						
Proof						

CLOTH MEASURE.

	8.			9.		
	yd	qr	nl	yd	qr	nl
From	17	2	1	9	0	2
Take	9	0	2	7	2	1
Rem.						
Proof						

LAND MEASURE.

	10.			11.		
	ac	ro	p	ac	ro	p
From	17	1	14	57	1	16
Take	16	2	8	22	3	29
Rem.						
Proof						

	WINE MEASURE.						ALE AND BEER MEASURE.					
	12.			13.			14.			15.		
	t	hd	gal	hd	gal	pt	hd	gal	pt	hd	gal	pt
From	17	2	23	5	0	4	14	29	3	71	16	5
Take	9	1	36	2	12	6	9	35	7	19	7	1
Rem.												
Proof												

	DRY MEASURE.						TIME.					
	16.			17.			18.			19.		
	la	qr	bu	bu	gal	pt	mo	we	da	ds	hrs	min
From	9	4	7	13	7	1	71	2	5	114	17	26
Take	6	3	5	9	2	7	17	1	6	72	10	37
Rem.												
Proof												

20. The line of defence in a certain polygon being 236 yards, and that part of it which is terminated by the curtain and shoulder being 146 yards 1 foot 4 inches; what then was the length of the face of the bastion? Ans. 89 yds 1 ft 8 in.

COMPOUND MULTIPLICATION.

Compound Multiplication shows how to find the amount of any given number of different denominations repeated a certain proposed number of times; which is performed by the following rule.

Set the multiplier under the lowest denomination of the multiplicand, and draw a line below it.—Multiply the number in the lowest denomination by the multiplier, and find how many units of the next higher denomination are contained in the product, setting down what remains.—In like manner, multiply the number in the next denomination, and to the product carry or add the units, before found, and find how many units of the next higher denomination are in this amount, which carry in like manner to the next product, setting down the overplus.—Proceed thus to the highest denomination proposed: so shall the last product, with the several remainders, taken as one compound number, be the whole amount required.—The method of Proof, and the reason of the Rule, are the same as in Simple Multiplication.

EXAMPLES OF MONEY.

1. To find the amount of 8 lb of Tea, at 5*s*. 8½*d*. per lb.

	s	*d*	
	5	8½	
		8	
£2	5	8	Answer.

			l	*s*	*d*
2.	4 lb of Tea, at 7*s* 8*d* per lb.	Ans.	1	10	8
3.	6 lb of Butter, at 9½*d* per lb.	Ans.	0	4	9
4.	7 lb of Tobacco, at 1*s* 8½*d* per lb.	Ans.	0	11	11½
5.	8 stone of Beef, at 2*s* 7½*d* per st.	Ans.	1	1	0
6.	10 cwt cheese, at 2*l* 17*s* 10*d* per cwt.	Ans.	28	18	4
7.	12 cwt of Sugar, at 3*l* 7*s* 4*d* per cwt.	Ans.	40	8	0

CONTRACTIONS.

I. If the multiplier exceed 12, multiply successively by its component parts, instead of the whole number at once.

EXAMPLES.

1. 15 cwt of Cheese, at 17*s* 6*d* per cwt.

l	*s*	*d*	
0	17	6	
		3	
2	12	6	
		5	
13	2	6	Answer.

			l	*s*	*d*
2.	20 cwt of Hops, at 4*l* 7*s* 2*d* per cwt.	Ans.	87	3	4
3.	24 tons of Hay, at 3*l* 7*s* 6*d* per ton.	Ans.	81	0	0
4.	45 ells of Cloth, at 1*s* 6*d* per ell.	Ans.	3	7	6

			l	s	d
Ex. 5.	63 gallons of Oil, at 2s 3d per gall.	Ans.	7	1	9
6.	70 barrels of Ale, at 1l 4s per barrel.	Ans.	84	0	0
7.	84 quarters of Oats, at 1l 12s 8d per qr.	Ans.	137	4	0
8.	96 quarters of Barley, at 1l 3s 4d per qr.	Ans.	112	0	0
9.	120 days' Wages, at 5s 9d per day.	Ans.	34	10	0
10.	144 reams of Paper, at 13s 4d per ream.	Ans.	96	0	0

II. If the multiplier cannot be exactly produced by the multiplication of simple numbers, take the nearest number to it, either greater or less, which can be so produced, and multiply by its parts, as before.—Then multiply the given multiplicand by the difference between this assumed number and the multiplier, and add the product to that before found, when the assumed number is less than the multiplier, but subtract the same when it is greater.

EXAMPLES.

1. 26 yards of Cloth, at 3s 0¾d per yard.

l	s	d	
0	3	0¾	
		5	
0	15	3¾	
		5	
3	16	6¾	
	3	0¾	add
£3	19	7½	Answer.

EXAMPLES OF WEIGHTS AND MEASURES.

2. 29 quarters of Corn, at 2l 5s 3¼d per qr. Ans. 65 12 10¼
3. 53 loads of Hay, at 3l 15s 2d per ld. Ans. 199 3 10
4. 79 bushels of Wheat, at 11s 5¾d per bush. Ans. 45 6 10¼
5. 97 casks of Beer, at 12s 2d per cask. Ans. 59 0 2
6. 114 stone of Meat, at 15s 3¾d per st. Ans. 87 5 7½

1.

lb	oz	dwt	gr
28	7	14	10
			5

2.

lb	oz	dr	sc	gr
2	6	3	2	10
				8

3.

cwt	qr	lb	oz
29	2	16	14
			12

4.

mls	fu	pls	yds
22	5	20	6
			4

5.

yds	qrs	na
126	3	1
		7

6.

ac	ro	po
28	3	27
		9

7.

tuns	hhd	gal	pts
20	2	26	2
			3

8.

we	qr	bu	pe
24	2	5	3
			6

9.

mo	we	da	ho	min
172	3	5	16	49
				10

COMPOUND DIVISION.

Compound Division teaches how to divide a number of several denominations by any given number, or into any number of equal parts; as follows:

Place the divisor on the left of the dividend, as in Simple Division.—Begin at the left-hand, and divide the number of the highest denomination by the divisor, setting down the quotient in its proper place.—If there be any remainder after this division, reduce it to the next lower denomination, which add to the number, if any, belonging to that denomination, and divide the sum by the divisor.—Set down again this quotient, reduce its remainder to the next lower denomination again, and so on through all the denominations to the last.

EXAMPLES OF MONEY.

1. Divide 237*l* 8*s* 6*d* by 2.

	l	*s*	*d*	
2)	237	8	6	
	£118	14	3	the Quotient.

		l	*s*	*d*			*l*	*s*	*d*
2.	Divide	432	12	1¾	by 3.	Ans.	144	4	0½
3.	Divide	507	3	5	by 4.	Ans.	126	15	10¼
4.	Divide	632	7	6¼	by 5.	Ans.	126	9	6
5.	Divide	690	14	3¼	by 6.	Ans.	115	2	4½
6.	Divide	705	10	2	by 7.	Ans.	100	15	8¾
7.	Divide	760	5	6	by 8.	Ans.	95	0	8¼
8.	Divide	761	5	7¾	by 9.	Ans.	84	11	8¾
9.	Divide	829	17	10	by 10.	Ans.	82	19	9¼
10.	Divide	937	8	8¾	by 11.	Ans.	85	4	5
11.	Divide	1145	11	4½	by 12.	Ans.	95	9	3¼

CONTRACTIONS.

I. If the divisor exceed 12, find what simple numbers, multiplied together, will produce it, and divide by them separately, as in Simple Division, as below.

EXAMPLES.

1. What is Cheese per cwt, if 16 cwt cost 25*l* 14*s* 8*d*?

	l	*s*	*d*
4)	25	14	8
4)	6	8	8
£	1	12	2 the Answer.

		l	*s*	*d*
2. If 20 cwt of Tobacco come to 150*l* 6*s* 8*d*, what is that per cwt?	Ans.	7	10	4
3. Divide 98*l* 8*s* by 36.	Ans.	2	14	8
4. Divide 71*l* 13*s* 10*d* by 56.	Ans.	1	5	7¼
5. Divide 44*l* 4*s* by 96.	Ans.	0	9	2½
6. At 31*l* 10*s* per cwt, how much per lb?	Ans.	0	5	7½

II. If the divisor cannot be produced by the multiplication of small numbers, divide by the whole divisor at once, after the manner of Long division, as follows.

EXAMPLES.

1. Divide 59*l* 6*s* $3\frac{3}{4}$*d* by 19.

```
     l    s   d       l   s   d
19) 59    6   3¾     (3   2   5¼  Ans.
    57
    ——
     2
    20
    ——
    46 (2
    38
    ——
     8
    12
    ——
    99 (5
    95
    ——
     4
     4
    ——
    19 (1
```

	l	*s*	*d*				*l*	*s*	*d*
2. Divide	39	14	$5\frac{1}{4}$	by	57.	Ans.	0	13	$11\frac{1}{4}$
3. Divide	125	4	9	by	48.	Ans.	2	18	3
4. Divide	542	7	10	by	97.	Ans.	5	11	10
5. Divide	123	11	$2\frac{1}{2}$	by	127.	Ans.	0	19	$5\frac{1}{2}$

EXAMPLES OF WEIGHTS AND MEASURES.

1. Divide 17 lb 9 oz 0 dwts 2 gr by 7.
   Ans. 2 lb 6 oz 8 dwts 14 gr.
2. Divide 17 lb 5 oz 2 dr 1 scr 4 gr by 12.
   Ans. 1 lb 5 oz 3 dr 1 scr 12 gr.
3. Divide 178 cwt 3 qrs 14 lb by 53. Ans. 3 cwt 1 qr 14 lb.
4. Divide 144 mi 4 fur 20 po 1 yd 2 ft 0 in by 39.
   Ans. 3 mi 5 fur 26 po 0 yds 2 ft 8 in.
5. Divide 534 yds 2 qrs 2 na by 47. Ans. 11 yds 1 qr 2 na.
6. Divide 77 ac 1 ro 33 po by 51. Ans. 1 ac 2 ro 3 po.
7. Divide 2 tu 0 hhds 47 gal 7 pi by 65. Ans. 27 gal 7 pi.
8. Divide 387 la 9 qr by 72. Ans. 5 la 3 qrs 7 bu.
9. Divide 206 mo 4 da by 26. Ans. 7 mo 3 we 5 ds.

THE GOLDEN RULE, OR RULE OF THREE.

THE RULE OF THREE teaches how to find a fourth proportional to three numbers given: for which reason it is sometimes called the Rule of Proportion. It is called the Rule of Three, because three terms or numbers are given, to find a fourth. And because of its great and extensive usefulness, it is often called the Golden Rule. This Rule is usually by practical men considered as of two kinds, namely, Direct and Inverse. The distinction, however, as well as the manner of stating, though retained here for practical purposes, does not well accord with the principles of proportion; as will be shown farther on.

The Rule of Three Direct is that in which more requires more, or less requires less. As in this; if three men dig 21 yards of trench in a certain time, how much will six men dig in the same time? Here more requires more, that is, 6 men, which are more than three men, will also perform more work, in the same time. Or when it is thus: if 6 men dig 42 yards, how much will 3 men dig in the same time? Here then, less requires less, or 3 men will perform proportionably less work than 6 men, in the same time. In both these cases then, the Rule, or the Proportion, is Direct; and the stating must be

thus, as 3 : 21 : : 6 : 42, or as 3 : 6 : : 21 : 42.
And, as 6 : 42 : : 3 : 21, or as 6 : 3 : : 42 : 21.

But the Rule of Three Inverse, is when more requires less, or less requires more. As in this: if 3 men dig a certain quantity of trench in 14 hours, in how many hours will 6 men dig the like quantity? Here it is evident that 6 men, being more than 3, will perform an equal quantity of work in less time, or fewer hours. Or thus: if 6 men perform a certain quantity of work in 7 hours, in how many hours will 3 men perform the same? Here less requires more, for 3 men will take more hours than 6 to perform the same work. In both these cases then the Rule, or the Proportion, is Inverse; and the stating must be

thus, as 6 : 14 : : 3 : 7, or as 6 : 3 : : 14 : 7.
And, as 3 : 7 : : 6 : 14, or as 3 : 6 : : 7 : 14.

And in all these statings, the fourth term is found, by multiplying the 2d and 3d terms together, and dividing the product by the 1st term.

Of the three given numbers: two of them contain the supposition, and the third a demand. And for stating and working questions of these kinds, observe the following general Rule:

State the question by setting down in a straight line the three given numbers, in the following manner, viz. so that the 2nd term be that number of supposition which is of the same kind that the answer or 4th term is to be; making the other number of supposition the 1st term, and the demanding number the 3d term, when the question is in direct proportion; but contrariwise, the other number of supposition the 3d term, and the demanding number the 1st term, when the question has inverse proportion.

Then, in both cases, multiply the 2d and 3d terms together, and divide the product by the 1st, which will give the answer, or 4th term sought, viz. of the same denomination as the second term.

Note, If the first and third terms consist of different denominations, reduce them both to the same: and if the second term be a compound number, it is mostly convenient to reduce it to the lowest denomination mentioned.—If, after division, there be any remainder, reduce it to the next lower denomination, and divide by the same divisor as before, and the quotient will be of this last denomination. Proceed in the same manner with all the remainders, till they be reduced to the lowest denomination which the second admits of, and the several quotients taken together will be the answer required.

Note also, The reason for the foregoing Rules will appear, when we come to treat of the nature of Proportions.—Sometimes two or more statings are necessary, which may always be known from the nature of the question.

EXAMPLES.

1. If 8 yards of Cloth cost 1*l* 4*s*, what will 96 yards cost?

```
   yds  l  s    yds   l  s
As  8 : 1  4 :: 96 : 14  8 the Answer.
          20
        ----
          24
          96
        ----
         144
        216
        ----
      8)2304
        ----
   2,0) 28,8s
        ----
        £14 8 Answer.
```

Ex. 2. An engineer having raised 100 yards of a certain work in 24 days with 5 men; how many men must he employ to finish a like quantity of work in 15 days?

```
   ds men   ds men
As 15 : 5 : : 24 : 8 Ans.
                5
              ----
         15) 120 (8 Answer.
             120
             ---
```

3. What will 72 yards of cloth cost, at the rate of 9 yards for 5*l* 12*s*? Ans. 44*l* 16*s*.

4. A person's annual income being 146*l*; how much is that per day? Ans. 8*s*.

5. If 3 paces or common steps of a certain person be equal to 2 yards, how many yards will 160 of his paces make? Ans. 106 yds 2 ft.

6. What length must be cut off a board, that is 9 inches broad, to make a square foot, or as much as 12 inches in length and 12 in breadth contains? Ans. 16 inches.

7. If 750 men require 22500 rations of bread for a month; how many rations will a garrison of 1000 men require? Ans. 30000.

8. If 7 cwt 1 qr. of sugar cost 26*l* 10*s* 4*d*; what will be the price of 43 cwt 2 qrs? Ans. 159*l* 2*s*.

9. The clothing of a regiment of foot of 750 men amounting to 2831*l* 5*s*; what will the clothing of a body of 3500 men amount to? Ans. 13212*l* 10*s*.

10. How many yards of matting, that is 3 ft broad, will cover a floor that is 27 feet long and 20 feet broad? Ans. 60 yards.

11. What is the value of six bushels of coals, at the rate of 1*l* 14*s*. 6*d* the chaldron? Ans. 5*s* 9*d*.

12. If 6352 stones of 3 feet long complete a certain quantity of walling; how many stones of 2 feet long will raise a like quantity? Ans. 9528.

13. What must be given for a piece of silver weighing 73 lb 5 oz 15 dwts, at the rate of 5*s* 9*d* per ounce? Ans. 253*l* 10*s* 0$\frac{3}{4}$*d*.

14. A garrison of 536 men having provision for 12 months; how long will those provisions last, if the garrison be increased to 1124 men? Ans. 174 days and $\frac{64}{1124}$.

15. What will be the tax upon 763*l* 15*s* at the rate of 3*s* 6*d* per pound sterling? Ans. 133*l* 13*s* 1$\frac{1}{2}$*d*.

16. A certain work being raised in 12 days, by working 4 hours each day; how long would it have been in raising by working 6 hours per day? Ans. 8 days.

17. What quantity of corn can I buy for 90 guineas, at the rate of 6*s* the bushel? Ans. 39 qrs 3 bu.

18. A person, failing in trade, owes in all 977*l*; at which time he has, in money, goods, and recoverable debts, 420*l* 6*s* $3\frac{1}{4}$*d*; now supposing these things delivered to his creditors, how much will they get per pound? Ans. 8*s* $7\frac{1}{4}$*d*.

19. A plain of a certain extent having supplied a body of 3000 horse with forage for 18 days; then how many days would the same plain have supplied a body of 2000 horse? Ans. 27 days.

20. Suppose a gentleman's income is 600 guineas a year, and that he spends 25*s* 6*d* per day, one day with another; how much will he have saved at the year's end? Ans. 164*l* 12*s* 6*d*.

21. What cost 30 pieces of lead, each weighing 1 cwt 12lb. at the rate of 16*s* 4*d* the cwt? Ans. 27*l* 2*s* 6*d*.

22. The governor of a besieged place having provision for 54 days, at the rate of $1\frac{1}{2}$lb of bread; but being desirous to prolong the siege to 80 days, in expectation of succour, in that case what must the ration of bread be? Ans. $1\frac{1}{80}$lb.

23. At half-a-guinea per week, how long can I be boarded for 20 pounds? Ans. $38\frac{12}{126}$ wks.

24. How much will 75 chaldrons 7 bushels of coals come to, at the rate of 1*l* 13*s* 6*d* per chaldron? Ans. 125*l* 19*s* $0\frac{1}{6}$*d*.

25. If the penny loaf weigh 8 ounces when the bushel of wheat costs 7*s* 3*d*, what ought the penny loaf to weigh when the wheat is at 8*s* 4*d*? Ans. 6 oz 15 $\frac{36}{100}$ dr.

26. How much a year will 173 acres 2 roods 14 poles of land give, at the rate of 1*l* 7*s* 8*d* per acre? Ans. 240*l* 2*s* $7\frac{1}{20}$*d*.

27 To how much amounts 73 pieces of lead, each weighing 1 cwt 3 qrs 7 lb, at 10*l* 4*s* per fother of $19\frac{1}{2}$ cwt? Ans. 69*l* 4*s* 2*d* $1\frac{3}{13}$q.

28. How many yards of stuff, of 3 qrs wide, will line a cloak that is $1\frac{3}{4}$ yards in length and $3\frac{1}{4}$ yards wide? Ans. 8 yds 0qrs $2\frac{2}{3}$ nl.

29. If 5 yards of cloth cost 14*s* 2*d*, what must be given for 9 pieces, containing each 21 yards 1 quarter? Ans. 27*l* 1*s* $10\frac{1}{2}$*d*.

30. If a gentleman's estate be worth 2107*l* 12*s* a year; what may he spend per day, to save 500*l* in the year? Ans. 4*l* 8*s* $1\frac{19}{365}$*d*.

31. Wanting just an acre of land cut off from a piece which is $13\frac{1}{2}$ poles in breadth, what length must the piece be? Ans. 11 po 4 yds 2 ft $0\frac{2}{3}$ in.

32. At 7*s* $9\frac{1}{2}$*d* per yard, what is the value of a piece of cloth containing 53 ells English 1 qr? Ans. 25*l* 18*s* $1\frac{3}{4}$*d*.

33. If the carriage of 5 cwt 14 lb for 96 miles be 1*l* 12*s* 6*d*; how far may I have 3 cwt 1 qr carried for the same money? Ans. 151 m 3 fur $3\frac{1}{12}$ pol.

34. Bought a silver tankard, weighing 1 lb 7 oz 14 dwts; what did it cost me at 6*s* 4*d* the ounce? Ans. 6*l* 4*s* $9\frac{1}{5}$*d*.

35. What is the half year's rent of 547 acres of land, at 15*s* 6*d* the acre? Ans. 211*l* 19*s* 3*d*.

36. A wall that is to be built to the height of 36 feet, was raised 9 feet high by 16 men in 6 days; then how many men must be employed to finish the wall in 4 days, at the same rate of working? Ans. 72 men.

37. What will be the charge of keeping 20 horses for a year, at the rate of $14\frac{1}{2}$*d* per day for each horse? Ans. 441*l* 0*s* 10*d*.

38. If 18 ells of stuff that is $\frac{3}{4}$ yard wide, cost 39*s* 6*d*; what will 50 ells, of the same goodness, cost, being yard wide? Ans. 7*l* 6*s* $3\frac{5}{9}$*d*.

39. How many yards of paper that is 30 inches wide, will hang a room that is 20 yards in circuit and 9 feet high? Ans. 72 yards.

40. If a gentleman's estate be worth 384*l* 16*s* a year, and the land-tax be assessed at 2*s* $9\frac{1}{2}$*d* per pound, what is his net annual income? Ans. 331*l* 1*s* $9\frac{1}{2}$*d*.

41. The circumference of the earth is about 25000 miles; at what rate per hour is a person at the middle of its surface carried round, one whole rotation being made in 23 hours 56 minutes? Ans. $1044\frac{816}{1436}$ miles.

42. If a person drink 20 bottles of wine per month, when it costs 8*s*. a gall; how many bottles per month may he drink, without increasing the expense, when wine costs 10*s* the gallon? Ans. 16 bottles.

43. What cost 43 qrs 5 bushels of corn, at 1*l* 8*s* 6*d* the quarter? Ans. 62*l* 3*s* $3\frac{3}{4}$*d*.

44. How many yards of canvas that is ell wide will line 50 yards of say that is 3 quarters wide? Ans. 30 yds.

45. If an ounce of gold cost 4 guineas, what is the value of a grain? Ans. $2\frac{1}{10}$*d*.

46. If 3 cwt of tea cost 40*l* 12*s*; at how much a pound must it be retailed, to gain 10*l* by the whole? Ans. $3\frac{4}{336}$*s*.

COMPOUND PROPORTION.

Compound Proportion is a rule by means of which the student may resolve such questions as require two or more statings in simple proportion.

The general rule for questions of this kind may be exhibited in the following precepts: viz.

1. Set down the terms that express the *conditions* of the question in one line.

2. Under each conditional term, set its corresponding one, in another line, putting the letter Q in the (otherwise) blank place of the term required.

3. Multiply the *producing terms* of one line, and the *produced* terms of the other line, continually, and take the result for a dividend.

4. Multiply the remaining terms continually, and let the product be a divisor.

5. The quotient of this division will be Q, the term required.*

Note. By *producing terms* are here meant whatever necessarily and jointly produce any effect; as the cause and the time; length, breadth, and depth; buyer and his money; things carried, and their distance, &c. all necessarily inseparable in producing their several effects.

In a question where a term is only understood, and not expressed, that term may always be expressed by unity.

A quotient is represented by the dividend put above a line, and the divisor put below it.

EXAMPLES.

1. How many men can complete a trench of 135 yards long in 8 days, when 16 men can dig 54 yards of the same trench in 6 days?

M	D	Yds
16	6	54
Q	8	135

---

* This rule, which is as applicable to *Simple* as to *Compound* Proportion, was given, in 1706, by *W. Jones, Esq.* F.R.S., the father of the late *Sir W. Jones.*

Here 16 men and 6 days, are the producing terms of the first line, and 135 yards, the produced term of the other. Therefore, by the rule,

$$Q = \frac{16\times6\times135}{8\times54} = \frac{2\times135}{9} = 30,$$

the number of men required.

ANOTHER QUESTION.

If a garrison of 3600 men have bread for 35 days, at 24 oz each a day: How much a day must be allowed to 4800 men, each for 45 days, that the same quantity of bread may serve?

men	oz	days	bread
3600 . .	24 . .	35 . .	1
4800 . .	Q . .	45 . .	1

$$Q = \frac{3600\times24\times35}{4800\times45} = 14 \text{ oz } \textit{per diem}.$$

AN EXAMPLE IN SIMPLE PROPORTION.

If 14 yards of cloth cost 21*l*, how many yards may be bought for 73*l* 10*s*?

man	£	yds.
1	21	14
1	$73\frac{1}{2}$	Q

$$Q = \frac{73\frac{1}{2}\times14}{21} = \tfrac{2}{3} \text{ of } 73\tfrac{1}{2} = 49 \text{ yards, Answer.}$$

2. If 100*l* in one year gain 5*l* interest, what will be the interest of 750*l* for seven years? Ans. 262*l* 10*s*.

3. If a family of 8 persons expend 200*l* in 9 months; how much will serve a family of 18 people 12 months? Ans. 600*l*.

4. If 27*s* be the wages of 4 men for 7 days; what will be the wages of 14 men for 10 days? Ans. 6*l* 15*s*.

If a footman travel 130 miles in 3 days, when the days are 12 hours long; in how many days, of 10 hours each, may he travel 360 miles? Ans. $9\frac{63}{65}$ days.

6. If 120 bushels of corn can serve 14 horses 56 days; how many days will 94 bushels serve 6 horses? Ans. $102\frac{16}{45}$ days.

7. If 3000 lbs of beef serve 340 men 15 days ; how many lbs will serve 120 men for 25 days ? Ans. 1764 lb 11 $\frac{5}{17}$ oz.

8. If a barrel of beer be sufficient to last a family of 8 persons 12 days ; how many barrels will be drank by 16 persons in the space of a year ? Ans. 60$\frac{5}{6}$ barrels.

9. If 180 men, in six days, of 10 hours each, can dig a trench 200 yards long, 3 wide, and 2 deep ; in how many days of 8 hours long, will 100 men dig a trench of 360 yards long, 4 wide, and 3 deep ? Ans. 48$\frac{3}{5}$ days.

OF VULGAR FRACTIONS.

A Fraction, or broken number, is an expression of a part, or some parts, of something considered as a whole.

It is denoted by two numbers, placed one below the other, with a line between them :

Thus, $\frac{3}{4}\left\{\begin{matrix}\text{numerator}\\\text{denominator}\end{matrix}\right\}$, which is named 3-fourths.

The denominator, or number placed below the line, shows how many equal parts the whole quantity is divided into ; and it represents the Divisor in Division.—And the Numerator, or number set above the line, shows how many of these parts are expressed by the Fraction : being the remainder after division.—Also, both these numbers are in general named the Terms of the Fraction.

Fractions are either Proper, Improper, Simple, Compound, Mixed, or Complex.

A Proper Fraction, is when the numerator is less than the denominator ; as, $\frac{1}{2}$, or $\frac{2}{3}$, or $\frac{3}{5}$, &c.

An Improper Fraction, is when the numerator is equal to, or exceeds, the denominator ; as, $\frac{3}{3}$, or $\frac{5}{4}$, or $\frac{7}{5}$, &c. In these cases the fraction is called *Improper*, because it is equal to, or exceeds unity.

A Simple Fraction, is a single expression, denoting any number of parts of the integer ; as, $\frac{2}{3}$, or $\frac{3}{5}$.

A Compound Fraction, is the fraction of a fraction, or two or more fractions connected with the word *of* between them ; as, $\frac{1}{2}$ of $\frac{2}{3}$, or $\frac{3}{5}$ of $\frac{5}{6}$ of 3, &c.

A Mixed Number, is composed of a whole number and a fraction together ; as, 3$\frac{1}{4}$, or 12$\frac{2}{3}$, &c.

A Complex Fraction, is one that has a fraction or a mixed number for its numerator, or its denominator, or both ; as, $\frac{\frac{1}{2}}{3}$, or $\frac{2}{\frac{3}{4}}$, or $\frac{\frac{2}{3}}{\frac{4}{5}}$, or $\frac{3\frac{1}{5}}{4}$, &c.

A whole or integer number may be expressed like a fraction, by writing 1 below it, as a denominator; so 3 is $\frac{3}{1}$, or 4 is $\frac{4}{1}$, &c.

A fraction denotes division; and its value is equal to the quotient obtained by dividing the numerator by the denominator: so $\frac{12}{4}$ is equal to 3, and $\frac{21}{5}$ is equal to $4\frac{1}{5}$.

Hence then, if the numerator be less than the denominator, the value of the fraction is less than 1. But if the numerator be the same as the denominator, the fraction is just equal to 1. And if the numerator be greater than the denominator, the fraction is greater than 1.

REDUCTION OF VULGAR FRACTIONS.

Reduction of Vulgar Fractions, is the bringing them out of one form or denomination into another; commonly to prepare them for the operations of Addition, Subtraction, &c.; of which there are several cases.

PROBLEM.

To find the Greatest Common Measure of Two or more Numbers.

The Common Measure of two or more numbers, is that number which will divide them all without remainder; so, 3 is a common measure of 18 and 24; the quotient of the former being 6, and of the latter 8. And the greatest number that will do this, is the greatest common measure: so 6 is the greatest common measure of 18 and 24; the quotient of the former being 3, and of the latter 4, which will not both divide further.

RULE.

If there be two numbers only, divide the greater by the less; then divide the divisor by the remainder; and so on, dividing always the last divisor by the last remainder, till nothing remains; so shall the last divisor of all be the greatest common measure sought.

When there are more than two numbers, find the greatest common measure of two of them, as before; then do the same for that common measure and another of the numbers;

and so on, through all the numbers; so will the greatest common measure last found be the answer.

If it happen that the common measure thus found is 1; then the numbers are said to be incommensurable, or not to have any common measure, or they are said to be prime to each other.

EXAMPLES.

1. To find the greatest common measure of 1908, 936, and 630.

```
936 ) 1908 ( 2           So that 36 is the greatest common
      1872                 measure of 1908 and 936.
      ----
        36 ) 936 (26.    Hence 36 ) 630 ( 17
             72                     36
             ---                    ---
             216                    270
             216                    252
             ---                    ---
                                    18 ) 36 ( 2
                                         36
                                         --
```

Hence 18 is the answer required.

2. What is the greatest common measure of 246 and 372? Ans. 6.

3. What is the greatest common measure of 324, 612, and 1032? Ans. 12.

CASE I.

To Abbreviate or Reduce Fractions to their Lowest Terms.

* Divide the terms of the given fraction by any number that will divide them without a remainder; then divide these

---

* That dividing both the terms of the fraction by the same number, whatever it be, will give another fraction equal to the former, is evident. And when these divisions are performed as often as can be done, or when the common divisor is the greatest possible, the terms of the resulting fraction must be the least possible

Note. 1. Any number ending with an even number, or a cipher, is divisible, or can be divided, by 2.

2. Any number, ending with 5, or 0, is divisible by 5.

3. If the right-hand place of any number be 0, the whole is divisible by 10; if there be two ciphers, it is divisible by 100; if three ciphers, by 1000: and so on; which is only cutting off those ciphers.

quotients again in the same manner ; and so on, till it appears that there is no number greater than 1 which will divide them ; then the fraction will be in its lowest terms.

Or, divide both the terms of the fraction by their greatest common measure at once, and the quotients will be the terms of the fraction required, of the same value as at first.

EXAMPLES.

1. Reduce $\frac{216}{288}$ to its least terms.

$\frac{216}{288} = \frac{72}{96} = \frac{36}{48} = \frac{12}{16} = \frac{6}{8} = \frac{3}{4}$, the answer.

Or thus :

```
216) 288 (1
     216
     ———
      72) 216 (3
          216
          ———
```

Therefore 72 is the greatest common measure ; and 72) $\frac{216}{288} = \frac{3}{4}$ the Answer, the same as before.

2. Reduce $\frac{105}{420}$ to its lowest terms. Ans. $\frac{1}{4}$.

3. Reduce $\frac{136}{204}$ to its lowest terms. Ans. $\frac{2}{3}$.

4. Reduce $\frac{525}{630}$ to its lowest terms. Ans. $\frac{5}{6}$.

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4. If the two right-hand figures of any number be divisible by 4, the whole is divisible by 4. And if the three right-hand figures be divisible by 8, the whole is divisible by 8. And so on.

5. If the sum of the digits in any number be divisible by 3, or by 9, the whole is divisible by 3, or by 9.

6. If the right-hand digit be even, and the sum of all the digits be divisible by 6, then the whole is divisible by 6.

7. A number is divisible by 11, when the sum of the 1st, 3d, 5th, &c. or all the odd places, is equal to the sum of the 2d, 4th, 6th, &c. or of all the even places of digits.

8. If a number cannot be divided by some quantity less than the square root of the same, that number is a prime, or cannot be divided by any number whatever.

9. All prime numbers, except 2 and 5, have either 1, 3, 7, or 9, in the place of units; and all other numbers are composite, or can be divided.

10. When numbers, with the sign of addition or subtraction between them, are to be divided by any number, then each of those numbers must be divided by it. Thus $\frac{10+8-4}{2} = 5+4-2 = 7$.

11. But if the numbers have the sign of multiplication between them, only one of them must be divided. Thus,

$$\frac{10\times8\times3}{6\times2} = \frac{10\times4\times3}{6\times1} = \frac{10\times4\times1}{2\times1} = \frac{10\times2\times1}{1\times1} = \frac{20}{1} = 20.$$

CASE II.

To Reduce a Mixed Number to its Equivalent Improper Fraction.

* MULTIPLY the integer or whole number by the denominator of the fraction, and to the product add the numerator; then set that sum above the denominator for the fraction required.

EXAMPLES.

1. Reduce $23\frac{2}{5}$ to a fraction.

$$\begin{array}{r} 23 \\ 5 \\ \hline 115 \\ 2 \\ \hline 117 \\ \hline 5 \\ \hline \end{array}$$

Or, thus,

$$\frac{(23\times5)+2}{5}=\frac{117}{5}, \text{ the Answer.}$$

2. Reduce $12\frac{7}{9}$ to a fraction. Ans. $\frac{115}{9}$.
3. Reduce $14\frac{7}{10}$ to a fraction. Ans. $\frac{147}{10}$.
4. Reduce $183\frac{5}{21}$ to a fraction. Ans. $\frac{3848}{21}$.

CASE III.

To Reduce an Improper Fraction to its Equivalent Whole or Mixed Number.

† DIVIDE the numerator by the denominator, and the quotient will be the whole or mixed number sought.

EXAMPLES.

1. Reduce $\frac{12}{3}$ to its equivalent number.
   Here $\frac{12}{3}$ or $12\div3=4$, the Answer.

---

* This is no more than first multiplying a quantity by some number, and then dividing the result back again by the same: which it is evident does not alter the value; for any fraction represents a division of the numerator by the denominator.

† This rule is evidently the reverse of the former; and the reason of it is manifest from the nature of Common Division.

2. Reduce $\frac{15}{7}$ to its equivalent number.

Here $\frac{15}{7}$ or $15 \div 7 = 2\frac{1}{7}$, the Answer.

3. Reduce $\frac{749}{17}$ to its equivalent number.

$$\begin{array}{r|l} \text{Thus, } 17\ )\ 749\ (\ 44\frac{1}{17} & \\ 68 & \\ \hline 69 & \text{So that } \frac{749}{17} = 44\frac{1}{17}\text{, the Answer.} \\ 68 & \\ \hline 1 & \end{array}$$

4. Reduce $\frac{56}{7}$ to its equivalent number. Ans. 8.
5. Reduce $\frac{1362}{25}$ to its equivalent number. Ans. $54\frac{12}{25}$.
6. Reduce $\frac{2918}{17}$ to its equivalent number. Ans. $171\frac{11}{17}$.

CASE IV.

To Reduce a Whole Number to an Equivalent Fraction, having a Given Denominator.

* Multiply the whole number by the given denominator; then set the product over the said denominator, and it will form the fraction required.

EXAMPLES.

1. Reduce 9 to a fraction whose denominator shall be 7.

Here $9 \times 7 = 63$: then $\frac{63}{7}$ is the Answer;

For $\frac{63}{7} = 63 \div 7 = 9$, the Proof.

2. Reduce 12 to a fraction whose denominator shall be 13. Ans. $\frac{156}{13}$.
3. Reduce 27 to a fraction whose denominator shall be 11. Ans. $\frac{297}{11}$.

CASE V.

To Reduce a Compound Fraction to an Equivalent Simple one.

† Multiply all the numerators together for a numerator, and all the denominators together for a denominator, and they will form the simple fraction sought.

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* Multiplication and Division being here equally used, the result must be the same as the quantity first proposed.

† The truth of this rule may be shown as follows: Let the compound fraction be $\frac{2}{3}$ of $\frac{4}{7}$. Now $\frac{1}{3}$ of $\frac{4}{7}$ is $\frac{4}{7} \div 3$, which is $\frac{4}{21}$; consequently

When part of the compound fraction is a whole or mixed number, it must first be reduced to a fraction by one of the former cases.

And, when it can be done, any two terms of the fraction may be divided by the same number, and the quotients used instead of them. Or, when there are terms that are common, they may be omitted, or cancelled.

EXAMPLES.

1. Reduce $\frac{1}{2}$ of $\frac{2}{3}$ of $\frac{3}{4}$ to a simple fraction.

Here $\frac{1\times2\times3}{2\times3\times4}=\frac{6}{24}=\frac{1}{4}$, the Answer.

Or, $\frac{1\times\not{2}\times\not{3}}{\not{2}\times\not{3}\times4}=\frac{1}{4}$, by cancelling the 2's and 3's.

2. Reduce $\frac{2}{3}$ of $\frac{3}{5}$ of $\frac{10}{11}$ to a simple fraction.

Here $\frac{2\times3\times10}{3\times5\times11}=\frac{60}{165}=\frac{12}{33}=\frac{4}{11}$, the Answer.

Or, $\frac{2\times\not{3}\times\overset{2}{\not{10}}}{\not{3}\times\not{5}\times11}=\frac{4}{11}$, the same as before, by cancelling the 3's, and dividing by 5's.

3. Reduce $\frac{3}{7}$ of $\frac{4}{5}$ to a simple fraction. Ans. $\frac{12}{35}$.
4. Reduce $\frac{2}{3}$ of $\frac{3}{5}$ of $\frac{5}{6}$ to a simple fraction. Ans. $\frac{3}{8}$.
5. Reduce $\frac{2}{5}$ of $\frac{5}{8}$ of $3\frac{1}{2}$ to a simple fraction. Ans. $\frac{7}{8}$.
6. Reduce $\frac{4}{7}$ of $\frac{5}{9}$ of $\frac{7}{8}$ of 4 to a simple fraction. Ans. $\frac{4}{9}$.
7. Reduce 2 and $\frac{2}{5}$ of $\frac{5}{6}$ to a fraction. Ans. 2.

CASE VI.

To Reduce Fractions of Different Denominations to Equivalent Fractions having a Common Denominator.

* MULTIPLY each numerator by all the denominators except its own for the new numerators: and multiply all the denominators together for a common denominator.

$\frac{2}{3}$ of $\frac{5}{7}$ will be $\frac{5}{21}\times2$ or $\frac{10}{21}$; that is, the numerators are multiplied together, and also the denominators, as in the Rule. When the compound fraction consists of more than two single ones; having first reduced two of them as above, then the resulting fraction and a third will be the same as a compound fraction of two parts; and so on to the last of all.

* This is evidently no more than multiplying each numerator and its denominator by the same quantity, and consequently the value of the fraction is not altered.

Note, It is evident, that in this and several other operations, when any of the proposed quantities are integers, or mixed numbers, or compound fractions, they must first be reduced, by their proper Rules, to the form of simple fractions.

EXAMPLES.

1. Reduce $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{4}$, to a common denominator.

$1 \times 3 \times 4 = 12$ the new numerator for $\frac{1}{2}$.
$2 \times 2 \times 4 = 16$ ditto $\frac{2}{3}$.
$3 \times 2 \times 3 = 18$ ditto $\frac{3}{4}$.
$2 \times 3 \times 4 = 24$ the common denominator.

Therefore the equivalent fractions are $\frac{12}{24}$, $\frac{16}{24}$, and $\frac{18}{24}$.

Or the whole operation of multiplying may often be performed mentally, only setting down the results and given fractions thus, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $= \frac{12}{24}$, $\frac{16}{24}$, $\frac{18}{24} = \frac{6}{12}$, $\frac{8}{12}$, $\frac{9}{12}$, by abbreviation.

2. Reduce $\frac{2}{7}$ and $\frac{5}{9}$ to fractions of a common denominator.
Ans. $\frac{18}{63}$, $\frac{35}{63}$.

3. Reduce $\frac{2}{3}$, $\frac{3}{5}$, and $\frac{3}{4}$ to a common denominator.
Ans. $\frac{40}{60}$, $\frac{36}{60}$, $\frac{45}{60}$.

4. Reduce $\frac{4}{5}$, $2\frac{1}{3}$, and 4 to a common denominator.
Ans. $\frac{24}{30}$, $\frac{70}{30}$, $\frac{120}{30}$.

Note 1. When the denominators of two given fractions have a common measure, let them be divided by it; then multiply the terms of each given fraction by the quotient arising from the other's denominator.

Ex. $\underset{5}{\frac{3}{25}}$ and $\underset{7}{\frac{2}{35}} = \frac{21}{175}$ and $\frac{10}{175}$, by multiplying the former by 7 and the latter by 5.

2. When the less denominator of two fractions exactly divides the greater, multiply the terms of that which has the less denominator by the quotient.

Ex. $\frac{3}{7}$ and $\underset{2}{\frac{5}{14}} = \frac{6}{14}$ and $\frac{5}{14}$, by mult. the former by 2.

3. When more than two fractions are proposed, it is sometimes convenient, first to reduce two of them to a common denominator; then these and a third; and so on till they be all reduced to their least common denominator.

Ex. $\frac{2}{3}$ and $\frac{3}{4}$ and $\frac{7}{8} = \frac{8}{12}$ and $\frac{9}{12}$ and $\frac{7}{8} = \frac{16}{24}$ and $\frac{18}{24}$ and $\frac{21}{24}$.

CASE VII.

To reduce Complex Fractions to single ones.

REDUCE the two parts both to simple fractions; then multiply the numerator of each by the denominator of the other; which is in fact only increasing each part by equal multiplications, which makes no difference in the value of the whole.

$$\text{So, } \frac{\frac{5}{2}}{3} = \frac{5}{6}. \text{ And } \frac{2\frac{1}{3}}{4} = \frac{7}{12}. \text{ Also } \frac{3\frac{2}{5}}{4\frac{1}{2}} = \frac{\frac{17}{5}}{\frac{9}{2}} = \frac{17}{5} \times \frac{2}{9} = \frac{34}{45}.$$

CASE VIII.

To find the value of a Fraction in Parts of the Integer.

MULTIPLY the integer by the numerator, and divide the product by the denominator, by Compound Multiplication and Division, if the integer be a compound quantity.

Or, if it be a single integer, multiply the numerator by the parts in the next inferior denomination, and divide the product by the denominator. Then, if any thing remains, multiply it by the parts in the next inferior denomination, and divide by the denominator, as before; and so on as far as necessary; so shall the quotients, placed in order, be the value of the fraction required.*

EXAMPLES.

1. What is the $\frac{4}{5}$ of 2*l* 6*s*?
By the former part of the Rule

```
      2l 6s
         4
     ------
   5) 9  4
Ans.  1l 16s 9d 2⅖q.
```

2. What is the value of $\frac{2}{3}$ of 1*l*?
By the 2d part of the Rule,

```
     2
    20
    --
3)  40 (13s 4d Ans.
    --
     1
    12
    --
3)  12 (4d
    --
```

3. Find the value of $\frac{3}{8}$ of a pound sterling. Ans. 7*s* 6*d*.
4. What is the value of $\frac{2}{9}$ of a guinea? Ans. 4*s* 8*d*.
5. What is the value of $\frac{3}{4}$ of a half crown? Ans 1*s* $10\frac{1}{2}d$.
6. What is the value of $\frac{2}{5}$ of 4*s* 10*d*? Ans. 1*s* $11\frac{1}{5}d$.
7. What is the value of $\frac{4}{5}$ lb troy? Ans. 9 oz 12 dwts.
8. What is the value of $\frac{5}{16}$ of a cwt? Ans. 1 qr 7 lb.
9. What is the value of $\frac{7}{8}$ of an acre? Ans. 3 ro 20 po.
10. What is the value of $\frac{3}{10}$ of a day? Ans. 7 hrs 12 min.

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* The numerator of a fraction being considered as a remainder, in Division, and the denominator as the divisor, this rule is of the same nature as Compound Division, or the valuation of remainders in the Rule of Three, before explained.

CASE IX.

To Reduce a Fraction from one Denomination to another.

* CONSIDER how many of the less denomination make one of the greater; then multiply the numerator by that number, if the reduction be to a less name, but multiply the denominator, if to a greater.

EXAMPLES.

1. Reduce $\frac{2}{3}$ of a pound to the fraction of a penny.
 $\frac{2}{3} \times \frac{20}{1} \times \frac{12}{1} = \frac{480}{3} = \frac{160}{1}$, the Answer.
2. Reduce $\frac{4}{7}$ of a penny to the fraction of a pound.
 $\frac{4}{7} \times \frac{1}{12} \times \frac{1}{20} = \frac{1}{420}$, the Answer.
3. Reduce $\frac{2}{15}l$ to the fraction of a penny. Ans. $\frac{32}{1}d$.
4. Reduce $\frac{2}{3}q$ to the fraction of a pound. Ans. $\frac{1}{1440}$.
5. Reduce $\frac{4}{7}$ cwt to the fraction of a lb. Ans. $\frac{64}{1}$.
6. Reduce $\frac{3}{5}$ dwt to the fraction of a lb troy. Ans. $\frac{1}{400}$.
7. Reduce $\frac{2}{5}$ crown to the fraction of a guinea. Ans. $\frac{2}{21}$.
8. Reduce $\frac{4}{9}$ half-crown to the fract. of a shilling. Ans. $\frac{10}{9}$.
9. Reduce 2*s* 6*d* to the fraction of a £. Ans. $\frac{1}{8}$.
10. Reduce 17*s* 7*d* $3\frac{2}{3}q$ to the fraction of a £. Ans. $\frac{2543}{2880}$.

ADDITION OF VULGAR FRACTIONS.

IF the fractions have a common denominator; add all the numerators together, then place the sum over the common denominator, and that will be the sum of the fractions required.

† If the proposed fractions have not a common denominator, they must be reduced to one. *Also compound fractions

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* This is the same as the Rule of Reduction in whole numbers from one denomination to another.

† Before fractions are reduced to a common denominator, they are quite dissimilar, as much as shillings and pence are, and therefore cannot be incorporated with one another, any more than these can. But when they are reduced to a common denominator, and made parts of the same thing, their sum, or difference, may then be as properly expressed by the sum or difference of the numerators, as the sum or difference of any two quantities whatever, by the sum or difference of

must be reduced to simple ones, and fractions of different denominations to those of the same denomination. Then add the numerators, as before. As to mixed numbers, they may either be reduced to improper fractions, and so added with the others; or else the fractional parts only added, and the integers united afterwards.

EXAMPLES.

1. To add $\frac{3}{5}$ and $\frac{4}{5}$ together.

   Here $\frac{3}{5}+\frac{4}{5} = \frac{7}{5} = 1\frac{2}{5}$, the Answer.

2. To add $\frac{3}{5}$ and $\frac{5}{6}$ together.

   $\frac{3}{5} + \frac{5}{6} = \frac{18}{30} + \frac{25}{30} = \frac{43}{30} = 1\frac{13}{30}$, the Answer.

3. To add $\frac{5}{8}$ and $7\frac{1}{2}$ and $\frac{1}{3}$ of $\frac{3}{4}$ together.

   $\frac{5}{8}+7\frac{1}{2}+\frac{1}{3}$ of $\frac{3}{4} = \frac{5}{8}+\frac{15}{2}+\frac{1}{4} = \frac{5}{8}+\frac{60}{8}+\frac{2}{8} = \frac{67}{8} = 8\frac{3}{8}$.

4. To add $\frac{3}{7}$ and $\frac{5}{7}$ together. Ans. $1\frac{1}{7}$.
5. To add $\frac{3}{4}$ and $\frac{5}{9}$ together. Ans. $1\frac{11}{36}$.
6. Add $\frac{2}{7}$ and $\frac{5}{14}$ together. Ans. $\frac{9}{14}$.

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their individuals. Whence the reason of the Rule is manifest, both for Addition and Subtraction.

When several fractions are to be collected, it is commonly best first to add two of them together that most easily reduce to a common denominator; then add their sum and a third, and so on.

Note 2. Taking any two fractions whatever, $\frac{7}{11}$ and $\frac{35}{55}$, for example, after reducing them to a common denominator, we judge whether they are equal or unequal, by observing whether the products 35×11, and 7×55, which constitute the new numerators, are equal or unequal. If, therefore, we have two equal products $35 \times 11 = 7 \times 55$, we may compose from them two equal fractions, as $\frac{35}{55} = \frac{7}{11}$, or $\frac{35}{7} = \frac{55}{11}$.

If, then, we take two equal fractions, such as $\frac{7}{11}$ and $\frac{35}{55}$, we shall have $35 \times 11 = 7 \times 55$; taking from each of these 7×11, there will remain $(35 - 7) \times 11 = (55 - 11) \times 7$, whence we have $\frac{35 - 7}{55 - 11} = \frac{7}{11}$, or $\frac{28}{44} = \frac{7}{11}$.

In like manner, if the terms of $\frac{7}{11}$ were respectively *added* to those of $\frac{35}{55}$, we should have $\frac{35 + 7}{55 + 11} = \frac{42}{66} = \frac{7}{11}$.

Or, generally, if $\frac{a}{b} = \frac{c}{d}$, it may in a similar way be shown, that $\frac{a \pm b}{c \pm d} = \frac{a}{b} = \frac{c}{d}$.

Hence, *when two fractions are of equal value, the fraction formed by taking the sum (or the difference) of their numerators respectively, and of their denominators respectively, is a fraction equal in value to each of the original fractions.* This proposition will be found useful in the doctrine of proportions.

7. What is the sum of $\frac{3}{5}$ and $\frac{2}{3}$ and $\frac{4}{7}$? Ans. $1\frac{88}{105}$.

8. What is the sum of $\frac{5}{6}$ and $\frac{3}{5}$ and $2\frac{1}{6}$? Ans. $3\frac{3}{5}$.

9. What is the sum of $\frac{3}{5}$ and $\frac{4}{5}$ of $\frac{1}{3}$, and $9\frac{3}{20}$? Ans. $10\frac{1}{60}$.

10. What is the sum of $\frac{2}{3}$ of a pound and $\frac{5}{9}$ of a shilling? Ans. $\frac{125}{9}s$ or $13s\ 10d\ 2\frac{2}{3}q$.

11. What is the sum of $\frac{3}{5}$ of a shilling and $\frac{4}{15}$ of a penny? Ans. $\frac{112}{15}d$ or $7d\ 1\frac{13}{15}q$.

12. What is the sum of $\frac{1}{7}$ of a pound, and $\frac{2}{9}$ of a shilling, and $\frac{5}{12}$ of a penny? Ans. $\frac{3139}{1008}s$ or $3s\ 1d\ 1\frac{10}{21}q$.

SUBTRACTION OF VULGAR FRACTIONS.

Prepare the fractions the same as for Addition, when necessary; then subtract the one numerator from the other, and set the remainder over the common denominator, for the difference of the fractions sought.

EXAMPLES.

1. To find the difference between $\frac{5}{6}$ and $\frac{1}{6}$.

Here $\frac{5}{6} - \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$, the Answer.

2. To find the difference between $\frac{3}{4}$ and $\frac{5}{9}$.

$\frac{3}{4} - \frac{5}{9} = \frac{27}{36} - \frac{20}{36} = \frac{7}{36}$, the Answer.

3. What is the difference between $\frac{5}{12}$ and $\frac{7}{12}$? Ans. $\frac{1}{6}$.

4. What is the difference between $\frac{3}{13}$ and $\frac{4}{39}$? Ans. $\frac{5}{39}$.

5. What is the difference between $\frac{5}{12}$ and $\frac{7}{13}$? Ans. $\frac{19}{156}$.

6. What is the diff. between $5\frac{3}{8}$ and $\frac{2}{7}$ of $4\frac{1}{6}$? Ans. $4\frac{31}{168}$.

7. What is the difference between $\frac{5}{9}$ of a pound, and $\frac{2}{3}$ of $\frac{3}{4}$ of a shilling? Ans. $\frac{191}{18}s$ or $10s\ 7d\ 1\frac{1}{3}q$.

8. What is the difference between $\frac{2}{7}$ of $5\frac{1}{6}$ of a pound, and $\frac{3}{5}$ of a shilling. Ans. $\frac{3037}{2100}l$ or $1l\ 8s\ 11\frac{3}{35}d$.

MULTIPLICATION OF VULGAR FRACTIONS.

* Reduce mixed numbers, if there be any, to equivalent

* Multiplication of any thing by a fraction, implies the taking some part or parts of the thing; it may therefore be truly expressed by a

fractions; then multiply all the numerators together for a numerator, and all the denominators together for a denominator, which will give the product required.

EXAMPLES.

1\. Required the product of $\frac{3}{4}$ and $\frac{2}{9}$.

Here $\frac{3}{4}\times\frac{2}{9}=\frac{6}{36}=\frac{1}{6}$, the Answer.

Or $\frac{3}{4}\times\frac{2}{9}=\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}$.

2\. Required the continued product of $\frac{2}{3}$, $3\frac{1}{4}$, 5, and $\frac{3}{4}$ of $\frac{3}{5}$.

Here $\frac{\not 2}{\not 3}\times\frac{13}{4}\times\frac{\not 5}{1}\times\frac{\not 3}{4}\times\frac{3}{\not 5}=\frac{13\times3}{4\times2}=\frac{39}{8}=4\frac{7}{8}$, Ans.

3\. Required the product of $\frac{2}{7}$ and $\frac{5}{8}$. Ans. $\frac{5}{28}$.

4\. Required the product of $\frac{4}{15}$ and $\frac{5}{24}$. Ans. $\frac{1}{18}$.

5\. Required the product of $\frac{2}{7}$, $\frac{1}{3}$, $\frac{14}{15}$. Ans. $\frac{4}{45}$.

6\. Required the product of $\frac{1}{2}$, $\frac{2}{3}$, and 3. Ans. 1.

7\. Required tho product of $\frac{7}{9}$, $\frac{3}{5}$, and $4\frac{5}{14}$. Ans. $2\frac{1}{30}$.

8\. Required the product of $\frac{5}{6}$, and $\frac{2}{3}$ of $\frac{6}{7}$. Ans. $\frac{10}{21}$.

9\. Required the product of 6, and $\frac{2}{3}$ of 5. Ans. 20.

10\. Required the product of $\frac{2}{5}$ of $\frac{3}{5}$, and $\frac{5}{6}$ of $3\frac{3}{7}$. Ans. $\frac{24}{35}$.

11\. Required the product of $3\frac{2}{7}$ and $4\frac{14}{33}$. Ans. $14\frac{124}{231}$.

12\. Required the product of 5, $\frac{2}{3}$, $\frac{3}{7}$ of $\frac{2}{5}$, and $4\frac{1}{6}$. Ans. $2\frac{8}{21}$.

DIVISION OF VULGAR FRACTIONS.

* PREPARE the fractions as before in Multiplication: then divide the numerator by the numerator, and the denominator by the denominator, if they will exactly divide: but if not,

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compound fraction; which is resolved by multiplying together the numerators and the denominators.

Note. A Fraction is best multiplied by an integer, by dividing the denominator by it; but if it will not exactly divide, then multiply the numerator by it.

* Division being the reverse of Multiplication, the reason of the rule is evident.

Note, A fraction is best divided by an integer, by dividing the numerator by it; but if it will not exactly divide, then multiply the denominator by it.

invert the terms of the divisor, and multiply the dividend by it, as in Multiplication.

EXAMPLES.

1. Divide $\frac{25}{9}$ by $\frac{5}{3}$.
   Here $\frac{25}{9} \div \frac{5}{3} = \frac{5}{3} = 1\frac{2}{3}$, by the first method.
2. Divide $\frac{5}{9}$ by $\frac{2}{15}$.
   Here $\frac{5}{9} \div \frac{2}{15} = \frac{5}{9} \times \frac{15}{2} = \frac{5}{3} \times \frac{5}{2} = \frac{25}{6} - 4\frac{1}{6}$.
3. It is required to divide $\frac{16}{25}$ by $\frac{4}{5}$. Ans. $\frac{4}{5}$.
4. It is required to divide $\frac{7}{16}$ by $\frac{3}{4}$. Ans. $\frac{7}{12}$.
5. It is required to divide $\frac{14}{9}$ by $\frac{7}{5}$. Ans. $1\frac{1}{9}$.
6. It is required to divide $\frac{5}{6}$ by $\frac{15}{7}$. Ans. $\frac{7}{18}$.
7. It is required to divide $\frac{12}{25}$ by $\frac{3}{5}$. Ans. $\frac{4}{5}$.
8. It is required to divide $\frac{3}{7}$ by $\frac{3}{5}$. Ans. $\frac{15}{21}$.
9. It is required to divide $\frac{9}{16}$ by 3. Ans. $\frac{3}{16}$.
10. It is required to divide $\frac{3}{8}$ by 2. Ans. $\frac{3}{16}$.
11. It is required to divide $7\frac{1}{3}$ by $9\frac{5}{9}$. Ans. $\frac{33}{43}$.
12. It is required to divide $\frac{3}{5}$ of $\frac{1}{3}$ by $\frac{4}{7}$ of $7\frac{3}{5}$. Ans. $\frac{7}{152}$.

RULE OF THREE IN VULGAR FRACTIONS.

Make the necessary preparations as before directed; then multiply continually together, the second and third terms, and the first with its parts inverted as in Division, for the answer*.

EXAMPLES.

1. If $\frac{3}{8}$ of a yard of velvet cost $\frac{2}{5}$ of a pound sterling; what will $\frac{5}{16}$ of a yard cost?

$$\frac{3}{8} : \frac{2}{5} :: \frac{5}{16} : \frac{8}{3} \times \frac{\not 2}{\not 5} \times \frac{\not 5}{\not 16} = \frac{1}{3}l = 6s\ 8d, \text{ Answer.}$$

2. What will $3\frac{3}{8}$ oz. of silver cost, at 6*s* 4*d* an ounce?
Ans. 1*l* 1*s* $4\frac{1}{2}d$.

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* This is only multiplying the 2d and 3d terms together, and dividing the product by the first, as in the Rule of Three in whole numbers.

3. If $\frac{3}{16}$ of a ship be worth 273*l* 2*s* 6*d*; what are $\frac{5}{32}$ of her worth? Ans. 227*l* 12*s* 1*d*.

4. What is the purchase of 1230*l* bank-stock, at 108⅝ per cent.? Ans. 1336*l* 1*s* 9*d*.

5. What is the interest of 273*l* 15*s* for a year, at $3\frac{1}{4}$ per cent.? Ans. 8*l* 17*s* $11\frac{1}{4}d$.

6. If ¼ of a ship be worth 73*l* 1*s* 3*d*; what part of her is worth 250*l* 10*s*? Ans. $\frac{6}{7}$.

7. What length must be cut off a board that is 7¾ inches broad, to contain a square foot, or as much as another piece of 12 inches long and 12 broad? Ans. $18\frac{18}{31}$ inches.

8. What quantity of shalloon that is ¾ of a yard wide, will line 9½ yards of cloth, that is 2½ yards wide? Ans. 31⅔ yds.

9. If the penny loaf weigh $6\frac{9}{10}$ oz. when the price of wheat is 5*s* the bushel; what ought it to weigh when the wheat is 8*s* 6*d* the bushel? Ans. $4\frac{1}{17}$ oz.

10. How much in length, of a piece of land that is $11\frac{11}{16}$ poles broad, will make an acre of land, or as much as 40 poles in length and 4 in breadth? Ans. $13\frac{129}{187}$ poles.

11. If a courier perform a certain journey in 35½ days, travelling 13⅗ hours a day; how long would he be in performing the same, travelling only $11\frac{9}{10}$ hours a day? Ans. $40\frac{68}{119}$ days.

12. A regiment of soldiers, consisting of 976 men, are to be new clothed; each coat to contain 2½ yards of cloth that is 1⅝ yard wide, and lined with shalloon ⅞ yard wide: how many yards of shalloon will line them? Ans. 4531 yds 1 qr $2\frac{6}{7}$ nails.

DECIMAL FRACTIONS.

A DECIMAL FRACTION is that which has for its denominator an unit (1), with as many ciphers annexed as the numerator has places; and it is usually expressed by setting down the numerator only, with a point before it, on the left-hand. Thus, $\frac{4}{10}$ is ·4, and $\frac{24}{100}$ is ·24, and $\frac{74}{1000}$ is ·074, and $\frac{124}{100000}$ is ·00124; where ciphers are prefixed to make up as many places as are ciphers in the denominator, when there is a deficiency in the figures.

A mixed number is made up of a whole number with some decimal fraction, the one being separated from the other by a point. Thus, 3·25 is the same as $3\frac{25}{100}$, or $\frac{325}{100}$.

Ciphers on the right-hand of decimals make no alteration in their value; for ·4, or ·40, or ·400 are decimals having all the same value, each being = $\frac{4}{10}$, or ⅖. But when they are placed on the left-hand, they decrease the value in a ten-fold proportion: Thus, ·4 is $\frac{4}{10}$, or 4 tenths; but ·04 is only $\frac{4}{100}$, or 4 hundredths, and ·004 is only $\frac{4}{1000}$, or 4 thousandths.

In decimals, as well as in whole numbers, the values of the places increase towards the left-hand, and decrease towards the right, both in the same tenfold proportion; as in the following Scale or Table of Notation.

millions	hundred thousands	ten thousands	thousands	hundreds	tens	units	tenth parts	hundredth parts	thousandth parts	ten thousandth parts	hundred thousandth parts	millionth parts
3	3	3	3	3	3	3 ·	3	3	3	3	3	3

ADDITION OF DECIMALS.

Set the numbers under each other according to the value of their places, as in whole numbers; in which state the decimal separating points will stand all exactly under each other. Then, beginning at the right hand, add up all the columns of numbers as in integers; and point off as many places for decimals, as are in the greatest number of decimal places in any of the lines that are added; or place the point directly below all the other points.

EXAMPLES.

1. To add together 29·0146, and 3146·5, and 2109, and ·62417, and 14·16.

```
  29·0146
3146·5
2109·
    ·62417
  14·16
---------
5299·29877 the Sum.
```

2. What is the sum of 276, 39·213, 72014·9, 417, and 5032? Ans. 77779·113.

3. What is the sum of 7530, 16·201, 3·0142, 957·13, 6·72119 and ·03014? Ans. 8513·09653.

4. What is the sum of 312·09, 3·5711, 7195·6, 71·498, 9739·215, 179, and ·0027? Ans. 17500·9718.

SUBTRACTION OF DECIMALS.

PLACE the numbers under each other according to the value of their places, as in the last Rule. Then, beginning at the right-hand, subtract as in whole numbers, and point off the decimals as in Addition.

EXAMPLES.

1. To find the difference between 91·73 and 2·138.

```
 91·73
  2·138
 ------
```

Ans. 89.592 the Difference.

2. Find the diff. between 1·9185 and 2·73. Ans. 0·8115.

3. To subtract 4·90142 from 214·81. Ans. 209·90858.

4. Find the diff. between 2714 and ·916. Ans. 2713·084.

MULTIPLICATION OF DECIMALS.

* PLACE the factors, and multiply them together the same as if they were whole numbers.—Then point off in the product just as many places of decimals as there are decimals in both the factors. But if there be not so many figures in the product, then supply the defect by prefixing ciphers.

---

* The rule will be evident from this example:—Let it be required to multiply ·12 by ·361; these numbers are equivalent to $\frac{12}{100}$ and $\frac{361}{1000}$; the product of which is $\frac{4332}{100000}$ = ·04432, by the nature of Notation, which consists of as many places as there are ciphers, that is, of as many places as there are in both numbers. And in like manner for any other numbers.

EXAMPLES.

1. Multiply ·321096
by ·2465

1605480
1926576
1284384
642192
Ans. ·0791501640 the Product.

2. Multiply 79·347 by 23·15. Ans. 1836·88305.
3. Multiply ·63478 by ·8204. Ans. ·520773512.
4. Multiply ·385746 by ·00464. Ans. ·00178986144.

CONTRACTION I.

To multiply Decimals by 1 *with any Number of Ciphers, as by* 10, *or* 100, *or* 1000, *&c.*

THIS is done by only removing the decimal point so many places farther to the right-hand, as there are ciphers in the multiplier; and subjoining ciphers if need be.

EXAMPLES.

1. The product of 51·3 and 1000 is 51300.
2. The product of 2·714 and 100 is
3. The product of ·916 and 1000 is
4. The product of 21·31 and 10000 is

CONTRACTION II.

To contract the Operation so as to retain only as many Decimals in the Product as may be thought necessary, when the Product would naturally contain several more Places.

SET the unit's place of the multiplier under the figure of the multiplicand whose place is the same as is to be retained for the last in the product; and dispose of the rest of the figures in the inverted or contrary order to what they are usually placed in.—Then, in multiplying, reject all the figures that are more to the right-hand than each multiplying figure, and set down the products, so that their right-hand figures may fall in a column straight below each other; but observe to increase the first figure of every line with what would arise from the figures omitted, in this manner namely 1 from 5 to 14, 2 from 15 to 24, 3 from 25 to 34, &c.; and the sum of all the lines will be the product as required, commonly to the nearest unit in the last figure.

EXAMPLES.

1. To multiply 27·14986 by 92·41035, so as to retain only four places of decimals in the product.

Contracted Way.	*Common Way.*	
27·14986	27·14986	
53014·29	92·41035	
24434874	13	574930
542997	81	44958
108599	2714	986
2715	108599	44
81	542997	2
14	24434874	
2508·9280	2508·9280	650510

2. Multiply 480·14936 by 2·72416, retaining only four decimals in the product.

3. Multiply 2490·3048 by ·573286, retaining only five decimals in the product.

4. Multiply 325·701428 by ·7218393, retaining only three decimals in the product.

DIVISION OF DECIMALS.

Divide as in whole numbers; and point off in the quotient as many places for decimals, as the decimal places in the dividend exceed those in the divisor*.

---

* The reason of this Rule is evident; for, since the divisor multiplied by the quotient gives the dividend, therefore the number of decimal places in the dividend, is equal to those in the divisor and quotient, taken together, by the nature of Multiplication; and consequently the quotient itself must contain as many as the dividend exceeds the divisor.

Another way to know the place for the decimal point is this : *The first figure of the quotient must be made to occupy the same place, of integers or decimals, as that figure of the dividend which stands over the unit's figure of the first product.*

When the places of the quotient are not so many as the Rule requires, the defect is to be supplied by prefixing ciphers.

When there happens to be a remainder after the division ; or when the decimal places in the divisor are more than those in the dividend ; then ciphers may be annexed to the dividend, and the quotient carried on as far as required.

EXAMPLES.

1.

```
178) ·48520998 (·00272589
       1292
        460
        1049
         1599
          1758
           156
```

1.

```
·2639) 27·00000 (102·3114
         6100
          8220
           3030
            3910
            12710
             2154
```

3. Divide 123·70536 by 54·25. Ans. 2·2802.
4. Divide 12 by ·7854. Ans. 15·278.
5. Divide 4195·68 by 100. Ans. 41·9568.
6. Divide ·8297592 by ·153. Ans. 5·4232.

CONTRACTION I.

When the divisor is an integer, with any number of ciphers annexed : cut off those ciphers, and remove the decimal point in the dividend as many places farther to the left as there are ciphers cut off, prefixing ciphers, if need be ; then proceed as before.

EXAMPLES.

1. Divide 45·5 by 2100.

```
21·00) ·455 (·0216, &c.
        35
        140
         14
```

2. Divide 41020 by 32000.
3. Divide 953 by 21600.
4. Divide 61 by 79000.

CONTRACTION II.

HENCE, if the divisor be 1 with ciphers, as 10, 100, or 1000, &c.; then the quotient will be found by merely moving the decimal point in the dividend so many places farther to the left, as the divisor hath ciphers; prefixing ciphers if need be.

EXAMPLES.

So, 217·3 ÷ 100 = 2·173	Ans. 419 ÷ 10 =
And 5·16 ÷ 100 =	Ans. ·21 ÷ 1000 =

CONTRACTION III.

WHEN there are many figures in the divisor; or when only a certain number of decimals are necessary to be retained in the quotient; then take only as many figures of the divisor as will be equal to the number of figures, both integers and decimals, to be in the quotient, and find how many times they may be contained in the first figures of the dividend, as usual.

Let each remainder be a new dividend; and for every such dividend, leave out one figure more on the right-hand side of the divisor; remembering to carry for the increase of the figures cut off, as in the 2d contraction in Multiplication.

Note. When there are not so many figures in the divisor as are required to be in the quotient, begin the operation with all the figures, and continue it as usual till the number of figures in the divisor be equal to those remaining to be found in the quotient; after which begin the contraction.

EXAMPLES.

1. Divide 2508·92806 by 92·41035, so as to have only four decimals in the quotient, in which case the quotient will contain six figures.

Contracted.	*Common.*
92·4103,5) 2508·928,06 (27·1498	92.4103,5) 2508·928,06 (27·1498
660721	66072106
13849	13848610
4608	46075750
912	91116100
80	79467850
6	5539570

2. Divide 4109·2351 by 230·409, so that the quotient may contain only four decimals. Ans. 17·8345.

3. Divide 37·10438 by 5713·96, that the quotient may contain only five decimals. Ans. ·00649.

4. Divide 913·08 by 2137·2, that the quotient may contain only three decimals.

REDUCTION OF DECIMALS.

CASE I.

To reduce a Vulgar Fraction to its equivalent Decimal.

DIVIDE the numerator by the denominator, as in Division of Decimals, annexing ciphers to the numerator as far as necessary; so shall the quotient be the decimal required*.

---

* The following method of throwing a vulgar fraction, whose denominator is a prime number, into a decimal consisting of a great number of figures, is given by Mr. *Colson* in page 162 of *Sir Isaac Newton's Fluxions*.

EXAMPLE.

Let $\frac{1}{29}$ be the fraction which is to be converted into an equivalent decimal.

Then, by dividing in the common way till the remainder becomes a single figure, we shall have $\frac{1}{29} = \cdot03448\frac{8}{29}$ for the complete quotient, and this equation being multiplied by the numerator 8, will give $\frac{8}{29} = \cdot27584\frac{64}{29}$, or rather $\frac{8}{29} = \cdot27586\frac{6}{29}$: and if this be substituted instead of the fraction in the first equation, it will make $\frac{1}{29} = \cdot0344827586\frac{6}{29}$. Again, let this equation be multiplied by 6, and it will give $\frac{6}{29} = \cdot2068965517\frac{7}{29}$; and then by substituting as before

$$\frac{1}{29} = \cdot03448275862068965517\frac{7}{29};$$

and so on, as far as may be thought proper; each fresh multiplication doubling the number of figures in the decimal value of the fraction.

In the present instance the decimal *circulates* in a complete period of 28 figures, i. e. one less than the denominator of the fraction. This, again, may be divided into equal periods, each of 14 figures, as below:

·03448275862068
·96551724137931

in which it will be found that each figure with the figure vertically below it makes 9; $0+9=9$; $3+6=9$; and so on. This circulate also comprehends all the separate values of $\frac{2}{29}$, $\frac{3}{29}$, $\frac{4}{29}$, &c. in corresponding circulates of 28 figures, only each beginning in a distinct place, easily ascertainable. Thus, $\frac{2}{29} = \cdot06896$, &c. beginning at the 12th place of the primitive circulate. $\frac{3}{29} = \cdot103448$, &c. beginning at the 28th place. So that, in fact, this circle includes 28 complete circles.

See, on this curious subject, Mr. Goodwyn's Tables of Decimal Circles, and the *Ladies' Diary* for 1824.

EXAMPLES.

1. Reduce $\frac{7}{24}$ to a decimal.

$24 = 4 \times 6$. Then

```
4) 7·
6) 1·750000
    ·291666 &c.
```

2. Reduce $\frac{1}{4}$, and $\frac{1}{2}$, and $\frac{3}{4}$, to decimals. Ans. ·25, and ·5, and ·75.
3. Reduce $\frac{5}{8}$ to a decimal. Ans. ·625.
4. Reduce $\frac{3}{25}$ to a decimal. Ans. ·12.
5. Reduce $\frac{6}{192}$ to a decimal. Ans. ·03125.
6. Reduce $\frac{550}{3842}$ to a decimal. Ans. ·143154 &c.

CASE II.

To find the Value of a Decimal in terms of the Inferior Denominations.

Multiply the decimal by the number of parts in the next lower denomination; and cut off as many places for a remainder to the right-hand, as there are places in the given decimal.

Multiply that remainder by the parts in the next lower denomination again, cutting off for another remainder as before.

Proceed in the same manner through all the parts of the integer; then the several denominations separated on the left-hand will make up the answer.

Note, This operation is the same as Reduction Descending in whole numbers.

EXAMPLES.

1. Required to find the value of ·775 pounds sterling.

```
    ·775
     20
  ------
s 15·500
     12
  ------
d  6·000     Ans. 15s 6d.
```

2. What is the value of ·625 shil? Ans. $7\frac{1}{2}d$.
3. What is the value of ·8635*l*? Ans. 17*s* 3·24*d*.
4. What is the value of ·0125 lb troy? Ans. 3 dwts.
5. What is the value of ·4694 lb troy?
Ans. 5 oz 12 dwts 15·744 gr.
6. What is the value of ·625 cwt? Ans. 2 qr 14 lb.
7. What is the value of ·009943 miles?
Ans. 17 yd 1 ft 5·98848 inc.
8. What is the value of ·6875 yd? Ans. 2 qr 3 nls.
9. What is the value of ·3375 acr? Ans. 1 rd 14 poles.
10. What is the value of ·2083 hhd of wine?
Ans. 13·4229 gal.

CASE III.

To reduce Integers or Decimals to Equivalent Decimals of Higher Denominations.

DIVIDE by the number of parts in the next higher denomination; continuing the operation to as many higher denominations as may be necessary, the same as in Reduction Ascending of whole numbers.

EXAMPLES.

1. Reduce 1 dwt to the decimal of a pound troy.

20	1 dwt
12	0·05 oz
	0·004166 &c. lb. Ans.

2. Reduce 9*d* to the decimal of a pound. Ans. ·0·375*l*.
3. Reduce 7 drams to the decimal of a pound avoird.
Ans. ·02734375 lb.
4. Reduce ·26*d* to the decimal of a *l*. Ans. ·0910833 &c. *l*.
5. Reduce 2·15 lb to the decimal of a cwt.
Ans. ·019196 + cwt.
6. Reduce 24 yards to the decimal of a mile.
Ans. ·013636 &c. mile.
7. Reduce ·056 pole to the decimal of an acre.
Ans. ·00035 ac.
8. Reduce 1·2 pint of wine to the decimal of a hhd.
Ans. ·00238 + hhd.
9. Reduce 14 minutes to the decimal of a day.
Ans. ·009722 &c. da.
10. Reduce ·21 pint to the decimal of a peck.
Ans. ·031325 pec.
11. Reduce 28″ 12‴ to the decimal of a minute.

NOTE, *When there are several numbers, to be reduced all to the decimal of the highest;*

Set the given numbers directly under each other, for dividends, proceeding orderly from the lowest denomination to the highest.

Opposite to each dividend, on the left-hand, set such a number for a divisor as will bring it to the next higher name; drawing a perpendicular line between all the divisors and dividends.

Begin at the uppermost, and perform all the divisions: only observing to set the quotient of each division, as decimal parts, on the right-hand of the dividend next below it; so shall the last quotient be the decimal required.

EXAMPLES.

1. Reduce 17*s* 9¾*d* to the decimal of a pound.

4	3·
12	9·75
20	17·8125
	£0·890625 Ans.

2. Reduce 19*l* 17*s* 3¼*d* to a *l*. Ans. 19·86354166 &c. *l*.
3. Reduce 15*s* 6*d* to the decimal of a *l*. Ans. ·775*l*.
4. Reduce 7½*d* to the decimal of a shilling. Ans. ·625*s*.
5. Reduce 5 oz 12 dwts 16 gr to lb. Ans. ·46944 &c. lb.

RULE OF THREE IN DECIMALS.

PREPARE the terms, by reducing the vulgar fractions to decimals, and any compound number either to decimals of the higher denominations, or to integers of the lower, also the first and third terms to the same name: Then multiply and divide as in whole numbers.

Note. Any of the convenient Examples in the Rule of Three or Rule of Five in Integers, or Vulgar Fractions, may be taken as proper examples to the same rules in Decimals. —The following example, which is the first in Vulgar Fractions, is wrought out here, to show the method.

If $\frac{3}{8}$ of a yard of velvet cost $\frac{2}{5}l$, what will $\frac{5}{16}$ yd cost?

$\frac{3}{8} = \cdot375$

$\frac{2}{5} = \cdot4$

$\frac{5}{16} = \cdot3125$

```
 yd     l      yd      l           s d
·375 : ·4 :: ·3125 : ·333 &c. or 6 8
               ·4
             ------
     ·375)  ·12500  (·333333 &c.
             1250       20
              125  -----------
                    s 6·66666 &c.
                         12
                   ---------------
Ans. 6s 8d.        d 7·99999 &c. = 8d.
```

DUODECIMALS.

Duodecimals, or Cross Multiplication, is a rule used by workmen and artificers, in computing the contents of their works.

Dimensions are usually taken in feet, inches, and quarters; any parts smaller than these being neglected as of no consequence. And the same in multiplying them together, or computing the contents. The method is as follows.

Set down the two dimensions to be multiplied together, one under the other, so that feet may stand under feet, inches under inches, &c.

Multiply each term in the multiplicand, beginning at the lowest, by the feet in the multiplier, and set the result of each straight under its corresponding term, observing to carry 1 for every 12, from the inches to the feet.

In like manner, multiply all the multiplicand by the inches and parts of the multiplier, and set the result of each term one place removed to the right-hand of those in the multiplicand; omitting, however, what is below parts of inches, only carrying to these the proper numbers of units from the lowest denomination.

Or, instead of multiplying by the inches, take such parts of the multiplicand as these are of a foot.

Then add the two lines together, after the manner of Compound Addition, carrying 1 to the feet for every 12 inches, when these come to so many.

EXAMPLES.

1. Multiply 4 f 7 inc.
   by 6 4

	27	6
	1	6⅓
Ans.	29	0⅓

2. Multiply 14 f 9 inc.
   by 4 6

	59	0
	7	4½
Ans.	66	4½

3. Multiply 5 feet 7 inches by 9 f 6 inc. Ans. 43 f 6½ inc.
4. Multiply 12 f 5 inc by 6 f 8 inc. Ans. 82 9⅓
5. Multiply 35 f 4½ inc by 12 f 3 inc. Ans. 433 4⅛
6. Multiply 64 f 6 inc by 8 f 9¼ inc. Ans. 565 8⅝

Note. The denomination which occupies the place of inches in these products, means not square inches, but *rectangles of an inch broad and a foot long.* Thus, the answer to the first example is 29 sq. feet, 4 sq. inches; to the second 66 sq. feet, 54 sq. inches.

INVOLUTION.

INVOLUTION is the raising of Powers from any given number, as a root.

A Power is a quantity produced by multiplying any given number, called the Root, a certain number of times continually by itself. Thus,

$$2 = 2 \text{ is the root, or 1st power of 2.}$$
$$2 \times 2 = 4 \text{ is the 2d power, or square of 2.}$$
$$2 \times 2 \times 2 = 8 \text{ is the 3d power, or cube of 2.}$$
$$2 \times 2 \times 2 \times 2 = 16 \text{ is the 4th power of 2, \&c.}$$

And in this manner may be calculated the following Table of the first nine powers of the first 9 numbers.

TABLE OF THE FIRST NINE POWERS OF NUMBERS.

1st	2d	3d	4th	5th	6th	7th	8th	9th
1	1	1	1	1	1	1	1	1
2	4	8	16	32	64	128	256	512
3	9	27	81	243	729	2187	6561	19683
4	16	64	256	1024	4096	16384	65536	262144
5	25	125	625	3125	15625	78125	390625	1953125
6	36	216	1296	7776	46656	279936	1679616	10077696
7	49	343	2401	16807	117649	823543	5764801	40353607
8	64	512	4096	32768	262144	2097152	16777216	134217728
9	81	729	6561	59049	531441	4782969	43046721	387420489

The Index or Exponent of a Power, is the number denoting the height or degree of that power ; and it is 1 more than the number of multiplications used in producing the same. So 1 is the index or exponent of the 1st power or root, 2 of the 2d power or square, 3 of the third power or cube, 4 of the 4th power, and so on.

Powers, that are to be raised, are usually denoted by placing the index above the root or first power.

So $2^2 = 4$ is the 2d power of 2.
$2^3 = 8$ is the 3d power of 2.
$2^4 = 16$ is the 4th power of 2.
540^4 is the 4th power of 540, &c.

When two or more powers are multiplied together, their product is that power whose index is the sum of the exponent of the factors or powers multiplied. Or the multiplication of the powers, answers to the addition of the indices. Thus, in the following powers of 2,

1st	2d	3d	4th	5th	6th	7th	8th	9th	10th
2	4	8	16	32	64	128	256	512	1024
or 2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}

Here, $4 \times 4 = 16$, and $2 + 2 = 4$ its index;
and $8 \times 16 = 128$, and $3 + 4 = 7$ its index;
also $16 \times 64 = 1024$, and $4 + 6 = 10$ its index.

OTHER EXAMPLES.

1. What is the 2d power of 45? Ans. 2025.
2. What is the square of 4·16? Ans. 17·3056.
3. What is the 3d power of 3·5? Ans. 42·875.
4. What is the 5th power of ·029? Ans. ·000000020511149.
5. What is the square of $\frac{2}{3}$? Ans. $\frac{4}{9}$.
6. What is the 3d power of $\frac{5}{7}$? Ans. $\frac{125}{343}$.
7. What is the 4th power of $\frac{3}{4}$? Ans. $\frac{81}{256}$.

EVOLUTION.

Evolution, or the reverse of Involution, is the extracting or finding the roots of any given powers.

The root of any number, or power, is such a number, as being multiplied into itself a certain number of times, will produce that power. Thus, 2 is the square root, or 2d root of 4, because $2^2 = 2 \times 2 = 4$; and 3 is the cube root or 3d root of 27, because $3^3 = 3 \times 3 \times 3 = 27$.

Any power of a given number or root may be found exactly, namely, by multiplying the number continually into itself. But there are many numbers of which a proposed root can never be exactly found. Yet, by means of decimals, we may approximate or approach towards the root, to any degree of exactness.

Those roots which only approximate, are called Surd Roots; but those which can be found quite exact, are called Rational Roots. Thus, the square root of 3 is a surd root; but the square root of 4 is a rational root, being equal to 2: also the cube root of 8 is rational, being equal to 2; but the cube root of 9 is surd or irrational.

Roots are sometimes denoted by writing the character $\sqrt{}$ before the power, with the index of the root against it. Thus, the 3d root of 20 is expressed by $\sqrt[3]{20}$; and the square root or 2d root of it is $\sqrt{20}$, the index 2 being always omitted, when only the square root is designed.

When the power is expressed by several numbers, with the sign + or − between them, a line is drawn from the top of the sign over all the parts of it: thus the third root of 45 − 12 is $\sqrt[3]{45-12}$, or thus, $\sqrt[3]{}(45-12)$, inclosing the numbers in parentheses.

But all roots are now often designed like powers, with fractional indices; thus, the square root of 8 is $8^{\frac{1}{2}}$, the cube root of 25 is $25^{\frac{1}{3}}$, and the 4th root of 45 − 18 is $(45-18)^{\frac{1}{4}}$.

TO EXTRACT THE SQUARE ROOT.

* Divide the given number into periods of two figures each, by setting a point over the place of units, another over the place of hundreds, and so on, over every second figure, both to the left-hand in integers, and to the right in decimals.

---

* The reason for separating the figures of the dividend into periods or portions of two places each, is, that the square of any single figure never consists of more than two places; the square of a number of two figures, of not more than four places, and so on. So that there will be as many figures in the root as the given number contains periods so divided or parted off.

And the reason of the several steps in the operation appears from the algebraic form of the square of any number of terms, whether two or three or more. Thus

$(a+b)^2 = a^2 + 2ab + b^2 = a^2 + (2a+b)\,b$, the square of two terms; where it appears that a is the first term of the root, and b the second term; also a the first divisor, and the new divisor is $2a+b$, or double the first term increased by the second. And hence the manner of extraction is thus:

$$\begin{array}{rl} \text{1st divisor } a) & a^2 + 2ab + b^2 \ (a+b \text{ the root.} \\ & a^2 \\ \hline \text{2d divisor } 2a+b & 2ab + b^2 \\ b & 2ab + b^2 \\ \hline \end{array}$$

Again, for a root of three parts, a, b, c, thus:

$$(a+b+c)^2 = a^2 + 2ab + b^2 + 2ac + 2bc + c^2 = a^2 + (2a+b)b + (2a+2b+c)c,$$ the square of three terms, where a is the first term of the root, b the second, and c the third term; also a the first divisor, $2a+b$ the second, and $2a+2b+c$ the third, each consisting of the double of the root increased by the next term of the same. And the mode of extraction agrees with the rule. See farther, Case 2, of Evolution in the Algebra.

For an approximation observe that $\sqrt{a^2+n} = a.\dfrac{4a^2+3n}{4a^2+n}$ nearly in all cases where n is small in respect of a.

Find the greatest square in the first period on the left-hand, and set its root on the right-hand of the given number, after the manner of a quotient figure in Division.

Subtract the square thus found from the said period, and to the remainder annex the two figures of the next following period, for a dividend.

Double the root above mentioned for a divisor; and find how often it is contained in the said dividend, exclusive of its right-hand figure; and set that quotient figure both in the quotient and divisor.

Multiply the whole augmented divisor by this last quotient figure, and subtract the product from the said dividend, bringing down to it the next period of the given number, for a new dividend.

Repeat the same process over again, viz. find another new divisor, by doubling all the figures now found in the root; from which, and the last dividend, find the next figure of the root as before; and so on through all the periods, to the last.

Note, The best way of doubling the root, to form the new divisors, is by adding the last figure always to the last divisor, as appears in the following examples.—Also, after the figures belonging to the given number are all exhausted, the operation may be continued into decimals at pleasure, by adding any number of periods of ciphers, two in each period.

EXAMPLES.

1. To find the square root of 20506624.

```
        29506624 (5432 the root.
        25
   ----------
   104 | 450
     4 | 416
   -----------
  1083 | 3466
     3 | 3249
  -------------
 10862 | 21724
     2 | 21724
  -------------
```

NOTE, *When the root is to be extracted to many places of figures, the work may be considerably shortened, thus:*

Having proceeded in the extraction after the common method, till there be found half the required number of figures in the root, or one figure more; then, for the rest, divide the last remainder by its corresponding divisor, after the manner of the third contraction in Division of Decimals; thus,

2. To find the root of 2 to nine places of figures.

```
                2 (1·41421356 the root.
                1
         ----------
         24 | 100
          4 |  96
         ------------
         281 |  400
           1 |  281
         --------------
         2824 | 11900
            4 | 11296
         -----------------
         28282 |  60400
             2 |  56564
         -----------------
         28284)    3836 (1356
          ....     1008
                    160
                     19
                      2
```

3. What is the square root of 2025? Ans. 45.
4. What is the square root of 17·3056? Ans. 4·16.
5. What is the square root of ·000729? Ans. ·027.
6. What is the square root of 3? Ans. 1·732050.
7. What is the square root of 5? Ans. 2·236068.
8. What is the square root of 6? Ans. 2·449489.
9. What is the square root of 7? Ans. 2·645751.
10. What is the square root of 10? Ans. 3·162277.
11. What is the square root of 11? Ans. 3·316624.
12. What is the square root of 12? Ans. 3·464101.

RULES FOR THE SQUARE ROOTS OF VULGAR FRACTIONS AND MIXED NUMBERS.

FIRST prepare all vulgar fractions, by reducing them to their least terms, both for this and all other roots. Then

1. Take the root of the numerator and of the denominator for the respective terms of the root required; which is the best way if the denominator be a complete power: but if it be not, then

2. Multiply the numerator and denominator together; take the root of the product: this root being made the numerator to the denominator of the given fraction, or made the denominator to the numerator of it, will form the fractional root required.

$$\text{That is, } \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{ab}}{b} = \frac{a}{\sqrt{ab}}.$$

This rule will serve, whether the root be finite or infinite.

3. Or reduce the vulgar fraction to a decimal, and extract its root.

4. Mixed numbers may be either reduced to improper fractions, and extracted by the first or second rule, or the vulgar fraction may be reduced to a decimal, then joined to the integer, and the root of the whole extracted.

EXAMPLES.

1. What is the root of $\frac{25}{36}$? Ans. $\frac{5}{6}$.
2. What is the root of $\frac{27}{147}$? Ans. $\frac{3}{7}$.
3. What is the root of $\frac{9}{12}$? Ans. 0·866025.
4. What is the root of $\frac{5}{12}$? Ans. 0·645497.
5. What is the root of $17\frac{3}{8}$? Ans. 4·168333.

By means of the square root also may readily be found the 4th root, or the 8th root, or the 16th root, &c. that is, the root of any power whose index is some power of the number 2; namely, by extracting so often the square root as is denoted by that power of 2; that is, two extractions for the 4th root, three for the 8th root, and so on.

So, to find the 4th root of the number 21035·8, extract the square root two times as follows:

```
21035·8000  ( 145·037237 ( 12·0431407 the 4th root.
1               1
---------       -----
24 | 110        22 | 45
 4 |  96         2 | 44
-----------     -----------
285 | 1435      2404  | 10372
  5 | 1425         4  |  9616
------------------------------------
29003 | 108000 24083 |  75637
    3 |  87009     3 |  72249
      --------        --------
        20991 ( 7237     3388 ( 1407
          687             980
          107              17
```

Ex. 2. What is the 4th root of 97·41 ?

TO EXTRACT THE CUBE ROOT.

I. *By the Common Rule**.

1. HAVING divided the given number into periods of three figures each (by setting a point over the place of units, and also over every third figure, from thence, to the left hand in whole numbers, and to the right in decimals), find the nearest less cube to the first period; set its root in the quotient, and subtract the said cube from the first period; to the remainder bring down the second period, and call this the resolvend.

2. To three times the square of the root, just found, add three times the root itself, setting this one place more to the right than the former, and call this sum the divisor. Then divide the resolvend, wanting the last figure, by the divisor, for the next figure of the root, which annex to the former;

---

* The reason for pointing the given number into periods of three figures each, is because the cube of one figure never amounts to more than three places. And, for a similar reason, a given number is pointed into periods of four figures for the 4th root, of five figures for the 5th root, and so on.

The reason for the other parts of the rule depends on the algebraic formation of a cube: for, if the root consist of the two parts $a+b$, then its cube is as follows: $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$; where a is the root of the first part a^3; the resolvend is $3a^2b + 3ab^2 + b^3$; which is also the same as the three parts of the subtrahend; also the divisor is $3a^2 + 3a$, by which dividing the first two terms of the resolvend $3a^2b + ab^2$, gives b for the second part of the root; and so on.

calling this last figure *e*, and the part of the root before found let be called *a*.

3. Add all together these three products, namely, thrice *a* square multiplied by *e*, thrice *a* multiplied by *e* square, and *e* cube, setting each of them one place more to the right than the former, and call the sum the subtrahend; which must not exceed the resolvend; but if it does, then make the last figure *e* less, and repeat the operation for finding the subtrahend, till it be less than the resolvend.

4. From the resolvend take the subtrahend, and to the remainder join the next period of the given number for a new resolvend; to which form a new divisor from the whole root now found; and from thence another figure of the root, as directed in article 2, and so on.

EXAMPLE.

To extract the cube root of 48228·544.

$3 \times 3^2 = 27$	48228·544 (36·4 root.
$3 \times 3 = \;\;09$	27
Divisor 279	21228 resolvend.

$3 \times 3^2 \times 6 = 162$
$3 \times 3 \times 6^2 = \;\;324$ } add
$6^3 = \;\;\;216$

$3 \times 36^2 = 3888$	19656 subtrahend.
$3 \times 36 = \;\;108$	
38988	1572544 resolvend.

$3 \times 36^2 \times 4 = 15552$
$3 \times 36 \times 4^2 = \;\;1728$ } add
$4^3 = \;\;\;\;64$

1572544 subtrahend.

0000000 remainder.

Ex. 2. Extract the cube root of 571482·19.

Ex. 3. Extract the cube root of 1628·1582.

Ex. 4. Extract the cube root of 1332.

II. *To extract the Cube Root by a short Way**.

1. By trials, or by the table of roots at p. 93, &c. take the nearest rational cube to the given number, whether it be greater or less; and call it the assumed cube.

2. Then say, by the Rule of Three, As the sum of the given number, and double the assumed cube, is to the sum of the assumed cube and double the given number, so is the root of the assumed cube, to the root required, nearly. Or, As the first sum is to the difference of the given and assumed cube, so is the assumed root to the difference of the roots nearly.

3. Again, by using, in like manner, the cube root of the last found as a new assumed cube, another root will be obtained still nearer. And so on as far as we please; using always the cube of the last found root, for the assumed cube.

EXAMPLE.

To find the cube root of 21034·8.

Here we soon find that the root lies between 20 and 30, and then between 27 and 28. Taking therefore 27, its cube is 19683, which is the assumed cube. Then

```
       19683        21035·8
           2              2
    --------       --------
       39366        42071·6
     21035·8          19683
    --------       --------
As   60401·8   :    61754·6  : : 27 : 27·6047.
                         27
                  ---------
                    4322822
                   1235092
                  ---------
       60401·8)   1667374·2  (27·6047 the root nearly.
                     459338
                      36525
                        284
                         42
```

* The method usually given for extracting the cube root, is so exceedingly tedious, and difficult to be remembered, that various other

Again, for a second operation, the cube of this root is 21035·318645155823, and the process by the latter method will be thus :

21035·318645 &c.

42070·637290	21035·8
21035·8	21035·318645 &c.

As 63106·43729 : diff. ·481355 : : 27·6047 : the diff. ·000210560.

conseq. the root req. is 27·604910560.

Ex. 2. To extract the cube root of ·67.

Ex. 3. To extract the cube root of ·01.

TO EXTRACT ANY ROOT WHATEVER*.

Let P be the given power or number, n the index of the power, A the assumed power, r its root, R the required root of P. Then say,

As the sum of $n + 1$ times A and $n - 1$ times P,
is to the sum of $n + 1$ times P and $n - 1$ times A ;
so is the assumed root r, to the required root R,

Or, as half the said sum of $n + 1$ times A and $n - 1$ times P, is to the difference between the given and assumed powers,

---

approximating rules have been invented, viz. by Newton, Raphson, Halley, De Lagny, Simpson, Emerson, and several other mathematicians ; but no one that I have yet seen is so simple in it form, or seems so well adapted for general use, as that above given. This rule is the same in effect as Dr. Halley's rational formula, but more commodiously expressed ; and the first investigation of it was given in my Tracts, p. 49. The algebraic form of it is this:

As $P + 2A : A + 2P :: r : R$. Or,
As $P + 2A : P \backsim A :: r : R \backsim r$;

where P is the given number, A is the assumed nearest cube, r the cube root of A, and R the root of P sought.

* This is a very general approximating rule, of which that for the cube root is a particular case, and is the best adapted for practice, and for memory, of any that I have yet seen. It was first discovered in this form by myself, and the investigation and use of it were given at large in my Tracts, p. 45, &c.

so is the assumed root r, to the difference between the true and assumed roots; which difference, added or subtracted, as the case requires, gives the true root nearly.

That is, $(n+1)\,A + (n-1)\,P : (n+1)\,P + (n-1)\,A :: r : R$.

Or, $(n+1)\,\frac{1}{2}A + (n-1)\,\frac{1}{2}P : P \backsim A :: r : R \backsim r$.

And the operation may be repeated as often as we please, by using always the last found root for the assumed root, and its nth power for the assumed power A.

EXAMPLE.

To extract the 5th root of 21035·8.

Here it appears that the 5th root is between 7·3 and 7·4. Taking 7·3, its 5th power is 20730 71593. Hence we have $P = 21035{\cdot}8$, $n = 5$, $r = 7{\cdot}3$, and $A = 20730{\cdot}71593$; then $n+1 \,.\, \frac{1}{2}A + n - 1 \,.\, \frac{1}{2}P : P \backsim A :: r : R \backsim r$, that is,
3×20730·71593 + 2×21053·8 : 305·084 : : 7·3 : ·0213605

3	2	7·3
62192·14779	42071·6	915252
42071·6		2135568
104263·74779		2227·1132(·0213605 = $R \backsim r$
		7·3 = r, add
		7·321360 = R, true to the last figure.

OTHER EXAMPLES.

1. What is the 3d root of 2? Ans. 1·259921.
2. What is the 3d root of 3214? Ans. 14·75758.
3. What is the 4th root of 2? Ans. 1·189207.
4. What is the 4th root of 97·41? Ans. 3·1415999.
5. What is the 5th root of 2? Ans. 1·148699.
6. What is the 6th root of 21035·8? Ans. 5·254037.
7. What is the 6th root of 2? Ans. 1·122462.
8. What is the 7th root of 21035·8? Ans. 4·145392.
9. What is the 7th root of 2? Ans. 1·104089.
10. What is the 8th root of 21035·8? Ans. 3·470323.

11. What is the 8th root of 2 ? Ans. 1.090508.
12. What is the 9th root of 21035·8 ? Ans. 3.022239.
13. What is the 9th root of 2 ? Ans. 1·080059.

The following is a Table of squares and cubes, and also the square roots and cube roots, of all numbers from 1 to 1000, which will be found very useful on many occasions, in numeral calculations, when roots or powers are concerned.

The use of this table may be greatly extended, either by the addition of ciphers, or by changing the places of the separating points. The following examples will suffice to suggest the method.

Root.	Square.	Cube.
36·	1296·	46656·
360·	129600·	46656000·
3600·	12960000·	46656000000·
546·	298116·	162771336·
54·6	2981·16	162771·336
·546	·298116	·162771336

For a simple and ingenious method of constructing tables of square and cube roots, and the reciprocals of numbers, see Dr. Hutton's *Tracts on Mathematical and Philosophical Subjects*, vol. i. Tract 24, pa. 459.

Number.	Square.	Cube.	Square Root.	Cube Root.
1	1	1	1·0000000	1·000000
2	4	8	1·4142136	1·259921
3	9	27	1·7320508	1·442250
4	16	64	2·0000000	1·587401
5	25	125	2·2360680	1·709976
6	36	216	2·4494897	1·817121
7	49	343	2·6457513	1·912931
8	64	512	2·8284271	2·000000
9	81	729	3·0000000	2·080084
10	100	1000	3·1622777	2·154435
11	121	1331	3·3166248	2·223980
12	144	1728	3·4641016	2·289428
13	169	2197	3·6055513	2·351335
14	196	2744	3·7416574	2·410142
15	225	3375	3·8729833	2·466212
16	256	4096	4·0000000	2·519842
17	289	4913	4·1231056	2·571282
18	324	5832	4·2426407	2·620741
19	361	6859	4·3588989	2·668402
20	400	8000	4·4721360	2·714418
21	441	9261	4·5825757	2·758924
22	484	10648	4·6904158	2·802039
23	529	12167	4·7958315	2·843867
24	576	13824	4·8989795	2·884499
25	625	15625	5·0000000	2·924018
26	676	17576	5·0990195	2·962496
27	729	19683	5·1961524	3·000000
28	784	21952	5·2915026	3·036589
29	841	24389	5·3851648	3·072317
30	900	27000	5·4772256	3·107232
31	961	29791	5·5677644	3·141381
32	1024	32768	5·6568542	3·174802
33	1089	35937	5·7445626	3·207534
34	1156	39304	5·8309519	3·239612
35	1225	42875	5·9160798	3·271066
36	1296	46656	6·0000000	3·301927
37	1369	50653	6·0827625	3·332222
38	1444	54872	6·1644140	3·361975
39	1521	59319	6·2449980	3·391211
40	1600	64000	6·3245553	3·419952
41	1681	68921	6·4031242	3·448217
42	1764	74088	6·4807407	3·476027
43	1849	79507	6·5574385	3·503398
44	1936	85184	6·6332496	3·530348
45	2025	91125	6·7082039	3·556893
46	2116	97336	6·7823300	3·583048
47	2209	103823	6·8556546	3·608826
48	2304	110592	6·9282032	3·634241
49	2401	117649	7·0000000	3·659306
50	2500	125000	7·0710678	3·684031

Number.	Square.	Cube.	Square Root.	Cube Root.
51	2601	132651	7·1414284	3·708430
52	2704	140608	7·2111026	3·732511
53	2809	148877	7·2801099	3·756286
54	2916	157464	7·3484692	3·779763
55	3025	166375	7·4161985	3·802953
56	3136	175616	7·4833148	3·825862
57	3249	185193	7·5498344	3·848501
58	3364	195112	7·6157731	3·870877
59	3481	205379	7·6811457	3·892996
60	3600	216000	7·7459667	3·914868
61	3721	226981	7·8102497	3·936497
62	3844	238328	7·8740079	3·957892
63	3969	250047	7·9372539	3·979057
64	4096	262144	8·0000000	4·000000
65	4225	274625	8·0622577	4·020726
66	4356	287496	8·1240384	4·041240
67	4489	300763	8·1853528	4·061548
68	4624	314432	8·2462113	4·081655
69	4761	328509	8·3066239	4·101566
70	4900	343000	8·3666003	4·121285
71	5041	357911	8·4261498	4·140818
72	5184	373248	8·4852814	4·160168
73	5329	389017	8·5440037	4·179339
74	5476	405224	8·6023253	4·198336
75	5625	421875	8·6602540	4·217163
76	5776	438976	8·7177979	4·235824
77	5929	456533	8·7749644	4·254321
78	6084	474552	8·8317609	4·272659
79	6241	493039	8·8881944	4·290841
80	6400	512000	8·9442719	4·308870
81	6561	531441	9·0000000	4·326749
82	6724	551368	9·0553851	4·344481
83	6889	571787	9·1104336	4·362071
84	7056	592704	9·1651514	4·379519
85	7225	614125	9·2195445	4·396830
86	7396	636056	9·2736185	4·414005
87	7569	658503	9·3273791	4·431047
88	7744	681472	9·3808315	4·447960
89	7921	704969	9·4339811	4·464745
90	8100	729000	9·4868330	4·481405
91	8281	753571	9·5393920	4·497941
92	8464	778688	9·5916630	4·514357
93	8649	804357	9·6436508	4·530655
94	8836	830584	9·6953597	4·546836
95	9025	857375	9·7467943	4·562903
96	9216	884736	9·7979590	4·578857
97	9409	912673	9·8488578	4·594701
98	9604	941192	9·8994949	4·610436
99	9801	970299	9·9498744	4·626065
100	10000	1000000	10·0000000	4·641589

Number.	Square.	Cube.	Square Root.	Cube Root.
101	10201	1030301	10·0498756	4·657010
102	10404	1061208	10·0995049	4·672329
103	10609	1092727	10·1488916	4·687548
104	10816	1124864	10·1980390	4·702669
105	11025	1157625	10·2469508	4·717694
106	11236	1191016	10·2956301	4·732624
107	11449	1225043	10·3440804	4·747459
108	11664	1259712	10·3923048	4·762203
109	11881	1295029	10·4403065	4·776856
110	12100	1331000	10·4880885	4·791420
111	12321	1367631	10·5356538	4·805896
112	12544	1404928	10·5830052	4·820284
113	12769	1442897	10·6301458	4·834588
114	12996	1481544	10·6770783	4·848808
115	13225	1520875	10·7238053	4·862944
116	13456	1560896	10·7703296	4·876999
117	13698	1601613	10·8166538	4·890973
118	13924	1643032	10·8627805	4·904868
119	14161	1685159	10·9087121	4·918685
120	14400	1728000	10·9544512	4·932424
121	14641	1771561	11·0000000	4·946088
122	14884	1815848	11·0453610	4·959676
123	15129	1860867	11·0905365	4·973190
124	15376	1906624	11·1355287	4·986631
125	15625	1953125	11·1803399	5·000000
126	15876	2000376	11·2249722	5·013298
127	16129	2048383	11·2694277	5·026526
128	16384	2097152	11·3137085	5·039684
129	16641	2146689	11·3578167	5·052774
130	16900	2197000	11·4017543	5·065797
131	17161	2248091	11·4455231	5·078753
132	17424	2299968	11·4891253	5·091643
133	17689	2352637	11·5325626	5·104469
134	17956	2406104	11·5758369	5·117230
135	18225	2460375	11·6189500	5·129928
136	18496	2515456	11·6619038	5·142563
137	18769	2571353	11·7046999	5·155137
138	19044	2628072	11·7473444	5·167649
139	19321	2685619	11·7898261	5·180101
140	19600	2744000	11·8321596	5·192494
141	19881	2803221	11·8743421	5·204828
142	20164	2863288	11·9163753	5·217103
143	20449	2924207	11·9582607	5·229321
144	20736	2985984	12·0000000	5·241483
145	21025	3048625	12·0415946	5·253588
146	21316	3112136	12·0830460	5·265637
147	21609	3176523	12·1243557	5·277632
148	21904	3241792	12·1655251	5·289572
149	22201	3307949	12·2065556	5·301459
150	22500	3375000	12·2474487	5·313293

Number.	Square.	Cube.	Square Root.	Cube Root.
151	22801	3442951	12·2882037	5·325074
152	23104	3511808	12·3288280	5·336803
153	23409	3581577	12·3693169	5·348481
154	23716	3652264	12·4096736	5·360108
155	24025	3723875	12·4498996	5·371685
156	24336	3796416	12·4899960	5·383213
157	24649	3869893	12·5299641	5·394691
158	24964	3944312	12·5698051	5·406120
159	25281	4019679	12·6095202	5·417501
160	25600	4096000	12·6491106	5·428835
161	25921	4173281	12·6885775	5·440122
162	26244	4251528	12·7279221	5·451362
163	26569	4330747	12·7671453	5·462556
164	26896	4410944	12·8062485	5·473704
165	27225	4492125	12·8452326	5·484806
166	27556	4574296	12·8840987	5·495865
167	27889	4657463	12·9228480	5·506879
168	28224	4741632	12·9614814	5·517848
169	28561	4826809	13·0000000	5·528775
170	28900	4913000	13·0384048	5·539658
171	29241	5000211	13·0766968	5·550499
172	29584	5088448	13·1148770	5·561298
173	29929	5177717	13·1529464	5·572055
174	30276	5268024	13·1909060	5·582770
175	30625	5359375	13·2287566	5·593445
176	30976	5451776	13·2664992	5·604079
177	31329	5545233	13·3041347	5·614673
178	31684	5639752	13·3416641	5·625226
179	32041	5735339	13·3790882	5·635741
180	32400	5832000	13·4164079	5·646216
181	32761	5929741	13·4536240	5·656653
182	33124	6028568	13·4907376	5·667051
183	33489	6128487	13·5277493	5·677411
184	33856	6229504	13·5646600	5·687734
185	34225	6331625	13·6014705	5·698019
186	34596	6434856	13·6381817	5·708267
187	34969	6539203	13·6747943	5·718479
188	35344	6644672	13·7113092	5·728654
189	35721	6751269	13·7477271	5·738794
190	36100	6859000	13·7840488	5·748897
191	36481	6967871	13·8202750	5·758965
192	36864	7077888	13·8564065	5·768998
193	37249	7189057	13·8924440	5·778996
194	37636	7301384	13·9283883	5·788960
195	38025	7414875	13·9642400	5·798890
196	38416	7529536	14·0000000	5·808786
197	38809	7645373	14·0356688	5·818648
198	39204	7762392	14·0712473	5·828476
199	39601	7880599	14·1067360	5·838272
200	40000	8000000	14·1421356	5·848035

Number.	Square.	Cube.	Square Root.	Cube Root.
201	40401	8120601	14·1774469	5·857766
202	40804	8242408	14·2126704	5·867464
203	41209	8365427	14·2478068	5·877130
204	41616	8489664	14·2828569	5·886765
205	42025	8615125	14·3178211	5·896368
206	42436	8741816	14·3527001	5·905941
207	42849	8869743	14·3874946	5·915482
208	43264	8998912	14·4222051	5·924992
209	43681	9123329	14·4568323	5·934473
210	44100	9261000	14·4913767	5·943922
211	44521	9393931	14·5258390	5·953342
212	44944	9528128	14·5602198	5·962731
213	45369	9663597	14·5945195	5·972091
214	45796	9800344	14·6287388	5·981426
215	46225	9938375	14·6628783	5·990727
216	46656	10077696	14·6969385	6·000000
217	47089	10218313	14·7309199	6·009244
218	47524	10360232	14·7648231	6·018463
219	47961	10503459	14·7986486	6·027650
220	48400	10648000	14·8323970	6·036811
221	48841	10793861	14·8660687	6·045943
222	49284	10941048	14·8996644	6·055048
223	49729	11089567	14·9331845	6·064126
224	50176	11239424	14·9666295	6·073178
225	50625	11390625	15·0000000	6·082201
226	51076	11543176	15·0332964	6·091199
227	51529	11697083	15·0665192	6·100170
228	51984	11852352	15·0996689	6·109115
229	52441	12008989	15·1327460	6·118033
230	52900	12167000	15·1657509	6·126925
231	53361	12326391	15·1986842	6·135792
232	53824	12487168	15·2315462	6·144634
233	54289	12649337	15·2643375	6·153449
234	54756	12812904	15·2970585	6·162239
235	55225	12977875	15·3297097	6·171005
236	55696	13144256	15·3622915	6·179747
237	56169	13312053	15·3948043	6·188463
238	56644	13481272	15·4272486	6·197154
239	57121	13651919	15·4596248	6·205822
240	57600	13824000	15·4919334	6·214465
241	58081	13997521	15·5241747	6·223084
242	58564	14172488	15·5563492	6·231679
243	59049	14348907	15·5884573	6·240251
244	59536	14526789	15·6204994	6·248800
245	60025	14706125	15·6524758	6·257325
246	60516	14886936	15·6843871	6·265826
247	61009	15069223	15·7162336	6·274305
248	61504	15252992	15·7480157	6·282760
249	62001	15438249	15·7797338	6·291195
250	62500	15625000	15·8113883	6·299605

Number.	Square.	Cube.	Square Root.	Cube Root.
251	63001	15813251	15·8429795	6·307994
252	63504	16003008	15·8745079	6·316359
253	64009	16194277	15·9059737	6·324704
254	64516	16387064	15·9373775	6·333026
255	65025	16581375	15·9687194	6·341326
256	65536	16777216	16·0000000	6·349604
257	66049	16974593	16·0312195	6·357861
258	66564	17173512	16·0623784	6·366095
259	67081	17373979	16·0934769	6·374311
260	67600	17576000	16·1245155	6·382504
261	68121	17779581	16·1554944	6·390676
262	68644	17984728	16·1864141	6·398828
263	69169	18191447	16·2172747	6·406958
264	69696	18399744	16·2480768	6·415008
265	70225	18609625	16·2788206	6·423158
266	70756	18821096	16·3095064	6·431228
267	71289	19034163	16·3401346	6·439277
268	71824	19248832	16·3707055	6·447305
269	72361	19465109	16·4012195	6·455315
270	72900	19683000	16·4316767	6·463304
271	73441	19902511	16·4620776	6·471274
272	73984	20123648	16·4924225	6·479224
273	74529	20346417	16·5227116	6·487154
274	75076	20570824	16·5529454	6·495065
275	75625	20796875	16·5831240	6·502956
276	76176	21024576	16·6132477	6·510830
277	76729	21253933	16·6433170	6·518684
278	77284	21484952	16·6733320	6·526519
279	77841	21717639	16·7032931	6·534335
280	78400	21952000	16·7332005	6·542133
281	78961	22188041	16·7630546	6·549912
282	79524	22425768	16·7928556	6·557672
283	80089	22665187	16·8226038	6·565415
284	80656	22906304	16·8522995	6·573139
285	81225	23149125	16·8819430	6·580844
286	81796	23393656	16·9115345	6·588532
287	82369	23639903	16·9410743	6·596202
288	82944	23887872	16·9705627	6·603854
289	83521	24137569	17·0000000	6·611489
290	84100	24389000	17·0293864	6·619106
291	84681	24642171	17·0587221	6·626705
292	85264	24897088	17·0880075	6·634287
293	85849	25153757	17·1172428	6·641852
294	86436	25412184	17·1464282	6·649399
295	87025	25672375	17·1755640	6·656930
296	87616	25934336	17·2046505	6·664444
297	88209	26198073	17·2336879	6·671940
298	88804	26463592	17·2626762	6·679420
299	89401	26730899	17·2916165	6·686882
300	90000	27000000	17·3205081	6·694329

Number.	Square.	Cube.	Square Root.	Cube Root.
301	90601	27270901	17·3493516	6·701759
302	91204	27543608	17·3781472	6·709173
303	91809	27818127	17·4068952	6·716570
304	92416	28094464	17·4355958	6·723951
305	93025	28372625	17·4642492	6·731316
306	93636	28652616	17·4928557	6·738665
307	94249	28934443	17·5214155	6·745997
308	94864	29218112	17·5499288	6·753313
309	95481	29503629	17·5783958	6·760614
310	96100	29791000	17·6068169	6·767899
311	96721	30080231	17·6351921	6·775169
312	97344	30371328	17·6635217	6·782423
313	97969	30664297	17·6918060	6·789661
314	98596	30959144	17·7200451	6·796884
315	99225	31255875	17·7482393	6·804092
316	99856	31554496	17·7763888	6·811284
317	100489	31855013	17·8044938	6·818462
318	101124	32157432	17·8325545	6·825624
319	101761	32461759	17·8605711	6·832771
320	102400	32768000	17·8885438	6·839904
321	103041	33076161	17·9164729	6·847021
322	103684	33386248	17·9443584	6·854124
323	104329	33698267	17·9722008	6·861212
324	104976	34012224	18·0000000	6·868285
325	105625	34328125	18·0277564	6·875344
326	106276	34645976	18·0554701	6·882388
327	106929	34965783	18·0831413	6·889419
328	107584	35287552	18·1107703	6·896435
329	108241	35611289	18·1383571	6·903436
330	108900	35937000	18·1659021	6·910423
331	109561	36264691	18·1934054	6·917396
332	110224	36594368	18·2208672	6·924355
333	110889	36926037	18·2482876	6·931301
334	111556	37259704	18·2756669	6·938232
335	112225	37595375	18·3030052	6·945149
336	112896	37933056	18·3303028	6·952053
337	113569	38272753	18·3575598	6·958943
338	114244	38614472	18·3847763	6·965819
339	114921	38958219	18·4119526	6·972683
340	115600	39304000	18·4390889	6·979532
341	116281	39651821	18·4661853	6·986368
342	116964	40001688	18·4932420	6·993191
343	117649	40353607	18·5202592	7·000000
344	118336	40707584	18·5472370	7·006796
345	119025	41063625	18·5741756	7·013579
346	119716	41421736	18·6010752	7·020349
347	120409	41781923	18·6279360	7·027106
348	121104	42144192	18·6547581	7·033850
349	121801	42508549	18·6815417	7·040581
350	122500	42875000	18·7082869	7·047298

Number.	Square.	Cube.	Square Root.	Cube Root.
351	123201	43243551	18·7349940	7·054004
352	123904	43614208	18·7616630	7·060696
353	124609	43986977	18·7882942	7·067376
354	125316	44361864	18·8148877	7·074044
355	126025	44738875	18·8414437	7·080699
356	126736	45118016	18·8679623	7·087341
357	127449	45499293	18·8944436	7·093971
358	128164	45882712	18·9208879	7·100588
359	128881	46268279	18·9472953	7·107194
360	129600	46656000	18·9736660	7·113786
361	130321	47045881	19·0000000	7·120367
362	131044	47437928	19·0262976	7·126936
363	131769	47832147	19·0525589	7·133492
364	132496	48228544	19·0787840	7·140037
365	133225	48627125	19·1049732	7·146569
366	133956	49027896	19·1311265	7·153090
367	134689	49430863	19·1572441	7·159599
368	135424	49836032	19·1833261	7·166096
369	136161	50243409	19·2093727	7·172580
370	136900	50653000	19·2353841	7·179054
371	137641	51064811	19·2613603	7·185516
372	138384	51478848	19·2873015	7·191966
373	139129	51895117	19·3132079	7·198405
374	139876	52313624	19·3390796	7·204832
375	140625	52734375	19·3649167	7·211248
376	141376	53157376	19·3907194	7·217652
377	142129	53582633	19·4164878	7·224045
378	142884	54010152	19·4422221	7·230427
379	143641	54439939	19·4679223	7·236797
380	144400	54872000	19·4935887	7·243156
381	145161	55306341	19·5192213	7·249504
382	145924	55742968	19·5448203	7·255841
383	146689	56181887	19·5703858	7·262167
384	147456	56623104	19·5959179	7·268482
385	148225	57066625	19·6214169	7·274786
386	148996	57512456	19·6468827	7·281079
387	149769	57960603	19·6723156	7·287362
388	150544	58411072	19·6977156	7·293633
389	151321	58863869	19·7230829	7·299894
390	152100	59319000	19·7484177	7·306143
391	152881	59776471	19·7737199	7·312383
392	153664	60236288	19·7989899	7·318611
393	154449	60698457	19·8242276	7·324829
394	155236	61162984	19·8494332	7·331037
395	156025	61629875	19·8746069	7·337234
396	156816	62099136	19·8997487	7·343420
397	157609	62570773	19·9248588	7·349597
398	158404	63044792	19·9499373	7·355762
399	159201	63521199	19·9749844	7·361918
400	160000	64000000	20·0000000	7·368063

Number.	Square.	Cube.	Square Root.	Cube Root.
401	160801	64481201	20·0249844	7·374198
402	161604	64964808	20·0499377	7·380322
403	162409	65450827	20·0748599	7·386437
404	163216	65939264	20·0997512	7·392542
405	164025	66430125	20·1246118	7·398636
406	164836	66923416	20·1494417	7·404720
407	165649	67419143	20·1742410	7·410795
408	166464	67911312	20·1990099	7·416859
409	167281	68417929	20·2237484	7·422914
410	168100	68921000	20·2484567	7·428959
411	168921	69426531	20·2731349	7·434994
412	169744	69934528	20·2977831	7·441019
413	170569	70444997	20·3224014	7·447034
414	171396	70957944	20·3469899	7·453040
415	172225	71473375	20·3715488	7·459036
416	173056	71991296	20·3960781	7·465022
417	173889	72511713	20·4205779	7·470999
418	174724	73034632	20·4450483	7·476966
419	175561	73560059	20·4694895	7·482924
420	176400	74088000	20·4939015	7·488872
421	177241	74618461	20·5182845	7·494811
422	178084	75151448	20·5426386	7·500741
423	178929	75686967	20·5669638	7·506661
424	179776	76225024	20·5912603	7·512571
425	180625	76765625	20·6155281	7·518473
426	181476	77308776	20·6397674	7·524365
427	182329	77854483	20·6639783	7·530248
428	183184	78402752	20·6881609	7·536121
429	184041	78953589	20·7123152	7·541986
430	184900	79507000	20·7364414	7·547842
431	185761	80062991	20·7605395	7·553688
432	186624	80621568	20·7846097	7·559526
433	187489	81182737	20·8086520	7·565355
434	188356	81746504	20·8326667	7·571174
435	189225	82312875	20·8566536	7·576985
436	190096	82881856	20·8806130	7·582786
437	190969	83453453	20·9045450	7·588579
438	191844	84027672	20·9284495	7·594363
439	192721	84604519	20·9523268	7·600138
440	193600	85184000	20·9761770	7·605905
441	194481	85766121	21·0000000	7·611662
442	195364	86350888	21·0237960	7·617412
443	196249	86938307	21·0475652	7·623152
444	197136	87528384	21·0713075	7·628884
445	198025	88121125	21·0950231	7·634607
446	198916	88716536	21·1187121	7·640321
447	199809	89314623	21·1423745	7·646027
448	200704	89915392	21·1660105	7·651725
449	201601	90518849	21·1896201	7·657414
450	202500	91125000	21·2132034	7·663094

Number.	Square.	Cube.	Square Root.	Cube Root.
451	203401	91733851	21·2367606	7·668766
452	204304	92345408	21·2602916	7·674430
453	205209	92959677	21·2837967	7·680086
454	206106	93576664	21·3072758	7·685733
455	207025	94196375	21·3307290	7·691372
456	207936	94818816	21·3541565	7·697002
457	208849	95443993	21·3775583	7·702625
458	209764	96071912	21·4009346	7·708239
459	210681	96702579	21·4242853	7·713845
460	211600	97336000	21·4476106	7·719442
461	212521	97972181	21·4709106	7·725032
462	213444	98611128	21·4941853	7·730614
463	214369	99252847	21·5174348	7·736188
464	215296	99897344	21·5406592	7·741753
465	216225	100544625	21·5638587	7·747311
466	217156	101194696	21·5870331	7·752861
467	218089	101847563	21·6101828	7·758402
468	219024	102503232	21·6333077	7·763936
469	219961	103161709	21·6564078	7·769462
470	220900	103823000	21·6794834	7·774980
471	221841	104487111	21·7025344	7·780490
472	222784	105154048	21·7255610	7·785993
473	223729	105823817	21·7485632	7·791487
474	224676	106496424	21·7715411	7·796974
475	225625	107171875	21·7944947	7·802454
476	226576	107850176	21·8174242	7·807925
477	227529	108531333	21·8403297	7·813389
478	228484	109215352	21·8632111	7·818846
479	229441	109902239	21·8860686	7·824294
480	230400	110592000	21·9089023	7·829735
481	231361	111284641	21·9317122	7·835169
482	232324	111980168	21·9544984	7·840595
483	233289	112678587	21·9772610	7·846013
484	234256	113379904	22·0000000	7·851424
485	235225	114084125	22·0227155	7·856828
486	236196	114791256	22·0454077	7·862224
487	237169	115501303	22·0680765	7·867613
488	238144	116214272	22·0907220	7·872994
489	239121	116930169	22·1133444	7·878368
490	240100	117649000	22·1359436	7·883735
491	241081	118370771	22·1585198	7·889095
492	242064	119095488	22·1810730	7·894447
493	243049	119823157	22·2036033	7·899792
494	244036	120553784	22·2261108	7·905129
495	245025	121287375	22·2485955	7·910460
496	246016	122023936	22·2710575	7·915783
497	247009	122763473	22·2934968	7·921100
498	248004	123505992	22·3159136	7·926408
499	249001	124251499	22·3383079	7·931710
500	250000	125000000	22·3606798	7·937005

Number.	Square.	Cube.	Square Root.	Cube Root.
501	251001	125751501	22·3830293	7·942293
502	252004	126506008	22·4053565	7·947574
503	253009	127263527	22·4276615	7·952848
504	254016	128024064	22·4499443	7·958114
505	255025	128787625	22·4722051	7·963374
506	256036	129554216	22·4944438	7·968627
507	257049	130323843	22·5166605	7·973873
508	258064	131096512	22·5388553	7·979112
509	259081	131872229	22·5610283	7·984344
510	260100	132651000	22·5831796	7·989570
511	261121	133432831	22·6053091	7·994788
512	262144	134217728	22·6274170	8·000000
513	263169	135005697	22·6495033	8·005205
514	264196	135796744	22·6715681	8·010403
515	265225	136590875	22·6936114	8·015595
516	266256	137388096	22·7156334	8·020779
517	267289	138188413	22·7376340	8·025957
518	268324	138991832	22·7596134	8·031129
519	269361	139798359	22·7815715	8·036293
520	270400	140608000	22·8035085	8·041451
521	271441	141420761	22·8254244	8·046603
522	272484	142236648	22·8473193	8·051748
523	273529	143055667	22·8691933	8·056886
524	274576	143877824	22·8910463	8·062018
525	275625	144703125	22·9128785	8·067143
526	276676	145531576	22·9346899	8·072262
527	277729	146363183	22·9564806	8·077374
528	278784	147197952	22·9782500	8·082480
529	279841	148035889	23·0000006	8·087579
530	280900	148877000	23·0217289	8·092672
531	281961	149721291	23·0434372	8·097759
532	283024	150568768	23·0651252	8·102839
533	284089	151419437	23·0867928	8·107913
534	285156	152273304	23·1084400	8·112980
535	286225	153130375	23·1300670	8·118041
536	287296	153990656	23·1516738	8·123096
537	288369	154854153	23·1732605	8·128145
538	289444	155720872	23·1948270	8·133187
539	290521	156590819	23·2163735	8·138223
540	291600	157464000	23·2379001	8·143253
541	292681	158340421	23·2594067	8·148276
542	293764	159220088	23·2808935	8·153294
543	294849	160103007	23·3023604	8·158305
544	295936	160989184	23·3238076	8·163310
545	297025	161878625	23·3452351	8·168309
546	298116	162771336	23·3666429	8·173302
547	299209	163667323	23·3880311	8·178289
548	300304	164566592	23·4093998	8·183269
549	301401	165469149	23·4307490	8·188244
550	302500	166375000	23·4520788	8·193213

Number.	Square.	Cube.	Square Root.	Cube Root.
551	303601	167284151	23·4733892	8·198175
552	304704	168196608	23·4946802	8·203132
553	305809	169112377	23·5159520	8·208082
554	306916	170031464	23·5372046	8·213027
555	308025	170953875	23·5584380	8·217966
556	309136	171879616	23·5796522	8·222898
557	310249	172808693	23·6008474	8·227825
558	311364	173741112	23·6220236	8·232746
559	312481	174676879	23·6431808	8·237661
560	313600	175616000	23·6643191	8·242571
561	314721	176558481	23·6854386	8·247474
562	315844	177504328	23·7065392	8·252371
563	316969	178453547	23·7276210	8·257263
564	318096	179406144	23·7486842	8·262149
565	319225	180362125	23·7697286	8·267029
566	320356	181321496	23·7907545	8·271904
567	321489	182284263	23·8117618	8·276773
568	322624	183250432	23·8327506	8·281635
569	323761	184220009	23·8537209	8·286493
570	324900	185193000	23·8746728	8·291344
571	326041	186169411	23·8956063	8·296190
572	327184	187149248	23·9165215	8·301030
573	328329	188132517	23·9374184	8·305865
574	329476	189119224	23·9582971	8·310694
575	330625	190109375	23·9791576	8·315517
576	331776	191102976	24·0000000	8·320335
577	332929	192100033	24·0208243	8·325147
578	334084	193100552	24·0416306	8·329954
579	335241	194104539	24·0624188	8·334755
580	336400	195112000	24·0831892	8·339551
581	337561	196122941	24·1039416	8·344341
582	338724	197137368	24·1246762	8·349126
583	339889	198155287	24·1453929	8·353905
584	341056	199176704	24·1660919	8·358678
585	342225	200201625	24·1867732	8·363446
586	343396	201230056	24·2074369	8·368209
587	344569	202262003	24·2280829	8·372967
588	345744	203297472	24·2487113	8·377719
589	346921	204336469	24·2693222	8·382465
590	348100	205379000	24·2899156	8·387206
591	349281	206425071	24·3104916	8·391942
592	350464	207474688	24·3310501	8·396673
593	351649	208527857	24·3515913	8·401398
594	352836	209584584	24·3721152	8·406118
595	354025	210644875	24·3926218	8·410833
596	355216	211708736	24·4131112	8·415542
597	356409	212776173	24·4335834	8·420246
598	357604	213847192	24·4540385	8·424945
599	358801	214921799	24·4744765	8·429638
600	360000	216000000	24·4948974	8·434327

Number.	Square.	Cube.	Square Root.	Cube Root.
601	361201	217081801	24·5153013	8·439010
602	362404	218167208	24·5356883	8·443688
603	363609	219256227	24·5560583	8·448360
604	364816	220348864	24·5764115	8·453028
605	366025	221445125	24·5967478	8·457691
606	367236	222545016	24·6170673	8·462348
607	368449	223648543	24·6373700	8·467000
608	369664	224755712	24·6576560	8·471647
609	370881	225866529	24·6779254	8·476289
610	372100	226981000	24·6981781	8·480926
611	373321	228099131	24·7184142	8·485558
612	374544	229220928	24·7386338	8·490185
613	375769	230346397	24·7588368	8·494806
614	376996	231475544	24·7790234	8·499423
615	378225	232608375	24·7991935	8·504035
616	379456	233744896	24·8193473	8·508642
617	380689	234885113	24·8394847	8·513243
618	381924	236029032	24·8596058	8·517840
619	383161	237176659	24·8797106	8·522432
620	384400	238328000	24·8997992	8·527019
621	385641	239483061	24·9198716	8·531601
622	386884	240641848	24·9399278	8·536178
623	388129	241804367	24·9599679	8·540750
624	389376	242970624	24·9799920	8·545317
625	390625	244140625	25·0000000	8·549879
626	391876	245314376	25·0199920	8·554437
627	393129	246491883	25·0399681	8·558990
628	394384	247673152	25·0599282	8·563538
629	395641	248858189	25·0798724	8·568081
630	396900	250047000	25·0998008	8·572619
631	398161	251239591	25·1197134	8·577152
632	399424	252435968	25·1396102	8·581681
633	400689	253636137	25·1594913	8·586205
634	401956	254840104	25·1793566	8·590724
635	403225	256047875	25·1992063	8·595238
636	404496	257259456	25·2190404	8·599747
637	405769	258474853	25·2388589	8·604252
638	407044	259694072	25·2586619	8·608753
639	408321	260917119	25·2784493	8·613248
640	409600	262144000	25·2982213	8·617739
641	410881	263374721	25·3179778	8·622225
642	412164	264609288	25·3377189	8·626706
643	413449	265847707	25·3574447	8·631183
644	414736	267089984	25·3771551	8·635655
645	416025	268336125	25·3968502	8·640123
646	417316	269586136	25·4165301	8·644585
647	418609	270840023	25·4361947	8·649044
648	419904	272097792	25·4558441	8·653497
649	421201	273359449	25·4754784	8·657946
650	422500	274625000	25·4950976	8·662391

Number.	Square.	Cube.	Square Root.	Cube Root.
651	423801	275894451	25·5147016	8·666831
652	425104	277167808	25·5342907	8·671266
653	426409	278445077	25·5538647	8·675697
654	427716	279726264	25·5734237	8·680124
655	429025	281011375	25·5929678	8·684546
656	430336	282300416	25·6124969	8·688963
657	431649	283593393	25·6320112	8·693376
658	432964	284890312	25·6515107	8·697784
659	434281	286191179	25·6709953	8·702188
660	435600	287496000	25·6904652	8·706587
661	436921	288804781	25·7099203	8·710983
662	438244	290117528	25·7203607	8·715373
663	439569	291434247	25·7487864	8·719759
664	440896	292754944	25·7681975	8·724141
665	442225	294079625	25·7875939	8·728518
666	443556	295408296	25·8069758	8·732892
667	444889	296740963	25·8263431	8·737260
668	446224	298077632	25·8456960	8·741624
669	447561	299418309	25·8650343	8·745985
670	448900	300763000	25·8843582	8·750340
671	450241	302111711	25·9036677	8·754691
672	451584	303464448	25·9229628	8·759038
673	452929	304821217	25·9422435	8·763381
674	454276	306182024	25·9615100	8·767719
675	455625	307546875	25·9807621	8·772053
676	456976	308915776	26·0000000	8·776383
677	458329	310288733	26·0192237	8·780708
678	459684	311665752	26·0384331	8·785029
679	461041	313046839	26·0576284	8·789346
680	462400	314432000	26·0768096	8·793659
681	463761	315821241	26·0959767	8·797968
682	465124	317214568	26·1151297	8·802272
683	466489	318611987	26·1342687	8·806572
684	467856	320013504	26·1533937	8·810868
685	469225	321419125	26·1725047	8·815160
686	470596	322828856	26·1916017	8·819447
687	471969	324242703	26·2106848	8·823731
688	473344	325660672	26·2297541	8·828009
689	474721	327082769	26·2488095	8·832285
690	476100	328509000	26·2678511	8·836556
691	477481	329939371	26·2868789	8·840823
692	478864	331373888	26·3058929	8·845085
693	480249	332812557	26·3248932	8·849344
694	481636	334255384	26·3438797	8·853598
695	483025	335702375	26·3628527	8·857849
696	484416	337153536	26·3818119	8·862095
697	485809	338608873	26·4007576	8·866337
698	487204	340068392	26·4196896	8·870576
699	488601	341532099	26·4386081	8·874810
700	490000	343000000	26·4575131	8·879040

Number.	Square.	Cube.	Square Root.	Cube Root.
701	491401	344472101	26·4764046	8·883266
702	492804	345948088	26·4952826	8·887488
703	494209	347428927	26·5141472	8·891706
704	495616	348913664	26·5329988	8·895920
705	497025	350402625	26·5518361	8·900130
706	498436	351895816	26·5706605	8·904336
707	499849	353393243	26·5894716	8·908538
708	501264	354894912	26·6082694	8·912737
709	502681	356400829	26·6270539	8·916931
710	504100	357911000	26·6458252	8·921121
711	505521	359425431	26·6645833	8·925308
712	506944	360944128	26·6833281	8·929490
713	508369	362467097	26·7020598	8·933668
714	509796	363994344	26·7207784	8·937843
715	511225	365525875	26·7394839	8·942014
716	512656	367061696	26·7581763	8·946181
717	514089	368601813	26·7768557	8·950344
718	515524	370146232	26·7955220	8·954503
719	516961	371694959	26·8141754	8·958658
720	518400	373248000	26·8328157	8·962809
721	519841	374805361	26·8514432	8·966957
722	521284	376367048	26·8700577	8·971101
723	522729	377933067	26·8886593	8·975240
724	524176	379503424	26·9072481	8·979376
725	525625	381078125	26·9258240	8·983509
726	527076	382657176	26·9443872	8·987637
727	528529	384240583	26·9629375	8·991762
728	529984	385828352	26·9814751	8·995883
729	531441	387420489	27·0000000	9·000000
730	532900	389017000	27·0185122	9·004113
731	534361	390617891	27·0370117	9·008223
732	535824	392223168	27·0554985	9·012328
733	537289	393832837	27·0739727	9·016431
734	538756	395446904	27·0924344	9·020529
735	540225	397065375	27·1108834	9·024624
736	541696	398688256	27·1293199	9·028715
737	543169	400315553	27·1477439	9·032802
738	544644	401947272	27·1661554	9·036886
739	546121	403583419	27·1845544	9·040965
740	547600	405224000	27·2029410	9·045041
741	549081	406869021	27·2213152	9·049114
742	550564	408518488	27·2396769	9·053183
743	552049	410172407	27·2580263	9·057248
744	553536	411830784	27·2763634	9·061310
745	555025	413493625	27·2946881	9·065367
746	556516	415160936	27·3130006	9·069422
747	558009	416832723	27·3313007	9·073473
748	559504	418508992	27·3495887	9·077520
749	561001	420189749	27·3678644	9·081563
750	562500	421875000	27·3861279	9·085603

Number.	Square.	Cube.	Square Root.	Cube Root.
751	564001	423564751	27·4043792	9·089639
752	565504	425259008	27·4226184	9·093672
753	567009	426957777	27·4408455	9·097701
754	568516	428661064	27·4590604	9·101726
755	570025	430368875	27·4772633	9·105748
756	571536	432081216	27·4954542	9·109766
757	573049	433798093	27·5136330	9·113781
758	574564	435519512	27·5317998	9·117793
759	576081	437245479	27·5499546	9·121801
760	577600	438976000	27·5680975	9·125805
761	579121	440711081	27·5862284	9·129806
762	580644	442450728	27·6043475	9·133803
763	582169	444194947	27·6224546	9·137797
764	583696	445943744	27·6405499	9·141788
765	585225	447697125	27·6586334	9·145774
766	586756	449455096	27·6767050	9·149757
767	588289	451217663	27·6947648	9·153737
768	589824	452984832	27·7128129	9·157714
769	591361	454756609	27·7308492	9·161686
770	592900	456533000	27·7488739	9·165656
771	594441	458314011	27·7668868	9·169622
772	595984	460099648	27·7848880	9·173585
773	597529	461889917	27·8028775	9·177544
774	599076	463684824	27·8208555	9·181500
775	600625	465484375	27·8388218	9·185453
776	602176	467288576	27·8567766	9·189402
777	603729	469097433	27·8747197	9·193347
778	605284	470910952	27·8926514	9·197289
779	606841	472729139	27·9105715	9·201229
780	608400	474552000	27·9284801	9·205164
781	609961	476379541	27·9463772	9·209096
782	611524	478211768	27·9642629	9·213025
783	613089	480048687	27·9821372	9·216950
784	614656	481890304	28·0000000	9·220873
785	616225	483736625	28·0178515	9·224791
786	617796	485587656	28·0356915	9·228707
787	619369	487443403	28·0535203	9·232619
788	620944	489303872	28·0713377	9·237528
789	622521	491169069	28·0891438	9·240433
790	624100	493039000	28·1069386	9·244335
791	625681	494913671	28·1247222	9·248234
792	627264	496793088	28·1424946	9·252130
793	628849	498677257	28·1602557	9·256022
794	630436	500566184	28·1780056	9·259911
795	632025	502459875	28·1957444	9·263797
796	633616	504358336	28·2134720	9·267680
797	635209	506261573	28·2311884	9·271559
798	636804	508169592	28·2488938	9·275435
799	638401	510082399	28·2665881	9·279308
800	640000	512000000	28·2842712	9·283178

Number.	Square.	Cube.	Square Root.	Cube Root.
801	641601	513922401	28·3019434	9·287044
802	643204	515849608	28·3196045	9·290907
803	644809	517781627	28·3372546	9·294767
804	646416	519718464	28·3548938	9·298624
805	648025	521660125	28·3725219	9·302477
806	649636	523606616	28·3901391	9·306328
807	651249	525557943	28·4077454	9·310175
808	652864	527514112	28·4253408	9·314019
809	654481	529475129	28·4429253	9·317860
810	656100	531441000	28·4604989	9·321697
811	657721	533411731	28·4780617	9·325532
812	659344	535387328	28·4956137	9·329363
813	660969	537366797	28·5131549	9·333192
814	662596	539353144	28·5306852	9·337017
815	664225	541343375	28·5482048	9·340838
816	665856	543338496	28·5657137	9·344657
817	667489	545338513	28·5832119	9·348473
818	669124	547343432	28·6006993	9·352286
819	670761	549353259	28·6181760	9·356095
820	672400	551368000	28·6356421	9·359902
821	674041	553387661	28·6530976	9·363705
822	675684	555412248	28·6705424	9·367505
823	677329	557441767	28·6879766	9·371302
824	678976	559476224	28·7054002	9·375096
825	680625	561515625	28·7228132	9·378887
826	682276	563559976	28·7402157	9·382675
827	683929	565609283	28·7576077	9·386460
828	685584	567663552	28·7749891	9·390242
829	687241	569722789	28·7923601	9·394020
830	688900	571787000	28·8097206	9·397796
831	690561	573856191	28·8270706	9·401569
832	692224	575930368	28·8444102	9·405339
833	693889	578009537	28·8617394	9·409105
834	695556	580093704	28·8790582	9·412369
835	697225	582182875	28·8963666	9·416630
836	698896	584277056	28·9136646	9·420387
837	700569	586376253	28·9309523	9·424142
838	702244	588480472	28·9482297	9·427894
839	703921	590589719	28·9654967	9·431642
840	705600	592704000	28·9827535	9·435388
841	707281	594823321	29·0000000	9·439131
842	708964	596947688	29·0172363	9·442870
843	710649	599077107	29·0344623	9·446607
844	712336	601211584	29·0516781	9·450341
845	714025	603351125	29·0688837	9·454072
846	715716	605495736	29·0860791	9·457800
847	717409	607645423	29·1032644	9·461525
848	719104	609800192	29·1204396	9·465247
849	720801	611960049	29·1376046	9·468966
850	722500	614125000	29·1547595	9·472682

Number.	Square.	Cube.	Square Root.	Cube Root.
851	724201	616295051	29·1719043	9·476395
852	725904	618470208	29·1890390	9·480106
853	727609	620650477	29·2061637	9·483813
854	729316	622835864	29·2232784	9·487518
855	731025	625026375	29·2403830	9·491220
856	732736	627222016	29·2574777	9·494919
857	734449	629422793	29·2745623	9·498615
858	736164	631628712	29·2916370	9·502308
859	737881	633839779	29·3087018	9·505998
860	739600	636056000	29·3257566	9·509685
861	741321	638277381	29·3428015	9·513370
862	743044	640503928	29·3598365	9·517051
863	744769	642735647	29·3768616	9·520730
864	746496	644972544	29·3938769	9·524406
865	748225	647214625	29·4108823	9·528079
866	749956	649461896	29·4278779	9·531749
867	751689	651714363	29·4448637	9·535417
868	753424	653972032	29·4618397	9·539082
869	755161	656234909	29·4788059	9·542744
870	756900	658503000	29·4957624	9·546403
871	758641	660776311	29·5127091	9·550059
872	760384	663054848	29·5296461	9·553712
873	762129	665338617	29·5465734	9·557363
874	763876	667627624	29·5634910	9·561011
875	765625	669921875	29·5803989	9·564656
876	767376	672221376	29·5972972	9·568298
877	769129	674526133	29·6141858	9·571938
878	770884	676836152	29·6310648	9·575574
879	772641	679151439	29·6479325	9·579208
880	774400	681472000	29·6647939	9·582840
881	776161	683797841	29·6816442	9·586468
882	777924	686128968	29·6984848	9·590094
883	779689	688465387	29·7153159	9·593716
884	781456	690807104	29·7321375	9·597337
885	783225	693154125	29·7489496	9·600955
886	784996	695506456	29·7657521	9·604570
887	786769	697864103	29·7825452	9·608182
888	788544	700227072	29·7993289	9·611791
889	790321	702595369	29·8161030	9·615398
890	792100	704969000	29·8328678	9·619002
891	793881	707347971	29·8496231	9·622603
892	795664	709732288	29·8663690	9·626201
893	797449	712121957	29·8831056	9·629797
894	799236	714516984	29·8998328	9·633390
895	801025	716917375	29·9165506	9·636981
896	802816	719323136	29·9332591	9·640569
897	804609	721734273	29·9499583	9·644154
898	806404	724150792	29·9666481	9·647737
899	808201	726572699	29·9833287	9·651316
900	810000	729000000	30·0000000	9·654894

Number.	Square.	Cube.	Square Root.	Cube Root.
901	811801	731432701	30·0166620	9·658468
902	813604	733870808	30·0333148	9·662040
903	815409	736314327	30·0499584	9·665609
904	817216	738763264	30·0665928	9·669176
905	819025	741217625	30·0832179	9·672740
906	820836	743677416	30·0998339	9·676302
907	822649	746142643	30·1164407	9·679860
908	824464	748613312	30·1330383	9·683416
909	826281	751089429	30·1496269	9·686970
910	828100	753571000	30·1662063	9·690521
911	829921	756058031	30·1827765	9·694069
912	831744	758550528	30·1993377	9·697615
913	833569	761048497	30·2158899	9·701158
914	835396	763551944	30·2324329	9·704699
915	837225	766060875	30·2489669	9·708237
916	839056	768575296	30·2654919	9·711772
917	840889	771095213	30·2820079	9·715305
918	842724	773620632	30·2985148	9·718835
919	844561	776151559	30·3150128	9·722363
920	846400	778688000	30·3315018	9·725888
921	848241	781229961	30·3479818	9·729411
922	850084	783777448	30·3644529	9·732931
923	851929	786330467	30·3809151	9·736448
924	853776	788889024	30·3973683	9·739963
925	855625	791453125	30·4138127	9·743476
926	857476	794022776	30·4302481	9·746986
927	859329	796597983	30·4466747	9·750493
928	861184	799178752	30·4630924	9·753998
929	863041	801765089	30·4795013	9·757500
930	864900	804357000	30·4959014	9·761000
931	866761	806954491	30·5122926	9·764497
932	868624	809557568	30·5286750	9·767992
933	870489	812166237	30·5450487	9·771484
934	872356	814780504	30·5614136	9·774974
935	874225	817400375	30·5777697	9·778462
936	876096	820025856	30·5941171	9·782946
937	877969	822656953	30·6104557	9·785429
938	879844	825293672	30·6267857	9·788909
939	881721	827936019	30·6431069	9·792386
940	883600	830584000	30·6594194	9·795861
941	885481	833237621	30·6757233	9·799334
942	887364	835896888	30·6920185	9·802804
943	889249	838561807	30·7083051	9·806271
944	891136	841232384	30·7245830	9·809736
945	893025	843908625	30·7408523	9·813199
946	894916	846590536	30·7571130	9·816659
947	896809	849278123	30·7733651	9·820117
948	898704	851971392	30·7896086	9·823572
949	900601	854670349	30·8058436	9·827025
950	902500	857375000	30·8220700	9·830476

Number.	Square.	Cube.	Square Root.	Cube Root.
951	904401	860085351	30·8382879	9·833924
952	906304	862801408	30·8544972	9·837369
953	908209	865523177	30·8706981	9·840813
954	910116	868250664	30·8868904	9·844254
955	912025	870983875	30·9030743	9·847692
956	913936	873722816	30·9192497	9·851128
957	915849	876467493	30·9354166	9·854562
958	917764	879217912	30·9515751	9·857993
959	919681	881974079	30·9677251	9·861422
960	921600	884736000	30·9838668	9·864848
961	923521	887503681	31·0000000	9·868272
962	925444	890277128	31·0161248	9·871694
963	927369	893056347	31·0322413	9·875113
964	929296	895841344	31·0483494	9·878530
965	931225	898632125	31·0644491	9·881945
966	933156	901428696	31·0805405	9·885357
967	935089	904231063	31·0966236	9·888767
968	937024	907039232	31·1126984	9·892175
969	938961	909853209	31·1287648	9·895580
970	940900	912673000	31·1448230	9·898983
971	942841	915498611	31·1608729	9·902383
972	944784	918330048	31·1769145	9·905782
973	946729	921167317	31·1929479	9·909178
974	948676	924010424	31·2089731	9·912571
975	950625	926859375	31·2249900	9·915962
976	952576	929714176	31·2409987	9·919351
977	954529	932574833	31·2569992	9·922738
978	956484	935441352	31·2729915	9·926122
979	958441	938313739	31·2889757	9·929504
980	960400	941192000	31·3049517	9·932884
981	962361	944076141	31·3209195	9·936261
982	964324	946966168	31·3368792	9·939636
983	966289	949862087	31·3528308	9·943009
984	968256	952763904	31·3687743	9·946380
985	970225	955671625	31·3847097	9·949748
986	972196	958585256	31·4006369	9·953114
987	974169	961504803	31·4165561	9·956477
988	976144	964430272	31·4324673	9·959839
989	978121	967361669	31·4483704	9·963198
990	980100	970299000	31·4642654	9·966555
991	982081	973242271	31·4801525	9·969909
992	984064	976191488	31·4960315	9·973262
993	986049	979146657	31·5119025	9·976612
994	988036	982107784	31·5277655	9·979960
995	990025	985074875	31·5436206	9·983305
996	992016	988047936	31·5594677	9·986649
997	994009	991026973	31·5753068	9·989990
998	996004	994011992	31·5911380	9·993329
999	998001	997002999	31·6069613	9·996666
1000	1000000	1000000000	31·6227766	10·000000

OF RATIOS, PROPORTIONS, AND PROGRESSIONS.

Numbers are compared to each other in two different ways: the one comparison considers the difference of the two numbers, and is named Arithmetical Relation; and the difference sometimes the Arithmetical Ratio: the other considers their quotient, which is called Geometrical Relation; and the quotient is the Geometrical Ratio. So, of these two numbers 6 and 3, the difference, or arithmetical ratio is $6-3$ or 3, but the geometrical ratio is $\frac{6}{3}$ or 2.

There must be two numbers to form a comparison: the number which is compared, being placed first, is called the Antecedent: and that to which it is compared, the Consequent. So, in the two numbers above, 6 is the antecedent, and 3 the consequent.

If two or more couplets of numbers have equal ratios, or equal differences, the equality is named Proportion, and the terms of the ratios Proportionals. So, the two couplets, 4, 2 and 8, 6, are arithmetical proportionals, because $4-2=8-6=2$; and the two couplets 4, 2 and 6, 3, are geometrical proportions, because $\frac{4}{2}=\frac{6}{3}=2$, the same ratio.

To denote numbers as being geometrically proportional, a colon is set between the terms of each couplet, to denote their ratio; and a double colon, or else a mark of equality, between the couplets or ratios. So, the four proportionals, 4, 2, 6, 3 are set thus, $4:2::6:3$, which means, that 4 is to 2 as 6 is to 3; or thus, $4:2=6:3$, or thus, $\frac{4}{2}=\frac{6}{3}$, both which mean, that the ratio of 4 to 2, is equal to the ratio of 6 to 3.

Proportion is distinguished into Continued and Discontinued. When the difference or ratio of the consequent of one couplet, and the antecedent of the next couplet, is not the same as the common difference or ratio of the couplets, the proportion is discontinued. So, 4, 2, 8, 6, are in discontinued arithmetical proportion, because $4-2=8-6=2$, whereas $8-2=6$: and 4, 2, 6, 3 are in discontinued geometrical proportion, because $\frac{4}{2}=\frac{6}{3}=2$, but $\frac{6}{2}=3$, which is not the same.

But when the difference or ratio of every two succeeding terms is the same quantity, the proportion is said to be Continued, and the numbers themselves make a series of Continued Proportionals, or a progression. So 2, 4, 6, 8 form an arithmetical progression, because $4 - 2 = 6 - 4 = 8 - 6 = 2$, all the same common difference; and 2, 4, 8, 16, a geometrical progression, because $\frac{4}{2} = \frac{8}{4} = \frac{16}{8} = 2$, all the same ratio.

When the following terms of a progression increase, or exceed each other, it is called an Ascending Progression, or Series; but when the terms decrease, it is a descending one.

So, 0, 1, 2, 3, 4, &c. is an ascending arithmetical progression, but 9, 7, 5, 3, 1, &c. is a descending arithmetical progression. Also 1, 2, 4, 8, 16, &c. is an ascending geometrical progression, and 16, 8, 4, 2, 1, &c. is a descending geometrical progression.

ARITHMETICAL PROPORTION AND PROGRESSION.

In Arithmetical Progression, the numbers or terms have all the same common difference. Also, the first and last terms of a Progression, are called the Extremes; and the other terms, lying between them, the Means. The most useful part of arithmetical proportion, is contained in the following theorems:

Theorem 1. When four quantities are in arithmetical proportion, the sum of the two extremes is equal to the sum of the two means. Thus, of the four 2, 4, 6, 8, here $2 + 8 = 4 + 6 = 10$.

Theorem 2. In any continued arithmetical progression, the sum of the two extremes is equal to the sum of any two means that are equally distant from them, or equal to double the middle term when there is an uneven number of terms.

Thus, in the terms 1, 3, 5, it is $1 + 5 = 3 + 3 = 6$.

And in the series 2, 4, 6, 8, 10, 12, 14, it is $2 + 14 = 4 + 12 = 6 + 10 = 8 + 8 = 16$.

Theorem 3. The difference between the extreme terms of an arithmetical progression, is equal to the common difference of the series multiplied by one less than the number of the terms. So, of the ten terms, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, the common difference is 2, and one less than the number of terms 9; then the difference of the extremes is $20 - 2 = 18$, and $2 \times 9 = 18$ also.

Consequently the greatest term is equal to the least term added to the product of the common difference multiplied by 1 less than the number of terms.

THEOREM 4. The sum of all the terms, of any arithmetical progression, is equal to the sum of the two extremes multiplied by the number of terms, and divided by 2; or the sum of the two extremes multiplied by the number of the terms, gives double the sum of all the terms in the series.

This is made evident by setting the terms of the series in an inverted order, under the same series in a direct order, and adding the corresponding terms together in that order. Thus,

in the series	1,	3,	5,	7,	9,	11,	13,	15;
ditto inverted	15,	13,	11,	9,	7,	5,	3,	1;
the sums are	16 +	16 +	16 +	16 +	16 +	16 +	16 +	16,

which must be double the sum of the single series, and is equal to the sum of the extremes repeated as often as are the number of the terms.

From these theorems may readily be found any one of these five parts; the two extremes, the number of terms, the common difference, and the sum of all the terms, when any three of them are given; as in the following problems:

PROBLEM I.

Given the Extremes, and the Number of Terms, to find the Sum of all the Terms.

ADD the extremes together, multiply the sum by the number of terms, and divide by 2.

EXAMPLES.

1. The extremes being 3 and 19, and the number of terms 9; required the sum of the terms?

$$\begin{array}{r} 19 \\ 3 \\ \hline 22 \\ 9 \\ \hline 2\,)\,198 \\ \hline \text{Ans. } 99 \\ \hline \end{array}$$

Or, $\frac{19+3}{2} \times 9 = \frac{22}{2} \times 9 = 11 \times 9 = 99$, the same answer.

2. It is required to find the number of all the strokes a common clock strikes in one whole revolution of the index, or in 12 hours. Ans. 78.

Ex. 3. How many strokes do the clocks of Venice strike in the compass of the day, which go continually on from 1 to 24 o'clock? Ans. 300.

4. What debt can be discharged in a year, by weekly payments in arithmetical progression, the first payment being 1*s*, and the last or 52d payment 5*l* 3*s*? Ans. 135*l* 4*s*.

PROBLEM II.

Given the Extremes, and the Number of Terms; to find the Common Difference.

Subtract the less extreme from the greater, and divide the remainder by 1 less than the number of terms, for the common difference.

EXAMPLES.

1. The extremes being 3 and 19, and the number of terms 9; required the common difference?

$$\begin{array}{r} 19 \\ 3 \\ \hline 8\,)\,16 \\ \hline \text{Ans.}\ \ 2 \\ \hline \end{array} \qquad \text{Or, } \frac{19-3}{9-1} = \frac{16}{8} = 2.$$

2. If the extremes be 10 and 70, and the number of terms 21; what is the common difference, and the sum of the series? Ans. the com. diff. is 3, and the sum is 840.

3. A certain debt can be discharged in one year, by weekly payments in arithmetical progression, the first payment being 1*s*, and the last 5*l* 3*s*; what is the common difference of the terms? Ans. 2.

PROBLEM III.

Given one of the Extremes, the Common Difference, and the Number of Terms; to find the other Extreme, and the Sum of the Series.

Multiply the common difference by 1 less than the number of terms, and the product will be the difference of the extremes: Therefore add the product to the less extreme to give the greater; or substract it from the greater, to give the less extreme.

EXAMPLES.

1. Given the least term 3, the common difference 2, of an arithmetical series of 9 terms; to find the greatest term, and the sum of the series.

Here $2 \times (9 - 1) + 3 = 19$, the greatest term. Theref. $(19 + 3) \frac{9}{2} = \frac{198}{2} = 99$, the sum of the series.

2. If the greatest term be 70, the common difference 3, and the number of terms 21, what is the least term, and the sum of the series?

Ans. The least term is 10, and the sum is 840.

3. A debt can be discharged in a year, by paying 1 shilling the first week, 3 shillings the second, and so on, always 2 shillings more every week; what is the debt, and what will the last payment be?

Ans. The last payment will be 5*l* 3*s*, and the debt is 135*l* 4*s*.

PROBLEM IV.

To find an Arithmetical Mean Proportional between two given terms.

ADD the two given extremes or terms together, and take half their sum for the arithmetical mean required.

EXAMPLE.

To find an arithmetical mean between the two numbers 4 and 14.

Here
14
4
———
2) 18
———
Ans. 9 the mean required.

PROBLEM V.

To find two Arithmetical Means between two given Extremes.

SUBTRACT the less extreme from the greater, and divide the difference by 3, so will the quotient be the common difference; which being continually added to the less extreme, or taken from the greater, will give the means.

EXAMPLE.

To find two arithmetical means between 2 and 3.

Here 8
2
―――
3) 6 — Then $2 + 2 = 4$ the one mean.
and $4 + 2 = 6$ the other mean.
―――
com. dif. 2

PROBLEM VI.

To find any Number of Arithmetical Means between two given Terms or Extremes.

SUBTRACT the less extreme from the greater, and divide the difference by 1 more than the number of means required to be found, which will give the common difference; then this being added continually to the least term, or subtracted from the greatest, will give the mean terms required.

EXAMPLE.

To find five arithmetical means between 2 and 14.

Here 14
2
―――
6) 12 — Then by adding this com. dif. continually, the means are found 4, 6, 8, 10, 12.
―――
com. dif. 2

See more of Arithmetical progression in the Algebra.

GEOMETRICAL PROPORTION AND PROGRESSION.

IF there be taken two ratios, as those of 6 to 3, and 14 to 7, which, by what has been already said (p. 113), may be expressed fractionally, $\frac{6}{3}$ and $\frac{14}{7}$; to judge whether they are equal or unequal, we must reduce them to a common denominator, and we shall have 6×7, and 14×3 for the two numerators. If these are equal, the fractions or ratios are equal. Therefore,

THEOREM I. If four quantities be in geometrical proportion, the product of the two extremes will be equal to the product of the two means.

And hence, if the product of the two means be divided by one of the extremes, the quotient will give the other extreme. So, of the above numbers, if the product of the means 42 be divided by 6, the quotient 7 is the other extreme; and if 42 be divided by 7, the quotient 6 is the first extreme. This is the foundation of the practice in the *Rule of Three.*

We see, also, that if we have four numbers, 6, 3, 14, 7, such, that the products of the means and of the extremes are equal, we may hence infer the equality of the ratios $\frac{6}{3} = \frac{14}{7}$, or the existence of the proportion 6 : 3 : : 14 : 7. Hence

THEOREM II. We may always form a proportion of the factors of two equal products.

If the two means are equal, as in the terms 3, 6, 6, 12, their product becomes a square. Hence

THEOREM III. The mean proportional between two numbers is the square root of their product.

We may, without destroying the accuracy of a proportion, give to its various terms all the changes which do not affect the equality of the products of the means and extremes.

Thus, with respect to the proportion 6 : 3 : : 14 : 7, which gives $6 \times 7 = 3 \times 14$, we may displace the extremes, or the means, an operation which is denoted by the word *Alternando.*

This will give 6 : 14 : : 3 : 7
or 7 : 3 : : 14 : 6
or 7 : 14 : : 3 : 6

Or, 2dly, we may put the extremes in the places of the means, called *Invertendo.*

Thus 3 : 6 : : 7 : 14.

Or, 3dly, we may multiply or divide the two antecedents, or the two consequents, by the same number, when proportionality will subsist.

As $6 \times 4 : 3 :: 14 \times 4 : 7$; viz. $24 : 3 :: 56 : 7$
and $6 \div 2 : 3 :: 14 \div 2 : 7$; viz. $3 : 3 :: 7 : 7$.

Also, applying the proposition in *note* 2, *Addition of Vulgar Fractions*, to the terms of a proportion, such as $30 : 6 :: 15 : 3$, or $\frac{30}{6} = \frac{15}{3}$, we shall have

$$\frac{30 \pm 15}{6 \pm 3} = \frac{15}{3} \text{ and } \frac{30+15}{6+3} = \frac{30-15}{6-3}. \quad \text{Hence}$$

THEOREM IV. The sum or the difference of the antecedents, is to that of the consequents, as any one of the antecedents is to its consequent.

THEOREM V. The sum of the antecedents is to their difference, as the sum of the consequents is to their difference.

In like manner, if there be a series of equal ratios, $\frac{6}{3} = \frac{10}{5} = \frac{14}{7} = \frac{30}{15}$; we have

$$\frac{6+10+14+30}{3+5+7+15} = \frac{14}{7} = \frac{30}{15} = \text{\&c.} \quad \text{Therefore,}$$

THEOREM VI. In any series of equal ratios, the sum of the antecedents is to that of the consequents, as any one antecedent is to its consequent.

THEOREM VII. If two proportions are multiplied, term by term, the products will constitute a proportional.

Thus, if $30 : 15 :: 6 : 3$
and $2 : 3 :: 4 : 6$.

Then $30 \times 2 : 15 \times 3 :: 6 \times 4 : 3 \times 6$
or $60 : 45 :: 24 : 18$; or $\frac{60}{45} = \frac{24}{18}$.

THEOREM VIII. If four quantities are in proportion, their squares, cubes, &c. will be in proportion.

For this will evidently be nothing else than assuming the proportionality of the products, term by term, of two, three, or more identical proportions.

The same properties hold with regard to surd or irrational expressions,

Thus, $\sqrt{720} : \sqrt{80} :: \sqrt{567} : \sqrt{63}$
and $\sqrt{12} : \sqrt{3} :: \sqrt{4} : \sqrt{1}$.

For $\frac{\sqrt{720}}{\sqrt{80}} = \frac{\sqrt{9 . 80}}{\sqrt{80}} = \frac{3}{1}$, and $\frac{\sqrt{567}}{\sqrt{63}} = \frac{\sqrt{9 \times 63}}{\sqrt{63}} = \frac{3}{1}$

$$\text{and } \frac{\sqrt{12}}{\sqrt{3}} = \sqrt{\frac{12}{3}} = \frac{\sqrt{4}}{1} = \frac{2}{1}.$$

THEOREM IX. The quotient of the extreme terms of a geometrical progression is equal to the common ratio of the series raised to the power denoted by 1 less than the number of the terms.

So, of the ten terms 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, the common ratio is 2, one less than the number of terms 9; then the quotient of the extremes is $\frac{1024}{2} = 512$, and $2^9 = 512$ also.

Consequently the greatest term is equal to the least term multiplied by the said power of the ratio whose index is 1 less than the number of terms.

THEOREM X. The sum of all the terms, of any geometrical progression, is found by adding the greatest term to the difference of the extremes divided by 1 less than the ratio.

So, the sum of 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, (whose ratio is 2) is $1024 + \frac{1024-2}{2-1} = 1024 + 1022 = 2046$.

This subject will be resumed in the Algebraic part of this work. A few examples may here be added.

EXAMPLES.

1. The least of ten terms, in geometrical progression, being 1, and the ratio 2; what is the greatest term, and the sum of all the terms?

Ans. The greatest term is 512, and the sum 1023.

2. What debt may be discharged in a year, or 12 months, by paying 1*l* the first month, 2*l* the second, 4*l* the third, and so on, each succeeding payment being double the last; and what will the last payment be?

Ans. The debt 4095*l*, and the last payment 2048*l*.

PROBLEM I.

To find one Geometrical Mean Proportional between any two Numbers.

MULTIPLY the two numbers together, and extract the square root of the product, which will give the mean proportional sought.

EXAMPLE.

To find a geometrical mean between the two numbers 3 and 12.

```
 12
  3
 --
 36 (6 the mean.
 --
 36
```

PROBLEM II.

To find two Geometrical Mean Proportionals between any two Numbers.

Divide the greater number by the less, and extract the cube root of the quotient, which will give the common ratio of the terms. Then multiply the least given term by the ratio for the first mean, and this mean again by the ratio for the second mean: or, divide the greater of the two given terms by the ratio for the greater mean, and divide this again by the ratio for the less mean.

EXAMPLE.

To find two geometrical means between 3 and 24.

Here 3) 24 (8 ; its cube root 2 is the ratio.
Then $3 \times 2 = 6$, and $6 \times 2 = 12$, the two means.
Or $24 \div 2 = 12$, and $12 \div 2 = 6$, the same.
That is, the two means between 3 and 24, are 6 and 12.

PROBLEM III.

To find any number of Geometrical Means between two Numbers.

Divide the greater number by the less, and extract such root of the quotient whose index is 1 more than the number of means required; that is, the 2d root for one mean, the 3d root for two means, the 4th root for three means, and so on; and that root will be the common ratio of all the terms.

Then, with the ratio, multiply continually from the first term, or divide continually from the last or greatest term.

EXAMPLE.

To find four geometrical means between 3 and 96.
Here 3) 96 (32; the 5th root of which is 2, the ratio.
Then $3\times2=6$, & $6\times2=12$, & $12\times2=24$, & $24\times2=48$.
Or $96\div2=48$, & $48\div2=24$, & $24\div2=12$, & $12\div2=6$.
That is, 6, 12, 24, 48, are the four means between 3 and 96.

OF HARMONICAL PROPORTION.

There is also a third kind of proportion, called Harmonical or musical, which being but of little or no common use, a very short account of it may here suffice.

Musical Proportion is when, of three numbers, the first has the same proportion to the third, as the difference between the first and second has to the difference between the second and third.

As in these three, 6, 8, 12;
where $6 : 12 :: 8-6 : 12-8$,
that is $6 : 12 :: 2 : 4$.

When four numbers are in musical proportion; then the first has the same ratio to the fourth, as the difference between the first and second has to the difference between the third and fourth.

As in these, 6, 8, 12, 18;
where $6 : 18 :: 8-6 : 18-12$,
that is $6 : 18 :: 2 : 6$.

When numbers are in musical progression, their reciprocals are in arithmetical progression; and the converse, that is, when numbers are in arithmetical progression, their reciprocals are in musical progression.

So in these musicals 6, 8, 12, their reciprocals, $\frac{1}{6}$, $\frac{1}{8}$, $\frac{1}{12}$, are in arithmetical progression; for $\frac{1}{6} + \frac{1}{12} = \frac{3}{12} = \frac{1}{4}$; and $\frac{1}{8} + \frac{1}{8} = \frac{2}{8} = \frac{1}{4}$; that is, the sum of the extremes is equal to double the mean, which is the property of arithmeticals.

The method of finding out numbers in musical proportion is best expressed by letters in Algebra.

FELLOWSHIP, OR PARTNERSHIP.

FELLOWSHIP is a rule, by which any sum or quantity may be divided into any number of parts, which shall be in any given proportion to one another.

By this rule are adjusted the gains or loss or charges of partners in company; or the effects of bankrupts, or legacies in case of a deficiency of assets or effects; or the shares of prizes; or the numbers of men to form certain detachments; or the division of waste lands among a number of proprietors.

Fellowship is either *Single* or *Double*. It is single, when the shares or portions are to be proportional each to one single given number only; as when the stocks of partners are all employed for the same time; and Double, when each portion is to be proportional to two or more numbers; as when the stocks of partners are employed for different times.

SINGLE FELLOWSHIP.

GENERAL RULE.

ADD together the numbers that denote the proportion of the shares. Then say,

As the sum of the said proportional numbers,
Is to the whole sum to be parted or divided,
So is each several proportional number,
To the corresponding share or part.

Or, as the whole stock, is to the whole gain or loss,
So is each man's particular stock,
To his particular share of the gain or loss.

TO PROVE THE WORK. Add all the shares or parts together, and the sum will be equal to the whole number to be shared, when the work is right.

EXAMPLES.

1. To divide the number 240 into three such parts, as shall be in proportion to each other as the three numbers 1, 2 and 3.

Here 1 + 2 + 3 = 6, the sum of the numbers.

Then, as 6 : 240 : : 1 : 40 the 1st part,
and as 6 : 240 : : 2 : 80 the 2d part,
also as 6 : 240 : : 3 : 120 the 3d part.

Sum of all 240, the proof.

2. Three persons, A, B, C, freighted a ship with 340 tuns of wine, of which A loaded 100 tuns, B 97, and C the rest : in a storm the seamen were obliged to throw overboard 85 tuns ; how much must each person sustain of the loss ?

Here 110 + 97 = 207 tuns, loaded by A and B ;
theref. 340 − 207 = 133 tuns, loaded by C.

Hence, as 340 : 85 : : 110
or as 4 : 1 : : 110 : $27\frac{1}{2}$ tuns = A's loss ;
and as 4 : 1 : : 97 : $24\frac{1}{4}$ tuns = B's loss ;
also as 4 : 1 : : 133 : $33\frac{1}{4}$ tuns = C's loss ;

Sum 85 tuns, the proof.

3. Two merchants, C and D, made a stock of 120*l* ; of which C contributed 75*l*, and D the rest : by trading they gained 30*l* ; what must each have of it ?

Ans. C 18*l* 15*s*, and D 11*l* 5*s*.

4. Three merchants, E, F, G, make a stock of 700*l*, of which E contributed 128*l*, F 358*l*, and G the rest : by trading they gain 125*l* 10*s* ; what must each have of it ?

Ans. E must have 22*l* 1*s* 0*d* $2\frac{2}{35}$*q*.
F - - - 64 3 8 $0\frac{32}{35}$.
G - - - 39 5 3 $1\frac{1}{35}$.

5. A General imposing a contribution* of 700*l* on four

* Contribution is a tax paid by provinces, towns, villages, &c. to excuse them from being plundered. It is paid in provisions or in money, and sometimes in both.

villages, to be paid in proportion to the number of inhabitants contained in each; the first containing 250, the 2d 350, the 3d 400, and the 4th 500 persons; what part must each village pay?

Ans. the 1st to pay 116*l* 13*s* 4*d*

the 2d - -	163	6	8
the 3d - -	186	13	4
the 4th - -	233	6	8

6. A piece of ground, consisting of 37 ac 2 ro 14 ps, is to be divided among three persons, L, M, and N, in proportion to their estates: now if L's estate be worth 500*l* a year, M's 320*l*, and N's 75*l*; what quantity of land must each one have?

Ans. L must have 20 ac 3 ro $39\frac{139}{179}$ ps.

M - - -	13	1	$30\frac{46}{179}$
N - - -	3	0	$23\frac{173}{179}$

7. A person is indebted to O 57*l* 15*s*, to P 108*l* 3*s* 8*d*, to Q 22*l* 10*d*, and to R 73*l*; but at his decease, his effects are found to be worth no more than 170*l* 14*s*; how must it be divided among his creditors?

Ans. O must have 37*l* 15*s* 5*d* $2\frac{5302}{10439}q$.

P - - -	70	15	2	$2\frac{7498}{10439}$.
Q - - -	14	8	4	$2\frac{4730}{10439}$.
R - - -	47	14	11	$2\frac{3358}{10439}$.

8. A ship, worth 900*l*, being entirely lost, of which $\frac{1}{8}$ belonged to S, $\frac{1}{4}$ to T, and the rest to V; what loss will each sustain, supposing 540*l* of her were insured?

Ans. S will lose 45*l*, T 90*l*, and V 225*l*.

9. Four persons, W, X, Y, and Z, spent among them 25*s*, and agree that W shall pay $\frac{1}{2}$ of it, X $\frac{1}{3}$, Y $\frac{1}{4}$, and Z $\frac{1}{5}$; that is, their shares are to be in proportion as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$: what are their shares?

Ans. W must pay 9*s* 8*d* $3\frac{41}{77}q$.

X - - -	6	5	$3\frac{53}{77}$.
Y - - -	4	10	$1\frac{59}{77}$.
Z - - -	3	10	$3\frac{1}{77}$.

10. A detachment, consisting of 5 companies, being sent into a garrison, in which the duty required 76 men a day; what number of men must be furnished by each company, in proportion to their strength; the 1st consisting of 54 men,

the 2d of 51 men, the 3d of 48 men, the 4th of 39, and the 5th of 36 men?

Ans. The 1st must furnish 18, the 2d 17, the 3d 16, the 4th 13, and the 5th 12 men*.

DOUBLE FELLOWSHIP.

Double Fellowship, as has been said, is concerned in cases in which the stocks of partners are employed or continued for different times.

Rule†.—Multiply each person's stock by the time of its continuance; then divide the quantity, as in Single Fellowship, into shares, in proportion to these products, by saying,

As the total sum of all the said products,
Is to the whole gain or loss, or quantity to be parted,
So is each particular product
To the correspondent share of the gain or loss.

EXAMPLES.

1. A had in company 50*l* for 4 months, and B had 60*l* for 5 months; at the end of which time they find 24*l* gained: how must it be divided between them?

Here	50	60
	4	5
	200 +	300 = 500

Then as 500 : 24 : : 200 : $9\frac{3}{5}$ = 9*l* 12*s* = A's share.
and as 500 : 24 : : 300 : $14\frac{2}{5}$ = 14 8 = B's share.

---

* Questions of this nature frequently occurring in military service, General Haviland, an officer of great merit, contrived an ingenious instrument, for more expeditiously resolving them; which is distinguished by the name of the inventor, being called a Haviland.

† The proof of this rule is as follows: When the times are equal, the shares of the gain or loss are evidently as the stocks, as in Single Fellowship; and when the stocks are equal, the shares are as the times; therefore, when neither are equal, the shares must be as their products.

2. C and D hold a piece of ground in common, for which they are to pay 54*l*. C put in 23 horses for 27 days, and D 21 horses for 39 days; how much ought each man to pay of the rent? Ans. C must pay 23*l* 5*s* 9*d*.

D must pay 30 14 3.

3. Three persons, E, F, G, hold a pasture in common, for which they are to pay 30*l* per annum; into which E put 7 oxen for 3 months, F put 9 oxen for 5 months, and G put in 4 oxen for 12 months; how much must each person pay of the rent? Ans. E must pay 5*l* 10*s* 6*d* $1\frac{6}{19}q$.

F - - 11 16 10 $0\frac{8}{19}$.

G - - 12 12 7 $2\frac{5}{19}$.

4. A ship's company take a prize of 1000*l*, which they agree to divide among them according to their pay and the time they have been on board: now the officers and midshipmen have been on board 6 months, and the sailors 3 months; the officers have 40*s* a month, the midshipmen 30*s*, and the sailors 22*s* a month; moreover, there are 4 officers, 12 midshipmen, and 110 sailors; what will each man's share be?

Ans. each officer must have 23*l* 2*s* 5*d* $0\frac{46}{173}q$.

each midshipman - 17 6 9 $3\frac{69}{173}$.

each seaman - - 6 7 2 $0\frac{58}{173}$.

5. H, with a capital of 1000*l*, began trade the first of January, and, meeting with success in business, took in I as a partner, with a capital of 1500*l*, on the first of March following. Three months after that they admit K as a third partner, who brought into stock 2800*l*. After trading together till the end of the year, they find there has been gained 1776*l* 10*s*; how must this be divided among the partners?

Ans. H must have 475*l* 9*s* $4\frac{1}{4}d$ $\frac{23}{233}q$.

I - - - 571 16 $8\frac{1}{4}$ $\frac{87}{233}$.

K - - - 747 3 $11\frac{1}{4}$ $\frac{123}{233}$.

6. X, Y, and Z made a joint-stock for 12 months; X at first put in 20*l*, and 4 months after 20*l* more; Y put in at first 30*l*, at the end of 3 months he put in 20*l* more, and 2 months after he put in 40*l* more; Z put in at first 60*l*, and 5 months after he put in 10*l* more, 1 month after which he took out 30*l*; during the 12 months they gained 50*l*; how much of it must each have?

Ans. X must have 10*l* 18*s* 6*d* 3[illegible]*q*.

Y - - - 22 8 1 0[illegible].

Z - - - 16 13 4 0.

SIMPLE INTEREST.

Interest is the premium or sum allowed for the loan, or forbearance of money. The money lent, or forborn, is called the Principal; and the sum of the principal and its interest added together, is called the Amount. Interest is allowed at so much per cent. per annum; which premium per cent. per annum, or interest of 100*l* for a year, is called the rate of interest:—So,

When interest is at 3 per cent. the rate is 3;
- - - 4 per cent. - - - 4;
- - - 5 per cent. - - - 5;
- - - 6 per cent. - - - 6.

But, by law, interest ought not to be taken higher than at the rate of 5 per cent.

Interest is of two sorts; Simple and Compound.

Simple Interest is that which is allowed for the principal lent or forborn only, for the whole time of forbearance. As the interest of any sum, for any time, is directly proportional to the principal sum, and also to the time of continuance; hence arises the following general rule of calculation.

As 100*l* is to the rate of interest, so is any given principal to its interest for one year. And again,

As 1 year is to any given time, so is the interest for a year, just found, to the interest of the given sum for that time.

Otherwise. Take the interest of 1 pound for a year, which multiply by the given principal, and this product again by the time of loan or forbearance, in years and parts, for the interest of the proposed sum for that time.

Note. When there are certain parts of years in the time, as quarters, or months, or days: they may be worked for, either by taking the aliquot or like parts of the interest of a year, or by the Rule of Three, in the usual way. Also, to divide by 100, is done by only pointing off two figures for decimals.

EXAMPLES.

1. To find the interest of 230*l* 10*s*, for 1 year, at the rate of 4 per cent. per annum.

Here, As 100 : 4 :: 230*l* 10*s* : 9*l* 4*s* 4¾*d*.

```
               4
         ---------
100 ) 9,22    0
        20
      ----
      4·40
        12
      ----
      4·80        Ans. 9l 4s 4¾d.
         4
      ----
      3·20
      ----
```

2. To find the interest of 547*l* 15*s*, for 3 years, at 5 per cent. per annum.

```
As 100 : 5 :: 547·75
Or  20 : 1 :: 547·75 : 27·3875 interest for 1 year.
                              3
                     ----------
                   l 82·1625 ditto for 3 years.
                          20
                     ----------
                    s 3·2500
                         12
                     -------
                    d 3·00   Ans. 82l 3s 3d.
                     -------
```

3. To find the interest of 200 guineas, for 4 years 7 months and 25 days, at 4½ per cent. per annum.

```
                          ds     l       ds
   210l             As 356 : 9·45 :: 25 :    l
     4½             or  73 : 9·45 ::  5 : ·6472
   ----                                5
   840                          -------
   105                   73) 47·25 (·6472
   ----                            345
   9·45 interest for 1 yr.         530
     4                              19
  -----
  37·80    ditto 4 years.
6 mo = ½ 4·725   ditto 6 months.
1 mo = ⅙  ·7875  ditto 1 month.
          ·6472  ditto 25 days.
  ---------
l 43·9597
       20
  ---------
s 19·1940
       12
  ---------
d  2·3280
        4       Ans. 43l 19s 2¼d.
  ---------
q  1·3120
  ---------
```

4. To find the interest of 450*l*, for a year, at 5 per cent. per annum. Ans. 22*l* 10*s*.

5. To find the interest of 715*l* 12*s* 6*d*, for a year, at 4½ per cent. per annum. Ans. 32*l* 4*s* 0¾*d*.

6. To find the interest of 720*l*, for 3 years, at 5 per cent. per annum. Ans. 108*l*.

7. To find the interest of 355*l* 15*s*, for 4 years, at 4 per cent. per annum. Ans. 56*l* 18*s* 4¾*d*.

8. To find the interest of 32*l* 5*s* 8*d*, for 7 years, at 4¼ per cent. per annum. Ans. 9*l* 12*s* 1*d*.

9. To find the interest of 170*l*, for 1½ year, at 5 per cent. per annum. Ans. 12*l* 15*s*.

10. To find the insurance on 205*l* 15*s*, for ¼ of a year, at 4 per cent. per annum. Ans. 2*l* 1*s* 1¾*d*.

11. To find the interest of 319*l* 6*d*, for 5¾ years, at 3¾ per cent. per annum. Ans. 68*l* 14*s* 9½*d*.

12. To find the insurance on 107*l*, for 117 days, at 4¾ per cent. per annum. Ans. 1*l* 12*s* 7*d*.

13. To find the interest of 17*l* 5*s*, for 117 days, at $4\frac{3}{4}$ per cent. per annum. Ans. 5*s* 3*d*.

14. To find the insurance on 712*l* 6*s*, for 8 months, at $7\frac{1}{2}$ per cent. per annum. Ans. 35*l* 12*s* $3\frac{1}{2}$*d*.

Note. The Rules for Simple Interest, serve also to calculate Insurances, or the Purchase of Stocks, or any thing else that is rated at so much per cent.

See also more on the subject of Interest, with the algebraical expression and investigation of the rules at the end of the Algebra.

COMPOUND INTEREST.

Compound Interest, called also Interest upon Interest, is that which arises from the principal and interest, taken together, as it becomes due, at the end of each stated time of payment. Though it be not lawful to lend money at Compound Interest, yet in purchasing annuities, pensions, or leases in reversion, it is usual to allow Compound Interest to the purchaser for his ready money.

Rules.—1. Find the amount of the given principal, for the time of the first payment, by Simple Interest. Then consider this amount as a new principal for the second payment, whose amount calculate as before. And so on through all the payments to the last, always accounting the last amount as a new principal for the next payment. The reason of which is evident from the definition of Compound Interest. *Or else,*

2. Find the amount of 1 pound for the time of the first payment, and raise or involve it to the power whose index is denoted by the number of payments. Then that power multiplied by the given principal, will produce the whole amount. From which the said principal being subtracted, leaves the Compound Interest of the same. As is evident from the first Rule.

EXAMPLES.

1. To find the amount of 720*l*, for 4 years, at 5 per cent. per annum.

Here 5 is the 20th part of 100, and the interest of 1*l* for a year is $\frac{1}{20}$ or ·05, and its amount 1·05. Therefore,

1. *By the 1st Rule.*

	l	s	d	
20)	720	0	0	1st yr's princip.
	36	0	0	1st yr's interest.
20)	756	0	0	2d yr's princip.
	37	16	0	2d yr's interest.
20)	793	16	0	3d yr's princip.
	39	13	9½	3d yr's interest.
20)	833	9	9½	4th yr's princip.
	41	13	5¾	4th yr's interest.
	£875	3	3¼	the whole amot. or ans. required.

2. *By the 2d Rule.*

1·05	amount of 1*l*.
1·05	
1·1025	2d power of it.
1·1025	
1·21550625	4th power of it.
720	
l 875·1645	
20	
s 3·2900	
12	
d 3·4800	

2. To find the amount of 50*l* in 5 years, at 5 per cent. per annum, compound interest. Ans. 63*l* 16*s* 8¼*d*.

3. To find the amount of 50*l* in 5 years, or 10 half-years, at 5 per cent. per annum, compound interest, the interest payable half-yearly. Ans. 64*l* 0*s* 1*d*.

4 To find the amount of 50*l* in 5 years, or 20 quarters, at 5 per cent. per annum, compound interest, the interest payable quarterly. Ans. 64*l* 2*s* 0¼*d*.

5. To find the compound interest of 370*l* forborn for 6 years, at 4 per cent. per annum. Ans. 98*l* 3*s* 4¼*d*.

6. To find the compound interest of 410*l* forborn for 2½ years, at 4½ per cent. per annum, the interest payable half-yearly. Ans. 48*l* 4*s* 11¼*d*.

7. To find the amount, at compound interest, of 217*l*, forborn at 2¼ years, at 5 per cent. per annum, the interest payable quarterly. Ans. 242*l* 13*s* 4½*d*.

ALLIGATION.

Alligation teaches how to compound or mix together several simples of different qualities, so that the composition may be of some intermediate quality, or rate. It is commonly distinguished into two cases, Alligation Medial, and Alligation Alternate.

ALLIGATION MEDIAL.

Alligation Medial is the method of finding the rate or quality of the composition, from having the quantities and rates or qualities of the several simples given. And it is thus performed:

* Multiply the quantity of each ingredient by its rate or quality; then add all the products together, and add also all the quantities together in another sum; then divide the former sum by the latter, that is, the sum of the products by the sum of the quantities, and the quotient will be the rate or quality of the composition required.

EXAMPLES.

1. If three sorts of gunpowder be mixed together, viz. 50lb at 12*d* a pound, 44lb at 9*d*, and 26lb at 8*d* a pound; how much a pound is the composition worth?

Here 50, 44, 26 are the quantities,
and 12, 9, 8 the rates or qualities;
then $50 \times 12 = 600$
$44 \times 9 = 396$
$26 \times 8 = 208$

$120) \quad 1204 \quad (10\frac{4}{120} = 10\frac{1}{30}$.

Ans. The rate or price is $10\frac{1}{30}d$ the pound.

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* *Demonstration.* The Rule is thus proved by Algebra.
Let a, b, c be the quantities of the ingredients,
and m, n, p their rates, or qualities, or prices;
then am, bn, cp are their several values,
and $am + bn + cp$ the sum of their values,
also $a + b + c$ is the sum of the quantities,
and if r denote the rate of the whole composition,
then $(a + b + c) \times r$ will be the value of the whole,
conseq. $(a + b + c) \times r = am + bn + cp$,
and $r = (am + bn + cp) \div (a + b + c)$, which is the Rule.

Note. If an ounce or any other quantity of pure gold be reduced into 24 equal parts, these parts are called Caracts; but gold is often mixed with some base metal, which is called the Alloy, and the mixture is said to be of so many caracts fine, according to the proportion of pure gold contained in it: thus, if 22 caracts of pure gold, and 2 of alloy be mixed together, it is said to be 22 caracts fine.

If any one of the simples be of little or no value with respect to the rest, its rate is supposed to be nothing; as water mixed with wine, and alloy with gold and silver.

2. A composition being made of 5lb of tea at 7*s* per lb, 9lb at 8*s* 6*d* per lb, and 14½lb at 5*s* 10*d* per lb; what is a lb of it worth? Ans. 6*s* 10½*d*.

3. Mixed 4 gallons of wine at 4*s* 10*d* per gall, with 7 gallons at 5*s* 3*d* per gall, and 9¾ gallons at 5*s* 8*d* per gall; what is a gallon of this composition worth? Ans. 5*s* 4¼*d*.

4. Having melted together 7 oz of gold of 22 caracts fine, 12 oz of 21 caracts fine, and 17 oz. of 19 caracts fine: I would know the fineness of the composition?

Ans. 20¼⅗ caracts fine.

ALLIGATION ALTERNATE.

Alligation Alternate is the method of finding what quantity of any number of simples, whose rates are given, will compose a mixture of a given rate. So that it is the reverse of Alligation Medial, and may be proved by it.

RULE I*.

1. Set the rates of the simples in a column under each other.—2. Connect, or link with a continued line, the rate

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* *Demonst.* By connecting the less rate with the greater, and placing the difference between them and the rate alternately, the quantities resulting are such, that there is precisely as much gained by one quantity as is lost by the other, and therefore the gain and loss upon the whole is equal, and is exactly the proposed rate: and the same will be true of any other two simples managed according to the Rule.

In like manner, whatever the number of simples may be, and with how many soever every one is linked, since it is always a less with a greater than the mean price, there will be an equal balance of loss and gain between every two, and consequently an equal balance on the whole. Q. E. D.

It is obvious, from the Rule, that questions of this sort admit of a great variety of answers; for, having found one answer, we may find as many more as we please, by only multiplying or dividing each of the quantities found, by 2, or 3, or 4, &c.: the reason of which is evident: for, if two quantities, of two simples, make a balance of loss and gain, with respect to the mean price, so must also the double or treble, the ½ or ⅓ part, or any other ratio of these quantities, and so on *ad infinitum*.

These kinds of questions are called by algebraists *indeterminate* or *unlimited* problems; and by an analytical process, theorems may be raised that will give all the *possible* answers.

of each simple, which is less than that of the compound, with one, or any number, of those that are greater than the compound; and each greater rate with one or any number of the less.—3. Write the difference between the mixture rate, and that of each of the simples, opposite the rate with which they are linked.—4. Then if only one difference stand against any rate, it will be the quantity belonging to that rate; but if there be several, their sum will be the quantity.

The examples may be proved by the rule for Alligation Medial.

EXAMPLES.

1. A merchant would mix wines at 16*s*, at 18*s*, and at 22*s* per gallon, so as that the mixture may be worth 20*s* the gallon; what quantity of each must be taken?

Here 20	16	2 at 16*s*
	18	2 at 18*s*
	22	4 + 2 = 6 at 22*s*

2. How much sugar at 4*d*, at 6*d*, and at 11*d* per lb, must be mixed together, so that the composition formed by them may be worth 7*d* per lb?

Ans. 1 lb, or 1 stone, or 1 cwt, or any other equal quantity of each sort.

3. How much corn at 2*s* 6*d*, 3*s* 8*d*, 4*s*, and 4*s* 8*d* per bushel must be mixed together, that the compound may be worth 3*s* 10*d* per bushel?

Ans. 2 at 2*s* 6*d*, 3 at 3*s* 8*d*, 3 at 4*s*, and 3 at 4*s* 8*d*.

RULE II.

When the whole composition is limited to a certain quantity: Find an answer as before by linking; then say, as the sum of the quantities, or differences thus determined, is to the given quantity; so is each ingredient, found by linking, to the required quantity of each.

EXAMPLE.

1. How much gold of 15, 17, 18, and 22 caracts fine, must be mixed together, to form a composition of 40 oz of 20 caracts fine?

Here 20	15	. . . 2
	17	. . . 2
	18	. . . 2
	22	5 + 3 + 2 = 10
		16

Then as 16 : 40 : : 2 : 5
and 16 : 40 : : 10 : 25

Ans. 5 oz of 15, of 17, and of 18 caracts fine, and 25 oz of 22 caracts fine*.

RULE III†.

WHEN one of the ingredients is limited to a certain quantity; Take the difference between each price, and the mean rate as before; then say, As the difference of that simple, whose quantity is given, is to the rest of the differences severally; so is the quantity given, to the several quantities required.

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* A great number of questions might be here given relating to the specific gravities of metals, &c. but one of the most curious may suffice.

Hiero, king of Syracuse, gave orders for a crown to be made entirely of pure gold; but suspecting the workmen had debased it by mixing it with silver or copper, he recommended the discovery of the fraud to the famous Archimedes, and desired to know the exact quantity of alloy in the crown.

Archimedes, in order to detect the imposition, procured two other masses, the one of pure gold, the other of silver or copper, and each of the same weight with the former; and by putting each separately into a vessel full of water, the quantity of water expelled by them determined their specific gravities: from which, and their given weights, the exact quantities of gold and alloy in the crown may be determined.

Suppose the weight of each crown to be 10lb, and that the water expelled by the copper or silver was 92lb, by the gold 52lb, and by the compound crown 64lb; what will be the quantities of gold and alloy in the crown?

The rates of the simples are 92 and 52, and of the compound 64; therefore

64	62	12 of copper
	52	28 of gold

And the sum of these is $12 + 28 = 40$, which should have been 10; therefore by the Rule,

40 : 10 : : 12 : 3lb of copper }
40 : 10 : : 28 : 7lb of gold } the answer

† In the very same manner questions may be wrought when several of the ingredients are limited to certain quantities, by finding first for one limit, and then for another. The two last Rules can need no demonstration, as they evidently result from the first, the reason of which has been already explained.

EXAMPLES.

1. How much wine at 5*s*, at 5*s* 6*d*, and 6*s* the gallon, must be mixed with 3 gallons at 4*s* per gallon, so that the mixture may be worth 5*s* 4*d* per gallon?

Here 64 { 48 — 8+2=10
60 — 8+2=10
66 — 16+4=20
72 — 16+4=20

Then 10 : 10 :: 3 : 3
10 : 20 :: 3 : 6
10 : 20 :: 3 : 6

Ans. 3 gallons at 5*s*, 6 at 5*s* 6*d*, and 6 at 6*s*.

2. A grocer would mix teas at 12*s*, 10*s*, and 6*s* per lb, with 20lb at 4*s* per lb: how much of each sort must he take to make the composition worth 8*s* per lb?

Ans. 20lb at 4*s*, 10lb at 6*s*, 10lb at 10*s*, and 20lb at 12*s*.

POSITION.

Position is a rule for performing certain questions, which cannot be resolved by the common direct rules. It is sometimes called False Position, or False Supposition, because it makes a supposition of false numbers, to work with the same as if they were the true ones, and by their means discovers the true numbers sought. It is sometimes also called Trial-and-Error, because it proceeds by *trials* of false numbers, and thence finds out the true ones by a comparison of the *errors*.—Position is either Single or Double.

SINGLE POSITION.

Single Position is that by which a question is resolved by means of one supposition only. Questions which have their result proportional to their supposition, belong to Single Position: such as those which require the multiplication or division of the number sought by any proposed number; or when it is to be increased or diminished by itself, or any parts of itself, a certain proposed number of times. The rule is as follows:

TAKE or assume any number for that which is required, and perform the same operations with it, as are described or performed in the question. Then say, As the result of the said operation, is to the position, or number assumed; so is the result in the question, to a fourth term, which will be the number sought*.

EXAMPLES.

1. A person after spending $\frac{1}{3}$ and $\frac{1}{4}$ of his money, has yet remaining 60*l*; what had he at first?

Suppose he had at first 120*l*.	Proof.
Now $\frac{1}{3}$ of 120 is 40	$\frac{1}{3}$ of 144 is 48
$\frac{1}{4}$ of it is 30	$\frac{1}{4}$ of 144 is 36
their sum is 70	their sum 84
which taken from 120	taken from 144
leaves 50	leaves 60 as
Then, 50 : 120 : : 60 : 144 the Answer.	per question.

2. What number is that, which, being increased by $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ of itself, the sum shall be 75? Ans. 36.

3. A general, after sending out a foraging $\frac{1}{2}$ and $\frac{1}{3}$ of his men, had yet remaining 1000; what number had he in command? Ans. 6000.

4. A gentleman distributed 52 pence among a number of poor people, consisting of men, women, and children; to each man he gave 6*d*, to each woman 4*d*, and to each child 2*d*: moreover there were twice as many women as men, and

* The reason of this Rule is evident, because it is supposed that the results are proportional to the suppositions.

Thus, $na : a : : nz : z$,

or $\frac{a}{n} : a : : \frac{z}{n} : z$,

or $\frac{a}{n} \pm \frac{a}{m}$ &c. $: a : : \frac{z}{n} \pm \frac{z}{m}$ &c. $: z$,

and so on.

thrice as many children as women. How many were there of each? Ans. 2 men, 4 women, and 12 children.

5. One being asked his age, said, if $\frac{3}{4}$ of the years I have lived, be multiplied by 7, and $\frac{2}{3}$ of them be added to the product, the sum will be 219. What was his age?

Ans. 45 years.

DOUBLE POSITION.

Double Position is the method of resolving certain questions by means of two suppositions of false numbers.

To the Double Rule of Position belong such questions as have their results not proportional to their positions: such are those in which the numbers sought, or their parts, or their multiples, are increased or diminished by some given absolute number, which is no known part of the number sought.

RULE*.

Take or assume any two convenient numbers, and proceed with each of them separately, according to the con-

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* *Demonstr.* The Rule is founded on this supposition, namely, that the first error is to the second, as the difference between the true and first supposed number, is to the difference between the true and second supposed number: when that is not the case, the exact answer to the question cannot be found by this Rule.—That the Rule is true, according to the assumption, may be thus proved.

Let a and b be the two suppositions, and A and B their results, produced by similar operation; also r and s their errors, or the differences between the results A and B from the true result N; and let x denote the number sought, answering to the true result N of the question.

Then is $N - A = r$, and $N - B = s$, or $B - A = r - s$. And, according to the supposition on which the Rule is founded, $r : s :: x - a : x - b$; hence, by multiplying extremes and means, $rx - rb = sx - sa$; then, by transposition, $rx - sx = rb - sa$; and, by division, $x = \frac{rb - sa}{r - s}$ = the number sought, which is the rule when the results are both too little.

If the results be both too great, so that A and B are both greater than N; then $N - A = -r$, and $N - B = -s$, or r and s are both negative; hence $-r : -s :: x - a : x - b$, but $-r : -s :: +r : +s$, therefore $r : s :: x - a : x - b$; and the rest will be exactly as in the former case.

But if one result A only be too little, and the other B too great, or one error r positive, and the other s negative, then the theorem becomes $x = \frac{rb + sa}{r + s}$, which is the rule in this case, or when the errors are unlike.

ditions of the question, as in Single Position; and find how much each result is different from the result mentioned in the question, calling these differences the *errors*, noting also whether the results are too great or too little.

Then multiply each of the said errors by the contrary supposition, namely, the first position by the second error, and the second position by the first error. Then,

If the errors are alike, divide the difference of the products by the difference of the errors, and the quotient will be the answer.

But if the errors are unlike, divide the sum of the products by the sum of the errors, for the answer.

Note, The errors are said to be alike, when they are either both too great or both too little; and unlike, when one is too great and the other too little.

EXAMPLE.

1. What number is that, which being multiplied by 6, the product increased by 18, and the sum divided by 9, the quotient should be 20?

Suppose the two numbers 18 and 30. Then,

	First Position.		Second Position.		Proof.
	18	Suppose	30		27
	6	mult.	6		6
	108		180		162
	18	add	18		18
	9) 126	div.	9) 198		9) 180
	14	results	22		20
	20	true res.	20		
	+6	errors unlike	−2		
2d pos.	30	mult.	18	1st pos.	
Er- 2	180		36		
rors 6	36				
sum 8)	216	sum of products			
	27	Answer sought.			

RULE II.

Find, by trial, two numbers, as near the true number as convenient, and work with them as in the question; marking the errors which arise from each of them.

Multiply the difference of the two numbers assumed, or found by trial, by one of the errors, and divide the product by the difference of the errors, when they are alike, but by their sum when they are unlike. Or thus, by proportion: As the difference of the errors, or of the results, (which is the same thing), is to the difference of the assumed numbers, so is either of the errors, to the correction of the assumed number belonging to that error.

Add the quotient, or correction, last found, to the number belonging to the said error, when that number is too little, but subtract it when too great, and the result will give the true quantity sought *.

EXAMPLES.

1. So, the foregoing example, worked by this 2d rule, will be as follows:

30 positions 18;		their diff.	12
−2 errors +6;		least error	2
		sum of errors 8)	24 (3 subtr.
		from the position	30
		leaves the answer	27

Or, as 22 − 14 : 30 − 18, or as 8 : 12 :: 2 : 3 the correction, as above.

2. A son asking his father how old he was, received this answer: Your age is now one-third of mine; but 5 years ago, your age was only one-fourth of mine. What then are their two ages? Ans. 15 and 45.

3. A workman was hired for 20 days, at 3*s* per day, for every day he worked; but with this condition, that for every day he did not work, he should forfeit 1*s*. Now it so hap-

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* For since, by the supposition, $r : s :: x - a : x - b$, therefore by division, $r - s : s :: b - a : x - b$, or as $B - A : b - a :: s : x - b$, for $B - A$ is $= r - s$: which is the 2d Rule.

pened, that upon the whole he had 2*l* 4*s* to receive. How many of the days did he work? Ans. 16.

4. A and B began to play together with equal sums of money: A first won 20 guineas, but afterwards lost back $\frac{2}{3}$ of what he then had; after which B had four times as much as A. What sum did each begin with? Ans. 100 guineas.

5. Two persons, A and B, have both the same income, A saves $\frac{1}{5}$ of his; but B, by spending 50*l* per annum more than A, at the end of 4 years finds himself 100*l* in debt. What does each receive and spend per annum?

Ans. They receive 125*l* per annum; also A spends 100*l*, and B spends 150*l* per annum.

PRACTICAL QUESTIONS IN ARITHMETIC.

Quest. 1. The swiftest velocity of a cannon-ball, is about 2000 feet in a second of time. Then in what time, at that rate, would such a ball move from the earth to the sun, admitting the distance to be 100 millions of miles, and the year to contain 365 days 6 hours? Ans. 8 $\frac{4808}{13149}$ years.

Quest. 2. What is the ratio of the velocity of light to that of a cannon-ball, which issues from the gun with a velocity of 1500 feet per second; light passing from the sun to the earth in 7$\frac{1}{2}$ minutes? Ans. the ratio of 782222$\frac{2}{9}$ to 1.

Quest. 3. The slow or parade-step being 70 paces per minute, at 28 inches each pace, it is required to determine at what rate per hour that movement is? Ans. 1$\frac{113}{132}$ miles.

Quest. 4. The quick-time or step, in marching, being 2 paces per second, or 120 per minute, at 28 inches each; at what rate per hour does a troop march on a rout, and how long will they be in arriving at a garrison 20 miles distant, allowing a halt of one hour by the way to refresh?

Ans. { the rate is 3$\frac{2}{11}$ miles an hour.
and the time 7$\frac{2}{7}$ hr, or 7h 17$\frac{1}{7}$ min. }

Quest. 5. A wall was to be built 700 yards long in 29 days. Now, after 12 men had been employed on it for 11 days, it was found that they had completed only 229 yards of the wall. It is required to determine how many men must be added to the former, that the whole number of them may just finish the wall in the time proposed, at the same rate of working. Ans. 4 men to be added.

Quest. 6. Determine how far 500 millions of guineas will reach, when laid down in a strait line touching one another; supposing each guinea to be an inch in diameter, as it is very nearly. Ans. 7891 miles, 728 yds, 2 ft. 8 in.

Quest. 7. Two persons, A and B, being on opposite sides of a wood, which is 536 yards about, they begin to go round it, both the same way, at the same instant of time; A goes at the rate of 11 yards per minute, and B 34 yards in 3 minutes; the question is, how many times will the wood be gone round before the quicker overtake the slower?

Ans. 17 times.

Quest. 8. A can do a piece of work alone in 12 days, and B alone in 14; in what time will they both together perform a like quantity of work? Ans. $6\frac{6}{13}$ days.

Quest. 9. A person who was possessed of a $\frac{3}{5}$ share of a copper mine, sold $\frac{3}{4}$ of his interest in it for 1800*l*; what was the reputed value of the whole at the same rate? Ans. 4000*l*.

Quest. 10. A person after spending 20*l* more than $\frac{1}{4}$ of his yearly income, had then remaining 30*l* more than the half of it; what was his income? Ans. 200*l*.

Quest. 11. The hour and minute hand of a clock are exactly together at 12 o'clock; when are they next together?

Ans. at $1\frac{1}{11}$ hr. or 1 hr. $5\frac{5}{11}$ min.

Quest. 12. If a gentleman whose annual income is 1500*l*, spend 20 guineas a week; whether will he save or run in debt, and how much in the year? Ans. save 408*l*.

Quest 13. A person bought 180 oranges at 2 a penny, and 180 more at 3 a penny; after which, selling them out again at 5 for 2 pence, whether did he gain or lose by the bargain? Ans. he lost 6 pence.

Quest. 14. If a quantity of provisions serves 1500 men 12 weeks, at the rate of 20 ounces a day for each man; how many men will the same provisions maintain for 20 weeks, at the rate of 8 ounces a day for each man? Ans. 2250 men.

Quest. 15. In the latitude of London, the distance round the earth, measured on the parallel of latitude, is about 15550 miles; now as the earth turns round in 23 hours 56 minutes, at what rate per hour is the city of London carried by this motion from west to east? Ans. $649\frac{259}{359}$ miles an hour.

Quest. 16. A father left his son a fortune, $\frac{1}{4}$ of which he ran through in 8 months: $\frac{3}{7}$ of the remainder lasted him 12 months longer; after which he had 820*l* left. What sum did the father bequeath his son? Ans. 1913*l* 6*s* 8*d*.

Quest. 17. If 1000 men, besieged in a town, with provisions for 5 weeks, allowing each man 16 ounces a day, be reinforced with 500 men more; and supposing that they cannot be relieved till the end of 8 weeks, how many ounces a day must each man have, that the provision may last that time? Ans. $6\frac{2}{3}$ ounces.

QUEST. 18. A younger brother received 8400*l*, which was just $\frac{7}{9}$ of his elder brother's fortune: What was the father worth at his death? Ans. 19200*l*.

QUEST. 19. A person, looking on his watch, was asked what was the time of the day, who answered, It is between 5 and 6; but a more particular answer being required, he said that the hour and minute hands were then exactly together: What was the time? Ans. $27\frac{3}{11}$ min. past 5.

QUEST. 20. If 20 men can perform a piece of work in 12 days, how many men will accomplish another thrice as large in one-fifth of the time? Ans. 300.

QUEST. 21. A father devised $\frac{7}{18}$ of his estate to one of his sons, and $\frac{7}{18}$ of the residue to another, and the surplus to his relict for life. The children's legacies were found to be 514*l* 6*s* 8*d* different: What money did he leave the widow the use of? Ans. 1270*l* 1*s* $9\frac{1}{49}$*d*.

QUEST. 22. A person, making his will, gave to one child $\frac{13}{20}$ of his estate, and the rest to another. When these legacies came to be paid, the one turned out 1200*l* more than the other: What did the testator die worth? Ans. 4000*l*.

QUEST. 23. Two persons, A and B, travel between London and Lincoln, distant 100 miles, A from London, and B from Lincoln, at the same instant. After 7 hours they meet on the road, when it appeared that A had rode $1\frac{1}{2}$ miles an hour more than B. At what rate per hour then did each of the travellers ride? Ans. A $7\frac{25}{28}$ and B $6\frac{11}{28}$ miles.

QUEST. 24. Two persons, A and B, travel between London and Exeter. A leaves Exeter at 8 o'clock in the morning, and walks at the rate of 3 miles an hour, without intermission; and B sets out from London at 4 o'clock the same evening, and walks for Exeter at the rate of 4 miles an hour constantly. Now, supposing the distance between the two cities to be 130 miles, whereabouts on the road will they meet? Ans. $69\frac{2}{7}$ miles from Exeter.

QUEST. 25. One hundred eggs being placed on the ground, in a straight line, at the distance of a yard from each other: How far will a person travel who shall bring them one by one to a basket, which is placed at one yard from the first egg? Ans. 10100 yards, or 5 miles and 1300 yds.

QUEST. 26. The clocks of Italy go on to 24 hours:

Then how many strokes do they strike in one complete revolution of the index? Ans. 300.

QUEST. 27. One Sessa, an Indian, having invented the game of chess, showed it to his prince, who was so delighted with it, that he promised him any reward he should ask; on which Sessa requested that he might be allowed one grain of wheat for the first square on the chess board, 2 for the second, 4 for the third, and so on, doubling continually, to 64, the whole number of squares. Now, supposing a pint to contain 7680 of these grains, and one quarter or 8 bushels to be worth 27*s* 6*d*, it is required to compute the value of all the corn? Ans. 64504682162851*l* 17*s* 3*d* $3\frac{33747}{33768}q$.

QUEST. 28. A person increased his estate annually by 100*l* more than the $\frac{1}{4}$ part of it; and at the end of 4 years found that his estate amounted to 10342*l* 3*s* 9*d*. What had he at first? Ans. 4000*l*.

QUEST. 29. Paid 1012*l* 10*s* for a principal of 750*l*, taken in 7 years before: at what rate per cent. per annum did I pay interest? Ans. 5 per cent.

QUEST. 30. Divide 1000*l* among A, B, C; so as to give A 120 more, and B 95 less than C.

Ans. A 445, B 230, C 325.

QUEST. 31. A person being asked the hour of the day, said, the time past noon is equal to $\frac{4}{5}$ths of the time till midnight. What was the time? Ans. 20 min. past 5.

QUEST. 32. Suppose that I have $\frac{3}{16}$ of a ship worth 1200*l*; what part of her have I left after selling $\frac{2}{5}$ of $\frac{4}{9}$ of my share, and what is it worth? Ans. $\frac{37}{240}$, worth 185*l*.

QUEST. 33. Part 1200 acres of land among A, B, C; so that B may have 100 more than A, and C 64 more than B.

Ans. A 312, B 412, C 476.

QUEST. 34. What number is that, from which if there be taken $\frac{2}{7}$ of $\frac{3}{8}$, and to the remainder be added $\frac{8}{15}$ of $\frac{5}{8}$, the sum will be 10? Ans. $9\frac{65}{84}$.

QUEST. 35. There is a number which, if multiplied by $\frac{2}{3}$ of $\frac{3}{4}$ of $1\frac{1}{2}$, will produce 1: what is the square of that number? Ans. $1\frac{7}{9}$.

QUEST. 36. What length must be cut off a board, $8\frac{1}{2}$ inches broad, to contain a square foot, or as much as 12 inches in length and 12 in breadth? Ans. $16\frac{16}{17}$ inches.

QUEST. 37. What sum of money will amount to 138*l* 2*s* 6*d*, in 15 months, at 5 per cent. per annum simple interest? Ans. 130*l*.

QUEST. 38. A father divided his fortune among his three sons, A, B, C, giving A 4 as often as B 3, and C 5 as often as B 6; what was the whole legacy, supposing A's share was 4000*l*? Ans. 9500*l*.

QUEST. 39. A young hare starts 40 yards before a greyhound, and is not perceived by him till she has been up 40 seconds; she scuds away at the rate of 10 miles an hour, and the dog, on view, makes after her at the rate of 18: how long will the course hold, and what ground will be run over, counting from the outsetting of the dog?

Ans. $60\frac{5}{11}$ sec. and 530 yards run.

QUEST. 40. Two young gentlemen, without private fortune, obtain commissions at the same time, and at the age of 18. One thoughtlessly spends 10*l* a year more than his pay; but, shocked at the idea of not paying his debts, gives his creditor a bond for the money, at the end of every year, and also insures his life for the amount; each bond costs him 30 shillings, besides the lawful interest of 5 per cent. and to insure his life costs him 6 per cent.

The other, having a proper pride, is determined never to run in debt; and, that he may assist a friend in need, perseveres in saving 10*l* every year, for which he obtains an interest of 5 per cent. which interest is every year added to his savings, and laid out, so as to answer the effect of compound interest.

Suppose these two officers to meet at the age of 50, when each receives from Government 400*l* per annum; that the one, seeing his past errors, is resolved in future to spend no more than he actually has, after paying the interest for what he owes, and the insurance on his life.

The other, having now something beforehand, means in future to spend his full income, without increasing his stock.

It is desirable to know how much each has to spend per annum, and what money the latter has by him to assist the distressed, or leave to those who deserve it?

Ans. The reformed officer has to spend 66*l* 19*s* $1\frac{3}{4}$·5389*d*. per annum.

The prudent officer has to spend 437*l* 12*s* $11\frac{3}{4}$·4379*d*. per annum, and

The latter has saved, to dispose of, 752*l* 19*s* 9·1896*d*.

OF LOGARITHMS *.

LOGARITHMS are made to facilitate troublesome calculations in numbers. This they do, because they perform multiplication by only addition, and division by subtraction, and raising of powers by multiplying the logarithm by the index of the power, and extracting of roots by dividing the logarithm of the number by the index of the root. For, logarithms are numbers so contrived, and adapted to other numbers, that the sums and differences of the former shall correspond to, and show the products and quotients of the latter, &c.

Or, more generally, logarithms are the numerical exponents of ratios; or they are a series of numbers in arith-

* The invention of Logarithms is due to Lord Napier, Baron of Merchiston, in Scotland, and is properly considered as one of the most useful inventions of modern times. A table of these numbers was first published by the inventor at Edinburgh, in the year 1614, in a treatise entitled *Canon Mirificum Logarithmorum;* which was eagerly received by all the learned throughout Europe. Mr. Henry Briggs, then professor of geometry at Gresham College, soon after the discovery, went to visit the noble inventor; after which, they jointly undertook the arduous task of computing new tables on this subject, and reducing them to a more convenient form than that which was at first thought of. But Lord Napier dying soon after, the whole burden fell upon Mr. Briggs, who, with prodigious labour and great skill, made an entire Canon, according to the new form, for all numbers from 1 to 20000, and from 90000 to 101000, to 14 places of figures, and published it at London in the year 1624, in a treatise entitled Arithmetica Logarithmica, with directions for supplying the intermediate parts.

This Canon was again published in Holland by Adrian Vlacq, in the year 1628, together with the Logarithms of all the numbers which Mr. Briggs had omitted; but he contracted them down to 10 places of decimals. Mr. Briggs also computed the Logarithms of the sines, tangents, and secants, to every degree, and centesm, or 100th part of a degree, of the whole quadrant; and annexed them to the natural sines, tangents, and secants, which he had before computed, to fifteen places of figures. These tables, with their construction and use, were first published in the year 1633, after Mr. Briggs's death, by Mr. Henry Gellibrand, under the title of Trigonometria Britannica.

metical progression, answering to another series of numbers in geometrical progression.

Thus	0,	1,	2,	3,	4,	5,	6,	Indices, or logarithms,
	1,	2,	4,	8,	16,	32,	64,	Geometric progression.
Or	0,	1,	2,	3,	4,	5,	6,	Indices, or logarithms,
	1,	3,	9,	27,	81,	243,	729,	Geometric progression.
Or	0,	1,	2,	3,	4,	5,		Indices, or logs.
	1,	10,	100,	1000,	10000,	100000,		Geom. progres.

Where it is evident, that the same indices serve equally for any geometric series ; and consequently there may be an endless variety of systems of logarithms, to the same common numbers, by only changing the second term, 2, 3, or 10, &c. of the geometrical series of whole numbers ; and by interpolation the whole system of numbers may be made to enter the geometric series, and receive their proportional logarithms, whether integers or decimals.

It is also apparent, from the nature of these series, that if any two indices be added together, their sum will be the index of that number which is equal to the product of the two terms, in the geometric progression, to which those indices belong. Thus the indices 2 and 3, being added together, make 5 ; and the numbers 4 and 8, or the terms corresponding to those indices, being multiplied together, make 32, which is the number answering to the index 5.

In like manner, if any one index be subtracted from another, the difference will be the index of that number which

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Benjamin Ursinus also gave a Table of Napier's Logs. and of sines, to every 10 seconds. And Chr. Wolf, in his Mathematical Lexicon, says that one Van Loser had computed them to every single second, but his untimely death prevented their publication. Many other authors have treated on this subject ; but as their numbers are frequently inaccurate and incommodiously disposed, they are now generally neglected. The Tables in most repute at present, are those of Gardiner in 4to, first published in the year 1742 ; and my own Tables in 8vo, first printed in the year 1785, where the Logarithms of all numbers may be easily found from 1 to 10800000 ; and those of the sines, tangents, and secants, to any degree of accuracy required.

Mr. Michael Taylor's Tables in large 4to, containing the common logarithms, and the logarithmic sines and tangents to every second of the quadrant, are very valuable. And, in France, the new book of logarithms by Callet ; the 2d edition of which, in 1795, has the tables still further extended, and are printed with what are called stereotypes, the types in each page beng soldered together into a solid mass or block.

Dodson's Antilogarithmic Canon is likewise a very elaborate work, and used for finding the numbers answering to any given logarithm, each to 11 places.

is equal to the quotient of the two terms to which those indices belong. Thus, the index 6, minus the index 4, is = 2; and the terms corresponding to those indices are 64 and 16, whose quotient is = 4, which is the number answering to the index 2.

For the same reason, if the logarithm of any number be multiplied by the index of its power, the product will be equal to the logarithm of that power. Thus, the index or logarithm of 4, in the above series, is 2; and if this number be multiplied by 3, the product will be = 6; which is the logarithm of 64, or the third power of 4.

And, if the logarithm of any number be divided by the index of its root, the quotient will be equal to the logarithm of that root. Thus, the index or logarithm of 64 is 6; and if this number be divided by 2, the quotient will be = 3; which is the logarithm of 8, or the square root of 64.

The logarithms most convenient for practice, are such as are adapted to a geometric series increasing in a tenfold proportion, as in the last of the above forms; and are those which are to be found, at present, in most of the common tables on this subject. The distinguishing mark of this system of logarithms is, that the index or logarithm of 10 is 1; that of 100 is 2; that of 1000 is 3; &c. And, in decimals, the logarithm of ·1 is — 1; that of ·01 is — 2; that of ·001 is — 3, &c. the log. of 1 being 0 in every system. Whence it follows, that the logarithm of any number between 1 and 10, must be 0 and some fractional parts; and that of a number between 10 and 100, will be 1 and some fractional parts; and so on, for any other number whatever. And since the integral part of a logarithm, usually called the Index, or Characteristic, is always thus readily found, it is commonly omitted in the tables; being left to be supplied by the operator himself, as occasion requires.

Another Definition of Logarithms is, that the logarithm of any number is the index of that power of some other number, which is equal to the given number. So, if there be $N = r^n$, then n is the log. of N; where n may be either positive or negative, or nothing, and the root or base r any number whatever, according to the different systems of logarithms. When n is $= 0$, then N is $= 1$, whatever the value of r is; which shows, that the log. of 1 is always 0, in every system of logarithms. When n is $= 1$, then N is $= r$;

so that the radix r is always that number whose log. is 1, in every system. When the radix r is = 2·718281828459 &c. the indices n are the hyperbolic or Napier's log. of the numbers N; so that n is always the hyp. log. of the number N or $(2.718 \text{ \&c.})^n$.

But when the radix r is = 10, then the index n becomes the common or Briggs's log. of the number N: so that the common log. of any number 10^n or N, is n the index of that power of 10 which is equal to the said number. Thus 100, being the second power of 10, will have 2 for its logarithm; and 1000, being the third power of 10, will have 3 for its logarithm: hence also, if 50 be = $10^{1\cdot69897}$, then is 1·69897 the common log. of 50. And, in general, the following decuple series of terms,

viz.	10^4,	10^3,	10^2,	10^1,	10^0,	10^{-1},	10^{-2},	10^{-3},	10^{-4},
or	10000,	1000,	100,	10,	1,	·1,	·01,	·001,	·0001,
have	4,	3,	2,	1,	0,	−1,	−2,	−3,	−4,

for their logarithms, respectively. And from this scale of numbers and logarithms, the same properties easily follow, as above mentioned.

PROBLEM.

To compute the Logarithm to any of the Natural Numbers 1, 2, 3, 4, 5, &c.

RULE I*.

TAKE the geometric series, 1, 10, 100, 1000, 10000, &c. and apply to it the arithmetic series, 0, 1, 2, 3, 4, &c. as logarithms.—Find a geometric mean between 1 and 10, or between 10 and 100, or any other two adjacent terms of the series, between which the number proposed lies.—In like manner, between the mean, thus found, and the nearest extreme, find another geometrical mean; and so on, till you arrive within the proposed limit of the number whose logarithm is sought.—Find also as many arithmetical means, in the same order as you found the geometrical ones, and these will be the logarithms answering to the said geometrical means.

* The reader who wishes to inform himself more particularly concerning the history, nature, and construction of Logarithms, may consult my Mathematical Tracts, vol. 1, lately published, where he will find his curiosity amply gratified.

EXAMPLE.

Let it be required to find the logarithm of 9.

Here the proposed number lies between 1 and 10.

First, then, the log. of 10 is 1, and the log. of 1 is 0;
theref. $(1 + 0) \div 2 = \frac{1}{2} = \cdot 5$ is the arithmetical mean,
and $\sqrt{(10 \times 1)} = \sqrt{10} = 3 \cdot 1622777$ the geom. mean;
hence the log. of 3·1622777 is ·5.

Secondly, the log. of 10 is 1, and the log. of 3·1622777 is ·5;
theref. $(1 + \cdot 5) \div 2 = \cdot 75$ is the arithmetical mean,
and $\sqrt{(10 \times 3 \cdot 1622777)} = 5 \cdot 6234132$ is the geom. mean;
hence the log. of 5·6234132 is ·75.

Thirdly, the log. of 10 is 1, and the log. of 5·6234132 is ·75;
theref. $(1 + \cdot 75) \div 2 = 875$ is the arithmetical mean,
and $\sqrt{(10 \times 5 \cdot 6234132)} = 7 \cdot 4989422$ the geom. mean;
hence the log. of 7·4989422 is ·875.

Fourthly, the log. of 10 is 1, and the log. of 7·4989422 is ·875;
theref. $(1 + \cdot 875) \div 2 = \cdot 9375$ is the arithmetical mean,
and $\sqrt{(10 \times 7 \cdot 4989422)} = 8 \cdot 6596431$ the geom. mean;
hence the log. of 8·6596431 is ·9375.

Fifthly, the log. of 10 is 1, and the log. of 8·6596431 is ·9375;
theref. $(1 + \cdot 9375) \div 2 = \cdot 96875$ is the arithmetical mean,
and $\sqrt{(10 \times 8 \cdot 6596431)} = 9 \cdot 3057204$ the geom. mean;
hence the log. of 9·3057204 is ·96875.

Sixthly, the log. of 8·6596431 is ·9375, and the log. of 9·3057204 is ·96875.
theref. $(\cdot 9375 + \cdot 96875) \div 2 = \cdot 953125$ is the arith. mean,
and $\sqrt{(8 \cdot 6596431 \times 9 \cdot 3057204)} = 8 \cdot 9768713$ the geometric mean;
hence the log. of 8·9768713 is ·953125.

And proceeding in this manner, after 25 extractions, it will be found that the logarithm of 8·9999998 is ·9542425; which may be taken for the logarithm of 9, as it differs so little from it, that it is sufficiently exact for all practical purposes. In this manner were the logarithms of almost all the prime numbers at first computed.

RULE II*.

LET b be the number whose logarithm is required to be found; and a the number next less than b, so that $b - a = 1$,

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* For the demonstration of this rule, see my Mathematical Tables, p. 109, &c. and my Tracts, vol. 1.

the logarithm of a being known; and let s denote the sum of the two numbers $a + b$. Then

1. Divide the constant decimal ·8685889638 &c. by s, and reserve the quotient: divide the reserved quotient by the square of s, and reserve this quotient; divide this last quotient also by the square of s, and again reserve the quotient: and thus proceed, continually dividing the last quotient by the square of s, as long as division can be made.

2. Then write these quotients orderly under one another, the first uppermost, and divide them respectively by the odd numbers, 1, 3, 5, 7, 9, &c. as long as division can be made; that is, divide the first reserved quotient by 1, the second by 3, the third by 5, the fourth by 7, and so on.

3. Add all these last quotients together, and the sum will be the logarithm of $b \div a$; therefore to this logarithm add also the given logarithm of the said next less number a, so will the last sum be the logarithm of the number b proposed.

That is,

$$\text{Log. of } b. \text{ is log. } a + \frac{n}{s} \times \left(1 + \frac{1}{3s^2} + \frac{1}{5s^4} + \frac{1}{7s^6} + \&c.\right.$$

where n denotes the constant given decimal ·8685889638 &c.

EXAMPLES.

Ex. 1. Let it be required to find the log. of the number 2. Here the given number b is 2, and the next less number a is 1, whose log. is 0; also the sum $2 + 1 = 3 = s$, and its square $s^2 = 9$. Then the operation will be as follows:

3)	·868588964	1)	·289529654 (	·289529654
9)	·289529654	3)	32169962 (	10723321
9)	32169962	5)	3574440 (	714888
9)	3574440	7)	397160 (	56737
9)	397160	9)	44129 (	4903
9)	44129	11)	4903 (	446
9)	4903	13)	545 (	42
9)	545	15)	61 (	4
9)	61			
			log. of $\frac{2}{1}$ -	·301029995
			add log. 1 -	·000000000
			log. of 2 -	·301029995

Ex. 2. To compute the logarithm of the number 3.

Here $b = 3$, the next less number $a = 2$, and the sum $a + b = 5 = s$, whose square s^2 is 25, to divide by which, always multiply by ·04. Then the operation is as follows :

5)	·868588964	1)	·173717793 (		·173717793
25)	173717793	3)	6948712 (		2316237
25)	6948712	5)	277948 (		55590
25)	277948	7)	11118 (		1588
25)	11118	9)	445 (		50
25)	445	11)	18 (		2
	18				
			log. of $\frac{3}{2}$	-	·176091260
			log. of 2 add		·301029995
			log. of 3 sought		·477121255

Then, because the sum of the logarithms of numbers, gives the logarithm of their product ; and the difference of the logarithms, gives the logarithm of the quotient of the numbers ; from the above two logarithms, and the logarithm of 10, which is 1, we may obtain a great many logarithms, as in the following examples :

EXAMPLE 3.

Because $2 \times 2 = 4$, therefore

to log. 2	-	·301029995$\frac{2}{3}$
add log. 2	-	·301029995$\frac{2}{3}$
sum is log. 4		·602059991$\frac{1}{3}$

EXAMPLE 4.

Because $2 \times 3 = 6$, therefore

to log. 2	-	·301029995
add log. 3	-	·477121255
sum is log. 6		·778151250

EXAMPLE 5.

Because $2^3 = 8$, therefore

log. 2	-	·301029995
mult. by 3		3
gives log. 8		·903089987

EXAMPLE. 6.

Because $3^2 = 9$, therefore

log. 3	-	·477121254$\frac{7}{10}$
mult. by 2		2
gives log. 9		·954242509

EXAMPLE 7.

Because $\frac{10}{2} = 5$, therefore

from log. 10	1·000000000
take log. 2	·301029995$\frac{2}{3}$
leaves log. 5	·698970004$\frac{1}{3}$

EXAMPLE 8.

Because $3 \times 4 = 12$, therefore

to log. 3	-	·477121255
add log. 4	-	·602059991
gives log. 12		1·079181246

And thus, computing by this general rule, the logarithms to the other prime numbers, 7, 11, 13, 17, 19, 23, &c. and then using composition and division, we may easily find as many logarithms as we please, or may speedily examine any logarithm in the table*.

Description and Use of the Table *of* Logarithms.

Having explained the manner of forming a table of the logarithms of numbers, greater than unity; the next thing to be done is, to show how the logarithms of fractional quantities may be found. In order to this, it may be observed, that as in the former case a geometric series is supposed to increase towards the left, from unity, so in the latter case it is supposed to decrease towards the right hand, still beginning with unit; as exhibited in the general description, page 152, where the indices being made negative, still show the logarithms to which they belong. Whence it appears, that as + 1 is the log. of 10, so − 1 is the log. of $\frac{1}{10}$ or ·1; and as + 2 is the log. of 100, so − 2 is the log. of $\frac{1}{100}$ or ·01: and so on.

Hence it appears in general, that all numbers which consist of the same figures, whether they be integral, or fractional, or mixed, will have the decimal parts of their logarithms the same, but differing only in the index, which will be more or less, and positive or negative, according to the place of the first figure of the number.

Thus, the logarithm of 2651 being 3·423410, the log. of $\frac{1}{10}$, or $\frac{1}{100}$, or $\frac{1}{1000}$, &c. part of it will be as follows:

Numbers.	Logarithms.
2 6 5 1	3 ·4 2 3 4 1 0
2 6 ·5 1	2 ·4 2 3 4 1 0
2 6 5 ·1	1 ·4 2 3 4 1 0
2 ·6 5 1	0 ·4 2 3 4 1 0
·2 6 5 1	−1 ·4 2 3 4 1 0
·0 2 6 5 1	−2 ·4 2 3 4 1 0
·0 0 2 6 5 1	−3 ·4 2 3 4 1 0

* There are, besides these, many other ingenious methods, which later writers have discovered for finding the logarithms of numbers, in a much easier way than by the original inventor; but, as they cannot be understood without a knowledge of some of the higher branches of the mathematics, it is thought proper to omit them, and to refer the reader to those works which are written expressly on the subject. It would likewise much exceed the limits of this compendium, to point out all the peculiar artifices that are made use of for constructing an

Hence it also appears, that the index of any logarithm, is always less by 1 than the number of integer figures which the natural number consists of: or it is equal to the distance of the first figure from the place of units, or first place of integers, whether on the left, or on the right, of it: and this index is constantly to be placed on the left-hand side of the decimal part of the logarithm.

When there are integers in the given number, the index is always affirmative; but when there are no integers, the index is negative, and is to be marked by a short line drawn before it, or else above it. Thus,

A number having 1, 2, 3, 4, 5, &c. integer places,
the index of its log. is 0, 1, 2, 3, 4, &c. or 1 less than those places.

And a decimal fraction having its first effective figure in the 1st, 2d, 3d, 4th, &c, place of the decimals, has always —1, —2, —3, —4, &c. for the index of its logarithm.

It may also be observed, that though the indices of fractional quantities are negative, yet the decimal parts of their logarithms are always affirmative. And the negative mark (—) may be set either before the index or over it.

I. TO FIND IN THE TABLE, THE LOGARITHM TO ANY NUMBER*.

1. *If the given Number be less than* 100, or consist of only two figures; its log. is immediately found by inspection in the first page of the table, which contains all numbers from 1 to 100, with their logs. and the index immediately annexed in the next column.

So the log. of 5 is 0·698970. The log. of 23 is 1·361728. The log. of 50. is 1·698970. And so on.

2. *If the Number be more than* 100 *but less than* 10000; that is, consisting of either three or four figures; the decimal part of the logarithm is found by inspection in the other pages of the table, standing against the given number in this manner; viz. the first three figures of the given number in the first column of the page, and the fourth figure one of those along the top line of it; then in the angle of meeting are the last four figures of the logarithm, and the first two figures of the same at the beginning of the same line in the second

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entire table of these numbers; but any information of this kind, which the learner may wish to obtain, may be found in my Tables. See also the article on Logarithms in the 2d volume, p. 340, &c.

* See the table of Logarithms, at the end of this volume.

column of the page: to which is to be prefixed the proper index which is always 1 less than the number of integer figures.

So the logarithm of 251 is 2·399674, that is, the decimal ·399674 found in the table, with the index 2 prefixed, because the given number contains three integers. And the log. of 34·09 is 1·532627, that is, the decimal ·532627 found in the table, with the index 1 prefixed, because the given number contains two integers.

2. *But if the given Number contain more than four figures;* take out the logarithm of the first four figures by inspection in the table, as before, as also the next greater logarithm, subtracting the one logarithm from the other, as also their corresponding numbers the one from the other. Then say,

As the difference between the two numbers,
Is to the difference of their logarithms,
So is the remaining part of the given number,
To the proportional part of the logarithm.

Which part being added to the less logarithm, before taken out, gives the whole logarithm sought very nearly.

EXAMPLE.

To find the logarithm of the number 34·0926.
The log. of 340900, as before, is 532627.
And log. of 341000 - - is 532754.
The diffs. are 100 and 127.

Then, as 100 : 127 : : 26 : 33, the proportional part.
This added to - - - 532627, the first log.
Gives, with the index, 1·532660 for the log. of 34·0926.

4. If the number consist both of integers and fractions, or is entirely fractional; find the decimal part of the logarithm the same as if all its figures were integral; then this, having prefixed to it the proper index, will give the logarithm required.

5. And if the given number be a proper vulgar fraction: subtract the logarithm of the denominator from the logarithm of the numerator, and the remainder will be the logarithm sought; which, being that of a decimal fraction, must always have a negative index.

6. But if it be a mixed number; reduce it to an improper fraction, and find the difference of the logarithm of the numerator and denominator, in the same manner as before.

EXAMPLES.

1. To find the log. of $\frac{37}{94}$.		2. To find the log. of $17\frac{14}{23}$.	
Log. of 37 -	1·568202	First, $17\frac{14}{23} = \frac{405}{23}$.	Then,
Log. of 94 -	1·973128	Log. of 405 -	2·607455
		Log. of 23 -	1·361728
Dif. log. of $\frac{37}{94}$	—1·595074		
		Dif. log. of $17\frac{14}{23}$	1·245727
Where the index 1 is negative			

II. TO FIND THE NATURAL NUMBER TO ANY GIVEN LOGARITHM.

This is to be found in the tables by the reverse method to the former, namely, by searching for the proposed logarithm among those in the table, and taking out the corresponding number by inspection, in which the proper number of integers are to be pointed off, viz. 1 more than the index. For, in finding the number answering to any given logarithm, the index always shows how far the first figure must be removed from the place of units, viz. to the left hand, or integers, when the index is affirmative; but to the right hand, or decimals, when it is negative.

EXAMPLES.

So, the number to the log. 1·532882 is 34·11.

And the number of the log. $\bar{1}$·532882 is ·3411.

But if the logarithm cannot be exactly found in the table; take out the next greater and the next less, subtracting the one of these logarithms from the other, as also their natural numbers the one from the other, and the less logarithm from the logarithm proposed. Then say,

As the difference of the first or tabular logarithms,
Is to the difference of their natural numbers,
So is the differ. of the given log. and the least tabular log.
To their corresponding numeral difference.

Which being annexed to the least natural number above taken, gives the natural number sought, corresponding to the proposed logarithm.

EXAMPLE.

So, to find the natural number answering to the given logarithm 1·532708.

Here the next greater and next less tabular logarithms, with their corresponding numbers, are as below:

Next greater	532754	its num.	341000;	given log.		532708
Next less	532627	its num.	340900;	next less		532627
Differences	127	—	100	—		81

Then, as 127 : 100 : : 81 : 64, nearly the numeral differ.

Therefore 34·0964 is the number sought, marking off two integers, because the index of the given logarithm is 1.

Had the index been negative, thus $\bar{1}$·532708, its corresponding number would have been ·340964, wholly decimal.

MULTIPLICATION BY LOGARITHMS.

RULE.

Take out the logarithms of the factors from the table, then add them together, and their sum will be the logarithm of the product required. Then, by means of the table, take out the natural number, answering to the sum, for the product sought.

Take care to add what is to be carried from the decimal part of the logarithm to the affirmative index or indices, or else subtract it from the negative.

Also, add the indices together when they are of the same kind, both affirmative or both negative; but subtract the less from the greater, when the one is affirmative and the other negative, and prefix the sign of the greater to the remainder.

EXAMPLES.

1. To multiply 23·14 by 5·062

	Numbers.	Logs.
	23·14	1·364363
	5·062	0·704322
Product	117·1347	2·068685

2. To multiply 2·581926 by 3·457291

	Numbers.	Logs.
	2·581926	0·411944
	3·457291	0·538736
Prod.	8·92648	0·950680

3. To mult. 3·902 and 597·16 and ·0314728 all together.

Numbers.		Logs.
3·902	-	0·591287
597·16	-	2·776091
·0314728		−2·497935
Prod. 73·3333	.	1·865313

Here the − 2 cancels the 2, and the 1 to carry from the decimals is set down.

4. To mult. 3·586, and 2·1046, and 0·8372, and 0·0294 all together.

Numbers.		Logs.
3 586	-	0·554610
2·1046	-	0·323170
0·8372		−1·922829
0·0294		−2·468347
Prod. 0·1857618		−1·268956

Here the 2 to carry cancels the - 2, and there remains the − 1 to set down.

DIVISION BY LOGARITHMS.

RULE.

From the logarithm of the dividend, subtract the logarithm of the divisor, and the number answering to the remainder will be the quotient required.

Change the sign of the index of the divisor, from affirmative to negative, or from negative to affirmative ; then take the sum of the indices if they be of the same name, or their difference when of different signs, with the sign of the greater, for the index to the logarithm of the quotient.

Also, when 1 is borrowed, in the left-hand place of the decimal part of the logarithm, add it to the index of the divisor when that index is affirmative, but subtract it when negative ; then let the sign of the index arising from hence be changed, and worked with as before.

EXAMPLES.

1. To divide 24163 by 4567.

	Numbers.		Logs.
Dividend	24163	.	4·383151
Divisor	4567	.	3·659631
Quot.	5·29078		0·723520

2. To divide 37·149 by 523·76.

	Numbers.		Logs.
Dividend	37·149	-	1·569947
Divisor	523·76	.	2·719132
Quot.	·0709275		−2·850815

3. Divide ·06314 by ·007241

	Numbers.	Logs.
Dividend	·06314	−2·800305
Divisor	·007241	−3·859799
Quot.	8·71979	0·940506

Here 1 carried from the decimals to the −3, makes it become −2, which taken from the other −2, leaves 0 remaining.

4. To divide ·7438 by 12·9476.

	Numbers.	Logs.
Divid.	·7438	−1·871456
Divisor	12·9476	1·112189
Quot.	·057447	−2·759267

Here the 1 taken from the −1, makes it become −2, to set down.

Note. The Rule-of-Three, or Rule of Proportion, is performed by adding the logarithms of the 2d and 3d terms, and subtracting that of the first term from their sum. Instances will occur in Plain Trigonometry.

INVOLUTION BY LOGARITHMS.

RULE.

Take out the logarithm of the given number from the table. Multiply the logarithm thus found, by the index of the power proposed. Find the number answering to the product, and it will be the power required.

Note. In multiplying a logarithm with a negative index, by an affirmative number, the product will be negative. But what is to be carried from the decimal part of the logarithm, will always be affirmative. And therefore their difference will be the index of the product, and is always to be made of the same kind with the greater.

EXAMPLES.

1. To square the number 2·5791.

	Numb.	Log.
Root	2·5791 - -	0·411468
	The index - -	2
Power	6·65174	0·822936

2. To find the cube of 3·07146.

	Numb.	Log.
Root	3·07146 - -	0·487345
	The index - -	3
Power	28·9758	1·462035

3. To raise ·09163 to the 4th power.

	Numb.	Log.
Root	·09163	—2·962038
	The index - -	4
Pow.	·000070494	—5·848152

Here 4 times the negative index being —8, and 3 to carry, the difference —5 is the index of the product.

4. To raise 1·0045 to the 365th power.

	Numb.	Log.
Root	1·0045 . -	0·001950
	The index - -	365
		9750
		11700
		5850
Power	5·14932	0·711750

EVOLUTION BY LOGARITHMS.

Take the log. of the given number out of the table. Divide the log. thus found by the index of the root. Then the number answering to the quotient will be the root.

Note. When the index of the logarithm, to be divided is negative, and does not exactly contain the divisor, without some remainder, increase the index by such a number as will make it exactly divisible by the index, carrying the units borrowed, as so many tens, to the left-hand place of the decimal, and then divide as in whole numbers.

EXAMPLES.

1. To find the square root of 365.

	Numb.	Log.
Power	365	2) 2·562293
Root	19·10496	1·281146½

2. To find the 3d root of 12345.

	Numb.	Log.
Power	12345	3) 4·091491
Root	23·1116	1·363830⅓

3. To find the 10th root of 2.

	Numb.	Log.
Power	2 -	10) 0·301030
Root	1·071773	0·030103

4. To find the 365th root of 1·045.

	Numb.	Log.
Power	1·045	365) 0·019116
Root	1·000121	0·000052⅓

5. To find $\sqrt{\cdot 093}$.

	Numb.	Log.
Power	·093 2)	—2·968483
Root	·304959	—1·484241½

Here the divisor 2 is contained exactly once in the negative index —2, and therefore the index of the quotient is —1.

6. To find the $\sqrt[3]{\cdot 00048}$.

	Numb.	Log.
Power	·00048 3)	—4·681241
Root	·0782973	—2·893747

Here the divisor 3, not being exactly contained in —4, it is augmented by 2, to make up 6, in which the divisor is contained just 2 times; then the 2, thus borrowed, being carried to the decimal figure 6, makes 26, which divided by 3, gives 8, &c.

7. To find $3\cdot1416 \times 82 \times \frac{7}{11}$.

8. To find $\cdot02916 \times 751\cdot3 \times \frac{8}{41}$.

9. As 7241 : 3·58 : : 20·46 : ?

10. As $\sqrt{724}$: $\sqrt{\frac{4}{13}}$: : 6·927 : ?

ALGEBRA.

DEFINITIONS AND NOTATION.

1. ALGEBRA is the science of investigation by means of symbols. It is sometimes also called Analysis; and is a general kind of arithmetic, or universal way of computation.

2. In this science, quantities of all kinds are represented by the letters of the alphabet. And the operations to be performed with them, as addition or subtraction, &c. are denoted by certain simple characters, instead of being expressed by words at length.

3. In algebraical inquiries, some quantities are known or given, viz. those whose values are known: and others unknown, or are to be found out, viz. those whose values are not known. The former of these are represented by the leading letters of the alphabet, a, b, c, d, &c.; and the latter, or unknown quantities, by the final letters, z, y, x, u, &c.

4. The characters used to denote the operations, are chiefly the following:

$+$ signifies addition, and is named *plus*.

$-$ signifies subtraction, and is named *minus*.

$\times$ or . signifies multiplication, and is named *into*.

$\div$ signifies division, and is named *by*.

$\sqrt{}$ signifies the square root; $\sqrt[3]{}$ the cube root; $\sqrt[4]{}$ the 4th root, &c.; and $\sqrt[n]{}$ the nth root.

: : : : signifies proportion.

$=$ signifies equality, and is named *equal to*.

And so on for other operations.

Thus $a + b$ denotes that the number represented by b is to be added to that represented by a.

$a - b$ denotes that the number represented by b is to be subtracted from that represented by a.

$a \backsim b$ denotes the difference of a and b, when it is not known which is the greater.

ab, or $a \times b$, or $a \cdot b$, expresses the product, by multiplication of the numbers represented by a and b.

$a \div b$, or $\frac{a}{b}$, denotes, that the number represented by a is to be divided by that which is expressed by b.

$a : b :: c : d$, signifies that a is in the same proportion to b, as c is to d.

$x = a - b + c$ is an equation, expressing that x is equal to the difference of a and b, added to the quantity c.

$\sqrt{a}$, or $a^{\frac{1}{2}}$, denotes the square root of a; $\sqrt[3]{a}$, or $a^{\frac{1}{3}}$, the cube root of a; and $\sqrt[3]{a^2}$ or $a^{\frac{2}{3}}$ the cube root of the square of a; also $\sqrt[m]{a}$, or $a^{\frac{1}{m}}$, is the mth root of a; and $\sqrt[m]{a^n}$ or $a^{\frac{n}{m}}$ is the nth power of the mth root of a, or it is a to the $\frac{n}{m}$ power.

a^2 denotes the square of a; a^3 the cube of a; a^4 the fourth power of a; and a^n the nth power of a.

$\overline{a+b} \times c$, or $(a+b)\,c$, denotes the product of the compound quantity $a+b$ multiplied by the simple quantity c. Using the bar ——, or the parenthesis ( ) as a vinculum, to connect several simple quantities into one compound.

$\overline{a+b} \div \overline{a-b}$, or $\frac{a+b}{a-b}$, expressed like a fraction, means the quotient of $a+b$ divided by $a-b$.

$\sqrt{ab+cd}$, or $(ab+cd)^{\frac{1}{2}}$, is the square root of the compound quantity $ab+cd$. And $c\sqrt{ab+cd}$, or $c\,(ab+cd)^{\frac{1}{2}}$, denotes the product of c into the square root of the compound quantity $ab+cd$.

$\overline{a+b-c}^3$, or $(a+b-c)^3$ denotes the cube, or third power, of the compound quantity $a+b-c$.

$3a$ denotes that the quantity a is to be taken 3 times, and $4(a+b)$ is 4 times $a+b$. And these numbers, 3 or 4, showing how often the quantities are to be taken, or multiplied, are called Co-efficients.

Also $\frac{3}{4}x$ denotes that x is multiplied by $\frac{3}{4}$; thus $\frac{3}{4} \times x$ or $\frac{3}{4}x$.

5. Like quantities, are those which consist of the same letters, and powers. As a and $3a$; or $2ab$ and $4ab$; or $3a^2bc$ and $-5a^2bc$.

6. Unlike Quantities, are those which consist of different letters, or different powers. As a and b; or $2a$ and a^2; or $3ab^2$ and $3abc$.

7. Simple Quantities are those which consist of one term only. As $3a$, or $5ab$, or $6abc^2$.

8. Compound Quantities are those which consist of two or more terms. As $a+b$, or $2a-3c$, or $a+2b-3c$.

9. And when the compound quantity consists of two terms, it is called a Binomial, as $a+b$; when of three terms, it is a Trinomial, as $a+2b-3c$; when of four terms, a Quadrinomial, as $2a-3b+c-4d$; and so on. Also a Multinomial or Polynomial, consists of many terms.

10. A Residual Quantity, is a binomial having one of the terms negative. As $a-2b$.

11. Positive or affirmative Quantities, are those which are to be added, or have the sign $+$. As a or $+a$, or ab: for when a quantity is found without a sign, it is understood to be positive, or have the sign $+$ prefixed.

12. Negative Quantities, are those which are to be subtracted. As $-a$, or $-2ab$, or $-3ab^2$.

13. Like Signs, are either all positive ($+$), or all negative ($-$).

14. Unlike Signs, are when some are positive ($+$), and others negative ($-$).

15. The Co-efficient of any quantity, as shown above, is the number prefixed to it. As 3, in the quantity $3ab$.

16. The power of a quantity (a), is its square (a^2), or cube (a^3), or biquadrate (a^4), &c.; called also, the 2d power, or 3d power, or 4th power, &c.

17. The Index or Exponent, is the number which denotes the power or root of a quantity. So 2 is the exponent of the square or second power a^2; and 3 is the index of the cube or 3d power; and $\frac{1}{2}$ is the index of the square root, $a^{\frac{1}{2}}$ or $\sqrt{a}$; and $\frac{1}{3}$ is the index of the cube root, $a^{\frac{1}{3}}$, or $\sqrt[3]{a}$.

18. A Rational Quantity, is that which has no radical sign ($\sqrt{\ }$) or index annexed to it. As a, or $3ab$.

19. An Irrational Quantity, or Surd, is that of which the value cannot be accurately expressed in numbers, as the square root of 2, 3, 5. Surds are commonly expressed by means of the radical sign $\sqrt{\ }$: as $\sqrt{2}$, or $\sqrt{a}$, or $\sqrt[3]{a^2}$, or $ab^{\frac{1}{2}}$.

20. The Reciprocal of any quantity, is that quantity inverted, or unity divided by it. So, the reciprocal of a, or $\frac{a}{1}$, is $\frac{1}{a}$, the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$, that of $\frac{a}{x+y}$ is $\frac{x+y}{a}$.

21. The letters by which any simple quantity is expressed, may be ranged according to any order at pleasure. So the product of a and b, may be either expressed by ab, or ba;

and the product of a, b, and c, by either abc, or acb, or bac, or bca, or cab, or cba; as it matters not which quantities are placed or multiplied first. But it will be sometimes found convenient in long operations, to place the several letters according to their order in the alphabet, as abc, which order also occurs most easily or naturally to the mind.

22. Likewise, the several members, or terms, of which a compound quantity is composed, may be disposed in any order at pleasure, without altering the value of the signification of the whole. Thus, $3a-2ab+4abc$ may also be written $3a+4abc-2ab$, or $4abc+3a-2ab$, or $-2ab+3a+4abc$, &c.; for all these represent the same thing, namely, the quantity which remains, when the quantity or term $2ab$ is subtracted from the sum of the terms or quantities $3a$ and $4abc$. But it is most usual and natural, to begin with a positive term, and with the first letters of the alphabet.

SOME EXAMPLES FOR PRACTICE.

In finding the numeral values of various expressions, or combinations, of quantities.

Supposing $a=6$, and $b=5$, and $c=4$, and $d=1$, and $e=0$. Then

1. Will $a^2+3ab-c^2=36+90-16=110$.
2. And $2a^3-3a^2b+c^3=432-540+64=-44$.
3. And $a^2\times(a+b)-2abc=36\times 11-240=156$.
4. And $\dfrac{a^3}{a+3c}+c^2=\dfrac{216}{18}+16=12+16=28$.
5. And $\sqrt{2ac+c^2}$ or $(2ac+c^2)^{\frac{1}{2}}=\sqrt{64}=8$.
6. And $\sqrt{c}+\dfrac{2bc}{\sqrt{(2ac+c^2)}}=2+\dfrac{40}{8}=7$.
7. And $\dfrac{a^2-\sqrt{(b^2-ac)}}{2a-\sqrt{(b^2+ac)}}=\dfrac{36-1}{12-7}=\dfrac{35}{5}=7$.
8. And $\sqrt{(b^2-ac)}+\sqrt{(2ac+c^2)}=1+8=9$.
9. And $\sqrt{b^2-ac+\sqrt{(2ac+c^2)}}=\sqrt{(25-24+8)}=3$.
10. And $a^2b+c-d=183$.
11. And $9ab-10b^2+c=24$.
12. And $\dfrac{a^2b}{c}\times d=45$.
13. And $\dfrac{a+b}{c}\times\dfrac{b}{d}=13\frac{3}{4}$.

14. And $\frac{a+b}{c} - \frac{a-b}{d} = 1\frac{3}{4}$.

15. And $\frac{a^2b}{c} + e = 45$.

16. And $\frac{a^2b}{c} \times e = 0$.

17. And $(b-c) \times (d-e) = 1$.

18. And $(a+b) - (c-d) = 8$.

19. And $(a+b) - c - d = 6$.

20. And $a^2c \times d^3 = 144$.

21. And $acd - d = 23$.

22. And $a^2e + b^2e + d^2 = 1$.

23. And $\frac{b-e}{d-e} \times \frac{a+b}{c-d} = 18\frac{1}{3}$.

24. And $\sqrt{a^2+b^2} - \sqrt{a^2-b^2} = 4{\cdot}4936249$.

25. And $3ac^2 + \sqrt[3]{a^3-b^3} = 292{\cdot}497942$.

26. And $4a^2 - 3a\sqrt{a^2 - \frac{2}{3}ab} = 72$.

ADDITION.

ADDITION, in Algebra, is the connecting the quantities together by their proper signs, and incorporating or uniting into one term or sum, such as are similar, and can be united. As $3a + 2b - 2a = a + 2b$, the sum.

The rule of addition in algebra, may be divided into three cases: one, when the quantities are like, and their signs like also; a second, when the quantities are like, but their signs unlike; and the third, when the quantities are unlike. Which was performed as follows*.

* The reasons on which these operations are founded, will readily appear, by a little reflection on the nature of the quantities to be added, or collected together. For, with regard to the first example, where the quantities are $3a$ and $5a$, whatever a represents in the one term, it will represent the same thing in the other; so that 3 times any thing and 5 times the same thing, collected together, must needs make 8 times that thing. As if a denote a shilling; then $3a$ is 3 shillings, and $5a$ is 5 shillings, and their sum 8 shillings. In like manner, $-2ab$ and $-7ab$, or -2 times any thing, and -7 times the same thing, make -9 times that thing.

CASE I.

When the Quantities are Like, and have Like Signs.

ADD the co-efficients together, and set down the sum; after which set the common letter or letters of the like quantities, and prefix the common sign + or —.

Thus, $3a$ added to $5a$, makes $8a$.
And $-2ab$ added to $-7ab$, makes $-9ab$.
And $5a+7b$ added to $7a+3b$, makes $12a+10b$.

OTHER EXAMPLES FOR PRACTICE.

$3a$	$-3bx$	bxy
$9a$	$-5bx$	$2bxy$
$5a$	$-4bx$	$5bxy$
$12a$	$-2bx$	bxy
a	$-7bx$	$3bxy$
$2a$	$-bx$	$6bxy$
$32a$	$-22bx$	$18bxy$

As to the second case, in which the quantities are like, but the signs unlike; the reason of its operation will easily appear, by reflecting, that addition means only the uniting of quantities together by means of the arithmetical operations denoted by their signs + and —, or of addition and subtraction; which being of contrary or opposite natures, the one co-efficient must be subtracted from the other, to obtain the incorporated or united mass.

As to the third case, where the quantities are unlike, it is plain that such quantities cannot be united into one, or otherwise added, than by means of their signs: thus, for example, if a be supposed to represent a crown, and b a shilling; then the sum of a and b can be neither $2a$ nor $2b$, that is, neither 2 crowns nor 2 shillings, but only 1 crown plus 1 shilling, that is $a+b$.

In this rule, the word *addition* is not very properly used; being much too limited to express the operation here performed. The business of this operation is to incorporate into one mass, or algebraic expression, different algebraic quantities, as far as an actual incorporation or union is possible; and to retain the algebraic marks for doing it, in cases where the former is not possible. When we have several quantities, some affirmative and some negative; and the relation of these quantities can in the whole or in part be discovered; such incorporation of two or more quantities into one, is plainly effected by the foregoing rules.

It may seem a paradox, that what is called addition in algebra, should sometimes mean addition, and sometimes subtraction. But the paradox wholly arises from the scantiness of the name given to the algebraic process; from employing an old term in a new and more enlarged sense. Instead of addition, call it *incorporation*, or *union*, or *striking a balance*, or give it any name to which a more extensive idea may be annexed, than that which is usually implied by the word addition: and the paradox vanishes.

$3z$	$3x^2+5xy$	$2ax-4y$
$2z$	$x^2+\ xy$	$4ax-\ y$
$4z$	$2x^2+4xy$	$ax-3y$
z	$5x^2+2xy$	$5ax-5y$
$5z$	$4x^2+3xy$	$7ax-2y$
$15z$	$15x^2+15xy$	$19ax-15y$

$5xy$	$-12y^2$	$4a-4b$
$14xy$	$-\ 7y^2$	$5a-5b$
$22xy$	$-\ 2y^2$	$6a-\ b$
$17xy$	$-\ 4y^2$	$3a-2b$
$1\frac{1}{2}xy$	$-\ y^2$	$2a-7b$
$\frac{1}{2}xy$	$-\ 3y^2$	$8a-\ b$

$30-13x^{\frac{1}{2}}-3xy$	$5xy-3x+4ab$
$23-10x^{\frac{1}{2}}-4xy$	$8xy-4x+3ab$
$14-14x^{\frac{1}{2}}-7xy$	$3xy-5x+5ab$
$10-16x^{\frac{1}{2}}-5xy$	$xy-2x+\ ab$
$16-20x^{\frac{1}{2}}-\ xy$	$4xy-\ x+7ab$

CASE II.

When the Quantities are alike, but have Unlike Signs.

Add all the affirmative co-efficients into one sum, and all the negative ones into another, when there are several of a kind. Then subtract the less sum, or the less co-efficient, from the greater, and to the remainder prefix the sign of the greater, and subjoin the common quantity or letters.

So $+5a$ and $-3a$, united, make $+2a$.
And $-5a$ and $-3a$, united, make $-2a$.

OTHER EXAMPLES FOR PRACTICE.

$$\begin{array}{rcr}
-5a & +3ax^2 & +8x^3+3y \\
+4a & +4ax^2 & -5x^3+4y \\
+6a & -8ax^2 & -16x^3+5y \\
-3a & -6ax^2 & +3x^3-7y \\
+a & +5ax^2 & +2x^3-2y \\
\hline
+3a & -2ax^2 & -8x^3+3y \\
\hline
\end{array}$$

$$\begin{array}{rrr}
-3a^2 & +3b^2y^3 & +4ab+4 \\
-5a^2 & +9b^2y^3 & -4ab+12 \\
-10a^2 & -10b^2y^3 & +7ab-14 \\
+10a^2 & -19b^2y^3 & +ab+8 \\
+14a^2 & -2b^2y^3 & -5ab-10 \\
\hline
 & & \\
\hline
\end{array}$$

$$\begin{array}{rrr}
-3ax^{\frac{1}{2}} & +10\sqrt{ax} & +3y+4ax^{\frac{1}{2}} \\
+ax^{\frac{1}{2}} & -3\sqrt{ax} & -y-5ax^{\frac{1}{2}} \\
+5ax^{\frac{1}{2}} & +4\sqrt{ax} & +4y+2ax^{\frac{1}{2}} \\
-6ax^{\frac{1}{2}} & -12\sqrt{ax} & -2y+6ax^{\frac{1}{2}} \\
\hline
 & & \\
\hline
\end{array}$$

CASE III.

When the Quantities are Unlike.

HAVING collected together all the like quantities, as in the two foregoing cases, set down those that are unlike, one after another, with their proper signs.

EXAMPLES.

$$\begin{array}{rcl}
3xy & 6xy-12x^2 & 4ax-130+3x^{\frac{1}{2}} \\
2ax & -4x^2+3xy & 5x^2+3ax+9x^2 \\
-5xy & +4x^2-2xy & 7xy-4x^{\frac{1}{2}}+90 \\
6ax & -3xy+4x^2 & \sqrt{x}+40-6x^2 \\
\hline
-2xy+8ax & 4xy-8x^2 & 7ax+8x^2+7xy \\
\hline
\end{array}$$

$9x^2y^2$	$14ax - 2x^2$	$9 + 10\sqrt{ax} - 5y$
$-7x^2y$	$5ax + 3xy$	$2x + 7\sqrt{xy} + 5y$
$+3axy$	$8y^2 - 4ax$	$5y + 3\sqrt{ax} - 4y$
$-4x^2y$	$3x^2 + 26$	$10 - 4\sqrt{ax} + 4y$

$4x^2y$	$4\sqrt{x} - 3y$	$3a^2 + 9 + x^{\frac{1}{2}} - 4$
$-6xy^2$	$2\sqrt{xy} + 14x$	$2a - 8 + 2a^2 - 3x$
$+3y^2x$	$3x + 2y$	$3x^2 - 2a^2 + 18 - 7$
$-7x^2y$	$-9 + 3\sqrt{xy}$	$-12 + a - 3x^2 - 2y$

Add $a+b$ and $3a-5b$ together.
Add $5a-8x$ and $3a-4x$ together.
Add $6x-5b+a+8$ to $-5a-4x+4b-3$.
Add $a+2b-3c-10$ to $3b-4a+5c+10$ and $5b-c$.
Add $a+b$ and $a-b$ together.
Add $8a+b-10$ to $c-d-a$ and $-4c+2a-3b-7$.
Add $3a^2+b^2-c$ to $2ab-3a^2+bc-b$.
Add $a^3+b^2c-b^2$ to $ab^2-abc+b^2$.
Add $9a-8b+10x-6d-7c+50$ to $2x-3a-5c+4b+6d-10$.

SUBTRACTION.

Set down in one line the first quantities from which the subtraction is to be made; and underneath them place all the other quantities composing the subtrahend; ranging the like quantities under each other, as in Addition.

Then change all the signs (+ and −) of the lower line, or conceive them to be changed; after which, collect all the terms together as in the cases of Addition*.

* This rule is founded on the consideration, that addition and subtraction are opposite to each other in their nature and operation, as are the signs + and −, by which they are expressed and represented. So that, since to unite a negative quantity with a positive one of the same kind, has the effect of diminishing it, or subducting an equal positive

EXAMPLES.

From	$7a^2-3b$	$9x^2-4y+8$	$8xy-3+6x-y$
Take	$2a^2-8b$	$6x^2+5y-4$	$4xy-7-6x-4y$
Rem.	$4a^2+5b$	$3x^2-9y+12$	$4xy+4+12x+3y$

From	$5xy-6$	$4y^2-3y-4$	$-20-6x-5xy$
Take	$-2xy+6$	$2y^2+2y+4$	$3xy-9x\times 8-2ay$
Rem.	$7xy-12$	$2y^2-5y-8$	$-28+3x-8xy+2ay$

From	$8x^2y+6$	$5\sqrt{xy}+2x\sqrt{xy}$	$7x^2+2\sqrt{x}-18+3b$
Take	$-2x^2y+2$	$7\sqrt{xy}+3-2xy$	$9x^2-12+5b+x^{\frac{1}{2}}$
Rem.			

$5xy-30$	$7x^3-2(a+b)$	$3xy^3+20a\sqrt{(xy+10)}$
$7xy-50$	$2x^3-4(a+b)$	$4x^2y^3+12a\sqrt{(xy+10)}$

From $a+b$, take $a-b$.
From $4a+4b$, take $b+a$.
From $4a-4b$, take $3a+5b$.
From $8a-12x$, take $4a-3x$.
From $2x-4a-2b+5$, take $8-5b+a+6x$.
From $3a+b+c-d-10$, take $c+2a-d$.
From $3a+b+c-d-10$, take $b-10+3a$.
From $2ab+b^2-4c+bc-b$, take $3a^2-c+b^2$.
From $a^3+3b^2c+ab^2-abc$, take b^2+ab^2-abc.
From $12x+6a-4b+40$, take $4b-3a+4x+6d-10$.
From $2x-3a+4b+6c-50$, take $9a+x+6b-6c-40$.
From $6a-4b-12c+12x$, take $2a-8a+4b-5c$.

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one from it, therefore to subtract a positive (which is the opposite of uniting or adding) is to add the equal negative quantity. In like manner, to subtract a negative quantity, is the same in effect as to add or unite an equal positive one. So that, changing the sign of a quantity from + to —, or from — to +, changes its nature from a subductive quantity to an additive one; and any quantity is in effect subtracted, by barely changing its sign.

MULTIPLICATION.

This consists of several cases, according as the factors are simple or compound quantities.

CASE I. *When both the Factors are Simple Quantities.*

First multiply the co-efficients of the two terms together, then to the product annex all the letters in those terms, which will give the whole product required.

*Note**. Like signs, in the factors, produce +, and unlike signs —, in the products.

EXAMPLES.

$10a$	$-2a$	$7a$	$-6x$
$2b$	$+2b$	$-4c$	$-4a$
$20ab$	$-6ab$	$-28ac$	$+24ax$

* That this rule for the signs is true, may be thus shown.

1. When $+a$ is to be multiplied by $+c$; the meaning is, that $+a$ is to be taken as many times as there are units in c; and since the sum of any number of positive terms is positive, it follows that $+a\times+c$ makes $+ac$.

2. When two quantities are to be multiplied together, the result will be exactly the same, in whatever order they are placed; for a times c is the same as c times a, and therefore, when $-a$ is to be multiplied by $+c$, or $+c$ by $-a$: this is the same thing as taking $-a$ as many times as there are units in $+c$; and as the sum of any number of negative terms is negative, it follows that $-a\times+c$, or $+a\times-c$ make or produce $-ac$.

3. When $-a$ is to be multiplied by $-c$: here $-a$ is to be subtracted as often as there are units in c: but subtracting negatives is the same thing as adding affirmatives, by the demonstration of the rule for subtraction; consequently the product is c times a, or $+ac$.

Otherwise. Since $a-a=0$, therefore $(a-a)\times-c$ is also $=0$, because 0 multiplied by any quantity, is still but 0; and since the first term of the product, or $a\times-c$ is $=-ac$, by the second case; therefore the last term of the product, or $-a\times-c$, must be $+ac$, to make the sum $=0$, or $-ac+ac=0$; that is, $-a\times-c=+ac$.

Other demonstrations upon the principles of proportion, or by means of geometrical diagrams, have also been given; but the above may suffice.

$4ac$	$9a^2x$	$-2x^2y$	$-4xy$
$-3ab$	$4x$	$3xy^2$	$-xy$
$-12a^2bc$	$36a^2x^2$	$-6x^3y^3$	$+4x^2y^2$

$-3ax$	$-ax$	$+3xy$	$-5xyz$
$4x$	$-6c$	-4	$-5ax$

CASE II.

When one of the Factors is a Compound Quantity.

Multiply every term of the multiplicand, or compound quantity, separately, by the multiplier, as in the former case; placing the products one after another, with the proper signs; and the result will be the whole product required.

EXAMPLES.

$5a - 3c$	$3ac - 4b$	$2a^2 - 3c + 5$
$2a$	$3a$	bc
$10a^2 - 6ac$	$9a^2c - 12ab$	$2a^2bc - 3bc^2 + 5bc$

$12x - 2ac$	$25c - 7b$	$4x - b + 3ab$
$4a$	$-2a$	$2ab$

$3c^2 + x$	$10x^2 - 3y^2$	$3a^2 - 2x^2 - 6b$
$4xy$	$-4x^2$	$2ax^2$

CASE III.

When both the Factors are Compound Quantities:

Multiply every term of the multiplier by every term of the multiplicand separately; setting down the products one after or under another, with their proper signs; and add the several lines of products all together for the whole product required.

$$\begin{array}{l} a+b \\ a+b \\ \hline a^2+ab \\ \quad +ab+b^2 \\ \hline a^2+2ab+b^2 \\ \hline \end{array} \qquad \begin{array}{l} 3x+2y \\ 4x-5y \\ \hline 12x^2+8xy \\ \qquad -15xy-10y^2 \\ \hline 12x^2-7xy-10y^2 \\ \hline \end{array} \qquad \begin{array}{l} 2x^2+xy-2y^2 \\ 3x-3y \\ \hline 6x^3+3x^2y-6xy^2 \\ \quad -6x^2y-3xy^2+6y^3 \\ \hline 6x^3-3x^2y-9xy^2+6y^3 \\ \hline \end{array}$$

$$\begin{array}{l} a+b \\ a-b \\ \hline a^2+ab \\ \quad -ab-b^2 \\ \hline a^2 \quad * \quad -b^2 \\ \hline \end{array} \qquad \begin{array}{l} x^2+y \\ x^2+y \\ \hline x^4+yx^2 \\ \quad +yx^2+y^2 \\ \hline x^4+2yx^2+y^2 \\ \hline \end{array} \qquad \begin{array}{l} a^2+ab+b^2 \\ a-b \\ \hline a^3+a^2b+ab^2 \\ \quad -a^2b-ab^2-b^3 \\ \hline a^3 \quad * \quad * \quad -b^3 \\ \hline \end{array}$$

Note. In the multiplication of compound quantities, it is the best way to set them down in order, according to the powers and the letters of the alphabet. And in the actual operation, begin at the left-hand side, and multiply from the left hand towards the right, in the manner that we write, which is contrary to the way of multiplying numbers. But in setting down the several products, as they arise, in the second and following lines, range them under the like terms in the lines above, when there are such like quantities; which is the easiest way for adding them up together.

In many cases, the multiplication of compound quantities is only to be performed by setting them down one after another, each within or under a vinculum, with a sign of multiplication between them. As $(a+b) \times (a-b) \times 3ab$, or $\overline{a+b} \,.\, \overline{a-b} \,.\, 3ab$.

EXAMPLES FOR PRACTICE.

1. Multiply $10ac$ by $2a$. Ans. $20a^2c$.
2. Multiply $3a^2 - 2b$ by $3b$. Ans. $9a^2b - 6b^2$.
3. Multiply $3a + 2b$ by $3a - 2b$. Ans. $9a^2 - 4b^2$.
4. Multiply $x^2 - xy + y^2$ by $x + y$. Ans. $x^3 + y^3$.
5. Multiply $a^3 + a^2b + ab^2 + b^3$ by $a - b$. Ans. $a^4 - b^4$.
6. Multiply $a^2 + ab + b^2$ by $a^2 - ab + b^2$.
7. Multiply $3x^2 - 2xy + 5$ by $x^2 + 2xy - 6$.
8. Multiply $3a^2 - 2ax + 5x^2$ by $3a^2 - 4ax - 7x^2$.
9. Multiply $3x^3 + 2x^2y^2 + 3y^3$ by $2x^3 - 3x^2y^2 + 3y^3$.
10. Multiply $a^2 + ab + b^2$ by $a - 2b$.

DIVISION.

DIVISION in Algebra, like that in numbers, is the converse of multiplication; and it is performed like that of numbers also, by beginning at the left-hand side, and dividing all the parts of the dividend by the divisor, when they can be so divided; or else by setting them down like a fraction, the dividend over the divisor, and then abbreviating the fraction as much as can be done. This may naturally be distinguished into the following particular cases.

CASE I.

When the Divisor and Dividend are both Simple Quantities:

SET the terms both down as in division of numbers, either the divisor before the dividend, or below it, like the denominator of a fraction. Then abbreviate these terms as much as can be done, by cancelling or striking out all the letters that are common to them both, and also dividing the one co-efficient by the other, or abbreviating them after the manner of a fraction, by dividing them by their common measure.

Note. Like signs in the two factors make $+$ in the quotient; and unlike signs make $-$; the same as in multiplication*.

---

* Because the divisor multiplied by the quotient, must produce the dividend. Therefore,

EXAMPLES.

1. To divide $6ab$ by $3a$.

Here $6ab \div 3a$, or $3a$) $6ab$ (or $\frac{6ab}{3a} = 2b$.

2. Also $c \div c = \frac{c}{c} = 1$; and $abx \div bxy = \frac{abx}{bxy} = \frac{a}{y}$.

3. Divide $16x^2$ by $8x$. Ans. $2x$.

4. Divide $12a^2x^2$ by $-3a^2x$. Ans. $-4x$.

5. Divide $-15ay^2$ by $3ay$. Ans. $-5y$.

6. Divide $-18ax^2y$ by $-8axz$. Ans. $\frac{9xy}{4z}$.

CASE. II.

When the Dividend is a Compound Quantity, and the Divisor a Simple one.

Divide every term of the dividend by the divisor, as in the former case.

EXAMPLES.

1. $(ab + b^2) \div 2b$, or $\frac{ab+b^2}{2b} = \frac{a+b}{2} = \frac{1}{2}a + \frac{1}{2}b$.

2. $(10ab + 15ax) \div 5a$, or $\frac{10ab+15ax}{5a} = 2b + 3x$.

3. $(30az - 48z) \div z$, or $\frac{30az-48z}{z} = 30a - 48$.

4. Divide $6ab - 8ax + a$ by $2a$.

5. Divide $3x^2 - 15 + 6x + 6a$ by $3x$.

---

1. When both the terms are $+$, the quotient must be $+$; because $+$ in the divisor $\times$ $+$ in the quotient, produces $+$ in the dividend.
2. When the terms are both $-$, the quotient is also $+$; because $-$ in the divisor $\times$ $-$ in the quotient, produces $+$ in the dividend.
3. When one term is $+$ and the other $-$, the quotient must be $-$; because $+$ in the divisor $\times$ $-$ in the quotient produces $-$ in the dividend, or $-$ in the divisor $\times$ $+$ in the quotient gives $-$ in the dividend.

So that the rule is general; viz. that like signs give $+$, and unlike signs give $-$, in the quotient.

6. Divide $6abc + 12abx - 9a^2b$ by $3ab$.
7. Divide $10a^2x - 15x^2 - 25x$ by $5x$.
8. Divide $15a^2bc - 15acx^2 + 5ad^2$ by $-5ac$.
9. Divide $15a + 3ay - 18y^2$ by $21a$.
10. Divide $-20a^2b^2 + 60ab^3$ by $-6ab$.

CASE III.

When the Divisor and Dividend are both Compound Quantities.

1. Set them down as in common division of numbers, the divisor before the dividend, with a small curved line between them, and range the terms according to the powers of some one of the letters in both, the higher powers before the lower.

2. Divide the first term of the dividend by the first term of the divisor, as in the first case, and set the result in the quotient.

3. Multiply the whole divisor by the term thus found, and subtract the result from the dividend.

4. To this remainder bring down as many terms of the dividend as are requisite for the next operation, dividing as before; and so on to the end, as in common arithmetic.

Note. If the divisor be not exactly contained in the dividend, the quantity which remains after the operation is finished may be placed over the divisor, like a vulgar fraction, and set down at the end of the quotient as in arithmetic.

EXAMPLES.

$$a - b)\ a^2 - 2ab + b^2\ (a - b$$
$$a^2 - ab$$
$$\overline{\quad - ab + b^2}$$
$$- ab + b^2$$
$$\overline{\qquad\qquad}$$

$$
\begin{array}{l}
a - c)\ a^3 - 4a^2c + 4ac^2 - c^3 - (a^2 - 3ac + c^2 \\
\quad a^3 - a^2c \\
\quad\overline{\quad - 3a^2c + 4ac^2} \\
\quad\quad - 3a^2c + 3ac^2 \\
\quad\quad\overline{\quad\quad ac^2 - c^3} \\
\quad\quad\quad ac^2 - c^3
\end{array}
$$

$$
\begin{array}{l}
a - 2)\ a^3 - 6a^2 + 12a - 8\ (a^2 - 4a + 4 \\
\quad a^3 - 2a^2 \\
\quad\overline{\quad - 4a^2 + 12a} \\
\quad\quad - 4a^2 + 8a \\
\quad\quad\overline{\quad\quad 4a - 8} \\
\quad\quad\quad 4a - 8
\end{array}
$$

$$
\begin{array}{l}
a + x)\ a^4 - 3x^4\ (a^3 - a^2x + ax^2 - x^3 - \dfrac{2x^4}{a + x} \\
\quad a^4 + a^3x \\
\quad\overline{\quad - a^3x - 3x^4} \\
\quad\quad - a^3x - a^2x^2 \\
\quad\quad\overline{\quad\quad a^2x^2 - 3x^4} \\
\quad\quad\quad a^2x^2 + ax^3 \\
\quad\quad\quad\overline{\quad - ax^3 - 3x^4} \\
\quad\quad\quad\quad - ax^3 - x^4 \\
\quad\quad\quad\quad\overline{\quad\quad - 2x^4}
\end{array}
$$

EXAMPLES FOR PRACTICE.

1. Divide $a^2 + 4ax + 4x^2$ by $a + 2x$. Ans. $a + 2x$.
2. Divide $a^3 - 3a^2z + 3az^2 - z^3$ by $a - z$. Ans. $a^2 - 2az + z^2$.
3. Divide 1 by $1 + a$. Ans. $1 - a + a^2 - a^3 +$ &c.
4. Divide $12x^4 - 192$ by $3x - 6$. Ans. $4x^3 + 8x^2 + 16x + 32$.
5. Divide $a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$ by $a^2 - 2ab + b^2$. Ans. $a^3 - 3a^2b + 3ab^2 - b^3$.

6. Divide $48x^3 - 96ax^2 - 64a^2x + 150a^3$ by $2x - 3a$.

7. Divide $b^6 - 3b^4x^2 + 3b^2x^4 - x^6$ by $b^3 - 3b^2x + 3bx^2 - x^3$.

8. Divide $a^7 - x^7$ by $a - x$.

9. Divide $a^3 + 5a\,x + 5ax^2 + x^3$ by $a + x$.

10. Divide $a^4 + 4a^2b^2 - 32b^4$ by $a + 2b$.

11. Divide $24a^4 - b^4$ by $3a - 2b$.

ALGEBRAIC FRACTIONS.

Algebraic Fractions have the same names and rules of operation, as numeral fractions in common arithmetic; as appears in the following Rules and Cases.

CASE I.

To reduce a Mixed Quantity to an Improper Fraction.

Multiply the integer by the denominator of the fraction, and to the product add the numerator, or connect it with its proper sign, + or —; then the denominator being set under this sum, will give the improper fraction required.

EXAMPLES.

1. Reduce $3\frac{4}{5}$, and $a - \frac{b}{x}$ to improper fractions.

First, $3\frac{4}{5} = \frac{(3 \times 5) + 4}{5} = \frac{15 + 4}{5} = \frac{19}{5}$ the Answer.

And, $a - \frac{b}{x} = \frac{(a \times x) - b}{x} = \frac{ax - b}{x}$ the Answer.

2. Reduce $a + \frac{a^2}{b}$ and $a - \frac{x^2 - a^2}{a}$ to improper fractions.

First, $a + \frac{a^2}{b} = \frac{(a \times b) + a^2}{b} = \frac{ab + a^2}{b}$ the Answer.

And, $a - \frac{x^2 - a^2}{a} = \frac{a^2 - x^2 + a^2}{a} = \frac{2a^2 - x^2}{a}$ the Answer.

3. Reduce $5\frac{3}{7}$ to an improper fraction. Ans. $\frac{38}{7}$.

4. Reduce $1 - \frac{3a}{x}$ to an improper fraction. Ans. $\frac{x - 3a}{x}$.

5. Reduce $2a - \frac{3ax + a^2}{4x}$ to an improper fraction.

6. Reduce $12 + \frac{4x - 18}{5x}$ to an improper fraction.

7. Reduce $x + \frac{1 - 3a - c}{c}$ to an improper fraction.

8. Reduce $4 + 2x - \frac{2x^3 - 3a}{5a}$ to an improper fraction.

CASE II.

To reduce an Improper Fraction to a Whole or Mixed Quantity.

DIVIDE the numerator by the denominator, for the integral part; and set the remainder, if any, over the denominator, for the fractional part; the two joined together will be the mixed quantity required.

EXAMPLES.

1. To reduce $\frac{16}{3}$ and $\frac{ab + a^2}{b}$ to mixed quantities.

First, $\frac{16}{3} = 16 \div 3 = 5\frac{1}{3}$, the answer required.

And, $\frac{ab + a^2}{b} = (ab + a^2) \div b = a + \frac{a^2}{b}$. Answer.

2. To reduce $\frac{2ac - 3a^2}{c}$ and $\frac{3ax + 4x^2}{a + x}$ to mixed quantities.

First, $\frac{2ac - 3a^2}{c} = (2ac - 3a^2) \div c = 2a - \frac{3a^2}{c}$. Answer.

And, $\frac{3ax + 4x^2}{a + x} = (3ax + 4x^2) \div (a + x) = 3x + \frac{x^2}{a + x}$. Ans.

3. Reduce $\frac{33}{5}$ and $\frac{2ax - 3x^2}{a}$ to mixed quantities.

Ans. $6\frac{3}{5}$, and $2x - \frac{3x^2}{a}$.

4. Reduce $\frac{4a^2x}{2a}$ and $\frac{2a^2 + 2b}{a - b}$ to whole or mixed quantities.

5. Reduce $\frac{3x^2 - 3y^2}{x+y}$ and $\frac{2x^3 - 2y^3}{x-y}$ to whole or mixed quantities.

6. Reduce $\frac{10a^2 - 4a + 6}{5a}$ to a mixed quantity.

7. Reduce $\frac{15a^3 + 5a^2}{3a^3 + 2a^2 - 2a - 4}$ to a mixed quantity.

CASE III.

To reduce Fractions to a Common Denominator.

Multiply every numerator, separately, by all the denominators except its own, for the new numerators; and all the denominators together, for the common denominator.

When the denominators have a common divisor, it will be better, instead of multiplying by the whole denominators, to multiply only by those parts which arise from dividing by the common divisor. Observing also the several rules and directions, as in Fractions in the Arithmetic.

EXAMPLES.

1. Reduce $\frac{a}{x}$ and $\frac{b}{z}$ to a common denominator.

Here $\frac{a}{x}$ and $\frac{b}{z} = \frac{az}{xz}$ and $\frac{bx}{xz}$, by multiplying the terms of the first fraction by z, and the terms of the 2d by x.

2. Reduce $\frac{a}{x}$, $\frac{x}{b}$, and $\frac{b}{c}$ to a common denominator.

Here $\frac{a}{x}$, $\frac{x}{b}$, and $\frac{b}{c} = \frac{abc}{bcx}$, $\frac{cx^2}{bcx}$, and $\frac{b^2x}{bcx}$, by multiplying the terms of the 1st fraction by bc, of the 2d by cx, and of the 3d by bx.

3. Reduce $\frac{2a}{x}$ and $\frac{3b}{2c}$ to a common denominator.

Ans. $\frac{4ac}{2cx}$ and $\frac{3bx}{2cx}$.

4. Reduce $\frac{2a}{b}$ and $\frac{3a+2b}{2c}$ to a common denominator.

Ans. $\frac{4ac}{2bc}$, and $\frac{3ab+2b^2}{2bc}$.

5. Reduce $\frac{5a}{3x}$ and $\frac{3b}{2c}$, and $4d$, to a common denominator.

Ans. $\frac{10ac}{6cx}$ and $\frac{9bx}{6cx}$ and $\frac{24cdx}{6cx}$.

6. Reduce $\frac{5}{6}$ and $\frac{3a}{4}$ and $2b+\frac{3a}{b}$, to fractions having a common denominator. Ans. $\frac{20b}{24b}$ and $\frac{18ab}{24b}$, and $\frac{48b^2+72a}{24b}$.

7. Reduce $\frac{1}{3}$ and $\frac{2a^2}{4}$ and $\frac{2a^2+b^2}{a+b}$ to a common denominator.

8. Reduce $\frac{3b}{4a^2}$ and $\frac{2c}{3a}$ and $\frac{d}{2a}$ to a common denominator.

CASE IV.

To find the greatest common Measure of the Terms of a Fraction.

Divide the greater term by the less, and the last divisor by the last remainder, and so on till nothing remains; then the divisor last used will be the common measure required; just the same as in common numbers.

But note, that it is proper to range the quantities according to the dimensions of some letters, as is shown in division. Note also, that all the letters or figures which are common to each term of the divisors, must be thrown out of them, or must divide them, before they are used in the operation.

EXAMPLES.

1. To find the greatest common measure of $\frac{ab+b^2}{ac^2+bc^2}$.

$$ab+b^2)\ ac^2+bc^2$$
$$\text{or } a+b\ )\ ac^2+bc^2\ (c^2$$
$$ac^2+bc^2$$

Therefore the greatest common measure is $a+b$.

2. To find the greatest common measure of $\frac{a^3-ab^2}{a^2+2ab+b^2}$.

$$
\begin{array}{l}
a^2+2ab+b^2)\ a^3-ab^2\ (a \\
\qquad\qquad a^3+2a^2b+ab^2 \\
\hline
\qquad\qquad -2a^2b-2ab^2)\ a^2+2ab+b^2 \\
\text{or} \qquad a+b\ )\ a^2+2ab+b^2\ (a+b \\
\qquad\qquad\qquad a^2+ab \\
\hline
\qquad\qquad\qquad\qquad ab+b^2 \\
\qquad\qquad\qquad\qquad ab+b^2
\end{array}
$$

Therefore $a+b$ is the greatest common divisor.

3. To find the greatest common divisor of $\frac{a^2-4}{ab+2b}$.

Ans. $a-2$.

4. To find the greatest common divisor of $\frac{a^5-a^3b^2}{a^4-b^4}$.

Ans. a^2-b^2.

5. Find the greatest com. measure of $\frac{a^3x+2a^2x^2+2ax^3+x^4}{5a^5+10a^4x+5a^3x^2}$.

CASE V.

To reduce a Fraction to its lowest Terms.

FIND the greatest common measure, as in the last problem. Then divide both the terms of the fraction by the common measure thus found, and it will reduce it to its lowest terms at once, as was required. Or divide the terms by any quantity which it may appear will divide them both as in arithmetical fractions.

EXAMPLES.

1. Reduce $\frac{ab+b^2}{ac^2+bc^2}$ to its lowest terms.

$$
\begin{array}{l}
ab+b^2)\ ac^2+bc^2 \\
\text{or} \quad a+b\ )\ ac^2+bc^2\ (c^2 \\
\qquad\qquad ac^2+bc^2 \\
\hline
\end{array}
$$

Here $ab+b^2$ is divided by the common factor b.

Therefore $a+b$ is the greatest common measure, and hence $a+b)\frac{ab+b^2}{ac^2+bc^2}=\frac{b}{c^2}$, is the fraction required.

2. To reduce $\frac{c^3-b^2c}{c^2+2bc+b^2}$ to its least terms.

Here, by a process similar to that of Ex. 2, Case IV., we find $c+b$ is the greatest common measure, and hence

$c+b)\ \dfrac{c^3-b^2c}{c^2+2bc+b^2}=\dfrac{c^2-bc}{c+b}$ is the fraction required.

3. Reduce $\dfrac{c^3-b^3}{c^4-b^2c^2}$ to its lowest terms. Ans. $\dfrac{c^2+bc+b^2}{c^3+bc^2}$.

4. Reduce $\dfrac{a^2-b^2}{a-b^4}$ to its lowest terms. Ans. $\dfrac{1}{a^2+b^2}$.

5. Reduce $\dfrac{a^4-b^4}{a^3-3a^2b+3ab^2-b^3}$ to its lowest terms.

6. Reduce $\dfrac{3a^5+6a^4c+3a^3c^2}{a^3c+3a^2c^2+3ac^3+c^4}$ to its lowest terms.

CASE VI.

To add Fractional Quantities together.

If the fractions have a common denominator, add all the numerators together; then under their sum set the common denominator, and it is done.

If they have not a common denominator, reduce them to one, and then add them as before.

EXAMPLES.

1. Let $\dfrac{a}{3}$ and $\dfrac{a}{4}$ be given, to find their sum.

Here $\dfrac{a}{3}+\dfrac{a}{4}=\dfrac{4a}{12}+\dfrac{3a}{12}=\dfrac{7a}{12}$ is the sum required.

2. Given $\dfrac{a}{b}$, $\dfrac{b}{c}$, and $\dfrac{c}{d}$, to find their sum.

Here $\dfrac{a}{b}+\dfrac{b}{c}+\dfrac{c}{d}=\dfrac{acd}{bcd}+\dfrac{bbd}{bcd}+\dfrac{bcc}{bcd}=\dfrac{acd+bbd+bcc}{bcd}$ the sum required.

* 3. Let $a-\dfrac{3x^2}{b}$ and $b+\dfrac{2ax}{c}$ be added together.

---

* In the addition of mixed quantities, it is best to bring the fractional parts only to a common denominator, and to annex their sum to the sum of the integers, with the proper sign. And the same rule may be observed for mixed quantities in subtraction also.

See, also, the note to Addition of Fractions in the Arithmetic.

Here $a - \frac{3x^2}{b} + b + \frac{2ax}{c} = a - \frac{3cx^2}{bc} + b + \frac{2abx}{bc}$

$= a + b + \frac{2abx - 3cx^2}{bc}$, the sum required.

4. Add $\frac{4x}{3a}$ and $\frac{2x}{5b}$ together. Ans. $\frac{20bx+6ax}{15ab}$.

5. Add $\frac{a}{3}$, $\frac{a}{4}$ and $\frac{a}{5}$ together. Ans. $\frac{47}{60}a$.

6. Add $\frac{2a-3}{4}$ and $\frac{5a}{8}$ together. Ans. $\frac{9a-6}{8}$.

7. Add $2a + \frac{a+3}{5}$ to $4a + \frac{2a-5}{4}$. Ans. $6a + \frac{14a-13}{20}$.

8. Add $6a$, and $\frac{3a^2}{4b}$ and $\frac{a+b}{3b}$ together.

9. Add $\frac{5a}{4}$, and $\frac{6a}{5}$ and $\frac{3a+2}{7}$ together.

10. Add $2a$, and $\frac{3a}{8}$ and $3 + \frac{a}{6}$ together.

11. Add $8a + \frac{3a}{4}$ and $2a - \frac{5a}{8}$ together.

CASE VII.

To subtract one Fractional Quantity from another.

Reduce the fractions to a common denominator, as in addition, if they have not a common denominator.

Subtract the numerators from each other, and under their difference set the common denominator, and the work is done.

EXAMPLES.

1. To find the difference of $\frac{3a}{4}$ and $\frac{4a}{7}$.

Here $\frac{3a}{4} - \frac{4a}{7} = \frac{21a}{28} - \frac{16a}{28} = \frac{5a}{28}$ is the differ. required.

2. To find the difference of $\frac{2a-b}{4c}$ and $\frac{3a-4b}{3b}$.

Here $\frac{2a-b}{4c}-\frac{3a-4b}{3b}=\frac{6ab-3bb}{12bc}-\frac{12ac-16bc}{12bc}=$ $\frac{6ab-3bb-12ac+16bc}{12bc}$ is the difference required.

3. Required the difference of $\frac{10a}{9}$ and $\frac{4a}{7}$.

4. Required the difference of $6a$ and $\frac{3a}{4}$.

5. Required the difference of $\frac{5a}{4}$ and $\frac{2a}{3}$.

6. Subtract $\frac{2b}{c}$ from $\frac{3a+c}{b}$.

7. Take $\frac{2a+6}{9}$ from $\frac{4a+8}{5}$.

8. Take $2a-\frac{a-3b}{c}$ from $4a+\frac{2a}{c}$.

CASE VIII.

To multiply Fractional Quantities together.

MULTIPLY the numerators together for a new numerator, and the denominators for a new denominator *.

EXAMPLES.

1. Required to find the product of $\frac{a}{8}$ and $\frac{2a}{5}$.

Here $\frac{a\times 2a}{8\times 5}=\frac{2a^2}{40}=\frac{a^2}{20}$ the product required.

---

* 1. When the numerator of one fraction, and the denominator of the other, can be divided by some quantity, which is common to both, the quotients may be used instead of them.

2. When a fraction is to be multiplied by an integer, the product is found either by multiplying the numerator, or dividing the denominator by it; and if the integer be the same with the denominator, the numerator may be taken for the product.

2. Required the product of $\frac{a}{3}$, $\frac{3a}{4}$, and $\frac{6a}{7}$.

$\frac{a \times 3a \times 6a}{3 \times 4 \times 7} = \frac{18a^3}{84} = \frac{3a^3}{14}$ the product required.

3. Required the product of $\frac{2a}{b}$ and $\frac{a+b}{2a+c}$.

Here $\frac{2a \times (a+b)}{b \times (2a+c)} = \frac{2aa+2ab}{2ab+bc}$ the product required.

4. Required the product of $\frac{4a}{3}$ and $\frac{6a}{5c}$.

5. Required the product of $\frac{3a}{4}$ and $\frac{4b^2}{3a}$.

6. To multiply $\frac{3a}{b}$, and $\frac{8ac}{b}$, and $\frac{4ab}{3c}$ together.

7. Required the product of $2a + \frac{ab}{2c}$ and $\frac{3a^2}{b}$.

8. Required the product of $\frac{2a^2 - 2b^2}{3bc}$ and $\frac{4a^2 + 2b^2}{a+b}$.

9. Required the product of $3a$, and $\frac{2a+1}{a}$ and $\frac{2a-1}{2a+b}$.

10. Multiply $a + \frac{x}{2a} - \frac{x^2}{4a}$ by $x - \frac{a}{2x} + \frac{a^2}{4x^2}$.

CASE IX.

To divide one Fractional Quantity by another.

DIVIDE the numerators by each other, and the denominators by each other, if they will exactly divide. But, if not, then invert the terms of the divisor, and multiply by it exactly as in multiplication*.

---

* 1. If the fractions to be divided have a common denominator, take the numerator of the dividend for a new numerator, and the numerator of the divisor for the new denominator.

2. When a fraction is to be divided by any quantity, it is the same thing whether the numerator be divided by it, or the denominator multiplied by it.

3. When the two numerators, or the two denominators, can be divided by some common quantity, let that be done, and the quotients used instead of the fractions first proposed.

EXAMPLES.

1. Required to divide $\frac{a}{4}$ by $\frac{3a}{8}$.

Here $\frac{a}{4} \div \frac{3a}{8} = \frac{a}{4} \times \frac{8}{3a} = \frac{8a}{12a} = \frac{2}{3}$ the quotient.

2. Required to divide $\frac{3a}{2b}$ by $\frac{5c}{4d}$.

Here $\frac{3a}{2b} \div \frac{5c}{4d} = \frac{3a}{2b} \times \frac{4d}{5c} = \frac{12ad}{10bc} = \frac{6ad}{5bc}$, the quotient.

3. To divide $\frac{2a+b}{3a-2b}$ by $\frac{3a+2b}{4a+b}$. Here,

$\frac{2a+b}{3a-2b} \times \frac{4a+b}{3a+2b} = \frac{8a^2+6ab+b^2}{9a^2-4b^2}$ the quotient required.

4. To divide $\frac{3a^2}{a^3+b^3}$ by $\frac{a}{a+b}$.

Here $\frac{3a^2}{a^3+b^3} \times \frac{a+b}{a} = \frac{3a^2 \times (a+b)}{(a^3+b^3) \times a} = \frac{3a}{a^2-ab+b^2}$ is the quotient required.

5. To divide $\frac{3x}{4}$ by $\frac{11}{12}$.

6. To divide $\frac{6x^2}{5}$ by $3x$.

7. To divide $\frac{3x+1}{9}$ by $\frac{4x}{3}$.

8. To divide $\frac{4x}{2x-1}$ by $\frac{x}{3}$.

9. To divide $\frac{4x}{5}$ by $\frac{3a}{5b}$.

10. To divide $\frac{2a-b}{4cd}$ by $\frac{5ac}{6d}$.

11. Divide $\frac{5a^4-5b^4}{2a^2-4ab+2b^2}$ by $\frac{6a^2+5ab}{4a-4b}$.

INVOLUTION.

Involution is the raising of powers from any proposed root; such as finding the square, cube, biquadrate, &c. of any given quantity. The method is as follows.

* **Multiply** the root or given quantity by itself, as many times as there are units in the index less one, and the last product will be the power required. Or, in literals, multiply the index of the root by the index of the power, and the result will be the power, the same as before.

Note. When the sign of the root is +, all the powers of it will be +; but when the sign is —, all the even powers will be +, and all the odd powers —; as is evident from multiplication.

EXAMPLES.

a, the root	a^2, the root
a^2 = square	a^4 = square
a^3 = cube	a^6 = cube
a^4 = 4th power	a^8 = 4th power
a^5 = 5th power	a^{10} = 5th power
&c.	&c.
$-2a$, the root	$-3ab^2$, the root
$+4a^2$ = square	$+9a^2b^4$ = square
$-8a^3$ = cube	$-27a^3b^6$ = cube
$+16a^4$ = 4th power	$+81a^4b^8$ = 4th power
$-32a^5$ = 5th power	$-243a^5b^{10}$ = 5th power
$-\frac{2ax^2}{3b}$, the root	$\frac{a}{2b}$, the root
$+\frac{4a^2x^4}{9b^2}$ = square	$\frac{a^2}{4b^2}$ = square
$-\frac{8a^3x^6}{27b^3}$ = cube	$\frac{a^3}{8b^3}$ = cube
$+\frac{16a^4x^8}{81b^4}$ 4th power	$\frac{a^3}{16b^4}$ = biquadrate

* Any power of the product of two or more quantities, is equal to the same power of each of the factors, multiplied together.

And any power of a fraction, is equal to the same power of the numerator, divided by the like power of the denominator.

$$\begin{array}{ll}
x-a=\text{root} & \\
x-a & \\
\hline
x^2-ax & \\
\quad -ax+a^2 & \\
\hline
x^2-2ax+a^2 & \text{square} \\
x-a & \\
\hline
x^3-2ax^2+a^2x & \\
\quad -ax^2+2a^2x-a^3 & \\
\hline
x^3-3ax^2+3a^2x-a^3 & \\
\hline
\end{array}
\qquad\qquad
\begin{array}{l}
x+a=\text{root} \\
x+a \\
\hline
x^2+ax \\
\quad +ax+a^2 \\
\hline
x^2+2ax+a^2 \\
x+a \\
\hline
x^3+2ax^2+a^2x \\
\quad +ax^2+2a^2x+a^3 \\
\hline
x^3+3ax^2+3a^2x+a^3 \\
\hline
\end{array}$$

the cubes, or third powers, of $x-a$ and $x+a$.

EXAMPLES FOR PRACTICE.

1. Required the cube or 3d power or $3a^2$.
2. Required the 4th power of $2a^2b$.
3. Required the 3d power of $-4a^2b^2$.
4. To find the biquadrate of $-\frac{a^2x}{2b^2}$.
5. Required the 5th power of $a-2x$.
6. To find the 6th power of $2a^{\frac{1}{2}}$.

SIR ISAAC NEWTON'S RULE *for raising a Binomial to any Power whatever**.

1. *To find the Terms without the Co-efficients.* The index of the first, or leading quantity, begins with the index of the given power, and in the succeeding terms decreases continually by 1, in every term to the last; and in the 2d or

---

Also, powers or roots of the same quantity, are multiplied by one another, by adding their exponents; or divided, by subtracting their exponents.

Thus, $a^3 \times a^2 = a^{3+2} = a^5$. And $a^3 \div a^2$ or $\frac{a^3}{a^2} = a^{3-2} = a$.

* This rule, expressed in general terms, is as follows:

$$(a+x)^n = a^n + n\,.\,a^{n-1}x + n\,.\,\frac{n-1}{2}a^{n-2}x^2 + n\,.\,\frac{n-1}{2}\,.\,\frac{n-2}{2}a^{n-3}x^3 \text{ \&c.}$$

$$(a-x)^n = a^n - n\,.\,a^{n-1}x + n\,.\,\frac{n-1}{2}a^{n-2}x^2 - n\,.\,\frac{n-1}{2}\,.\,\frac{n-2}{3}a^{n-3}x^3 \text{ \&c.}$$

Note. The sum of the co-efficients, in every power, is equal to the following quantity, the indices of the terms are 0, 1, 2, 3, 4, &c. increasing always by 1. That is, the first term will contain only the 1st part of the root with the same index, or of the same height as the intended power: and the last term of the series will contain only the 2d part of the given root, when raised also to the same height of the intended power: but all the other or intermediate terms will contain the products of some powers of both the members of the root, in such sort, that the powers or indices of the 1st or leading member will always decrease by 1, while those of the 2d member always increase by 1.

2. *To find the Co-efficients.* The first co-efficient is always 1, and the second is the same as the index of the intended power; to find the 3d co-efficient, multiply that of the 2d term by the index of the leading letter in the same term, and divide the product by 2; and so on, that is, multiply the co-efficient of the term last found by the index of the leading quantity in that term, and divide the product by the number of terms to that place, and it will give the co-efficient of the term next following; which rule will find all the co-efficients, one after another.

Note. The whole number of terms will be 1 more than the index of the given power: and when both terms of the root are +, all the terms of the power will be +; but if the second term be —, all the odd terms will be +, and all the even terms —, which causes the terms to be + and — alternately. Also the sum of the two indices, in each term, is always the same number, viz. the index of the required power; and counting from the middle of the series, both ways, or towards the right and left, the indices of the two terms are the same figures at equal distances, but with mutually changed places. Moreover, the co-efficients are the same numbers at equal distances from the middle of the series, towards the right and left; so by whatever numbers they increase to the middle, by the same in the reverse order they decrease to the end.

---

number 2, when raised to that power. Thus $1+1=2$ in the first power; $1+2+1=4=2^2$ in the square; $1+3+3+1=8=2^3$ in the cube, or third power: and so on.

A trinomial or a quadrinomial may be expanded in the same manner. Thus, to raise $a-b+c-d$ to the 6th power, put $a-b=x$, $c-d=z$, and raise $x+z$ to the 6th power; after which substitute for the powers of x and y their corresponding values in terms of $a-b$, and $c-d$, and their powers respectively.

EXAMPLES.

1. Let $a + x$ be involved to the 5th power.

The terms without the co-efficients, by the 1st rule, will be

$$a^5,\ a^4x,\ a^3x^2,\ a^2x^3,\ ax^4,\ x^5,$$

and the co-efficients, by the 2d rule, will be

$$1,\ 5,\ \frac{5 \times 4}{2},\ \frac{10 \times 3}{3},\ \frac{10 \times 2}{4},\ \frac{5 \times 1}{5};$$

$$\text{or, } 1,\ 5,\ 10,\ 10,\ 5,\ 1;$$

Therefore the 5th power altogether is

$$a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + 5ax^4 + x^5.$$

But it is best to set down both the co-efficients and the powers of the letters at once, in one line, without the intermediate lines in the above example, as in the example here below. The operation is very easily effected by performing the division first.

2. Let $a - x$ be involved to the 6th power.

The terms with the co-efficients will be

$$a^6 - 6a^5x + 15a^4x^2 - 20a^3x^3 + 15a^2x^4 - 6ax^5 + x^6.$$

3. Required the 4th power of $a - x$.

Ans. $a^4 - 4a^3x + 6a^2x^2 - 4ax^3 + x^4$.

And thus any other powers may be set down at once, in the same manner, which is the best way.

4. Involve $a - z$ to the ninth power; $x - y$ to the tenth power, and $a + b - c$ to the fourth power.

EVOLUTION.

Evolution is the reverse of Involution, being the method of finding the square root, cube root, &c. of any given quantity, whether simple or compound.

CASE I. *To find the Roots of Simple Quantities.*

Extract the root of the co-efficient, for the numeral part; and divide the index of the letter or letters, by the index of the power, and it will give the root of the literal part; then annex this to the former, for the whole root sought*.

EXAMPLES.

1. The square root of $4a^2$, is $2a$.
2. The cube root of $8a^3$, is $2a^{\frac{3}{3}}$ or $2a$.
3. The square root of $\frac{5a^2b^2}{9c^2}$, or $\sqrt{\frac{5a^2b^2}{9c^2}}$, is $\frac{ab}{3c}\sqrt{5}$.
4. The cube root of $-\frac{16a^4b^6}{27c^3}$, is $-\frac{2ab^2}{3c}\sqrt[3]{2a}$.
5. To find the square root of $2a^2b^4$. Ans. $ab^2\sqrt{2}$.
6. To find the cube root of $-64a^3b^6$. Ans. $-4ab^2$.
7. To find the square root of $\frac{8a^2b^2}{3c^3}$. Ans. $\frac{2ab}{c}\sqrt{\frac{2}{3c}}$.
8. To find the 4th root of $81a^4b^6$. Ans. $3ab\sqrt{b}$.
9. To find the 5th root of $-32a^5b^6$. Ans. $-2ab\sqrt[5]{b}$.

CASE II.

To find the square root of a Compound Quantity.

This is performed like as in numbers, thus:

1. Range the quantities according to the dimensions of one of the letters, and set the root of the first term in the quotient.

2. Subtract the square of the root, thus found, from the first term, and bring down the next two terms to the remainder for a dividend; and take double the root for a divisor.

---

* Any even root of an affirmative quantity, may be either + or −: thus the square root of $+a^2$ is either $+a$, or $-a$; because $+a \times +a = +a^2$, and $-a \times -a = +a^2$ also.

But an odd root of any quantity will have the same sign as the quantity itself: thus the cube root of $+a^3$ is $+a$, and the cube root of $-a^3$ is $-a$; for $+a \times +a \times +a = +a^3$, and $-a \times -a \times -a = -a^3$.

Any even root of a negative quantity is impossible; for neither $+a \times +a$, nor $-a \times -a$ can produce $-a^2$.

Any root of a product is equal to the like root of each of the factors multiplied together. For the root of a fraction, take the root of the numerator, and the root of the denominator.

3. Divide the dividend by the divisor, and annex the result both to the quotient and to the divisor.

4. Multiply the divisor, thus increased, by the term last set in the quotient, and subtract the product from the dividend.

And so on, always the same, as in common arithmetic.

EXAMPLES.

1. Extract the square root of $a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$.

$$\begin{array}{l}
a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4 \;(a^2 - 2ab + b^2 \text{ the root.} \\
\hline
2a^2 - 2ab)\; -4a^3b + 6a^2b^2 \\
\qquad\qquad\quad -4a^3b + 4a^2b^2 \\
\hline
2a^2 - 4ab + b^2)\; 2a^2b^2 - 4ab^3 + b^4 \\
\qquad\qquad\qquad\quad 2a^2b^2 - 4ab^3 + b^4 \\
\hline
\end{array}$$

2. Find the root of $a^4 + 4a^3b + 10a^2b^2 + 12ab^3 + 9b^4$.

$$\begin{array}{l}
a^4 + 4a^3b + 10a^2b^2 + 12ab^3 + 9b^4 \;(a^2 + 2ab + 3b^2. \\
a^4. \\
\hline
2a^2 + 2ab)\; 4a^3b + 10a^2b^2 \\
\qquad\qquad\quad 4a^3b + 4a^2b^2 \\
\hline
2a^2 + 4ab + 3b^2)\; 6a^2b^2 + 12ab^3 + 9b^4 \\
\qquad\qquad\qquad\quad 6a^2b^2 + 12ab^3 + 9b^4 \\
\hline
\end{array}$$

3. To find the square root of $a^4 + 4a^3 + 6a^2 + 4a + 1$.
Ans. $a^2 + 2a + 1$.

4. Extract the square root of $a^4 - 2a^3 + 2a^2 - a + \frac{1}{4}$.
Ans. $a^2 - a + \frac{1}{2}$.

5. It is required to find the square root of $a^2 - ab$.

Ans. $a - \frac{b}{2} - \frac{b^2}{8a} - \frac{b^3}{16a^2} -$ &c.

CASE III.

To find the Roots of any Powers in general.

This is also done like the same roots in numbers, thus:

Find the root of the first term, and set it in the quotient. —Subtract its power from that term, and bring down the second term for a dividend.—Involve the root, last found, to the next lower power, and multiply it by the index of the given power, for a divisor.—Divide the dividend by the divisor, and set the quotient as the next term of the root.—Involve now the whole root to the power to be extracted; then subtract the power thus arising from the given power, and divide the first term of the remainder by the divisor first found; and so on till the whole is finished *.

EXAMPLES.

1. To find the square root of $a^4-2a^3b+3a^2b^2-2ab^3+b^4$.

$a^4-2a^3b+3a^2b^2-2ab^3+b^4\ (a^2-ab+b^2$.
a^4

$2a^2)-2a^3b$

$a^4-2a^3b+a^2b^2=(a^2-ab)^2$

$2a^2)\quad 2a^2b^2$

$a^4-2a^3b+3a^2b^2-2ab^3+b^4=(a^2-ab+b^2)^2$.

2. Find the cube root of $a^6-6a^5+21a^4-44a^3+63a^2-54a+27$.

$a^6-6a^5+21a^4-44a^3+63a^2-54a+27\ (a^2-2a+3$.

a^6

$3a^4)-6a^5$

$a^6-6a^5+12a^4-8a^3=(a^2-2a)^3$

$3a^4)+9a^4$

$a^6-6a^5+21a^4-44a^3+63a^2-54a+27=(a^2-2a+3)^3$.

* As this method, in high powers, may be thought too laborious, it will not be improper to observe, that the roots of compound quantities may sometimes be easily discovered, thus:

Extract the roots of some of the most simple terms, and connect them together by the sign + or —, as may be judged most suitable for the purpose.—Involve the compound root, thus found, to the proper power; then, if this be the same with the given quantity, it is the root required.—But if it be found to differ only in some of the signs, change them from + to —, or from — to +, till its power agrees with the given one throughout.

Thus, in the 5th example, the root $3a-2b$, is the difference of the roots of the first and last terms; and in the 3d example, the root $a-b+x$, is the sum of the roots of the 1st, 4th, and 6th terms. The same may also be observed of the 6th example, where the root is found from the first and last terms.

3. To find the square root of $a^2 - 2ab + 2ax + b^2 - 2bx + x^2$. Ans. $a - b + x$.

4. Find the cube root of $a^6 - 3a^5 + 9a^4 - 13a^3 + 18a^2 - 12a + 8$. Ans. $a^2 - a + 2$.

5. Find the 4th root of $81a^4 - 216a^3b + 216a^2b^2 - 96ab. + 16b^4$. Ans. $3a - 2b$.

6. Find the 5th root of $a^5 - 10a^4 + 40a^3 - 80a^2 + 80a - 32$. Ans. $a - 2$.

7. Required the square root of $1 - x^2$.

8. Required the cube root of $1 - x^3$.

SURDS.

Surds are such quantities as have no exact root; and are usually expressed by fractional indices, or by means of the radical sign $\sqrt{}$. Thus, $3^{\frac{1}{2}}$, or $\sqrt{3}$, denotes the square root of 3; and $2^{\frac{2}{3}}$, or $\sqrt[3]{2^2}$, or $\sqrt[3]{4}$, the cube root of the square of 2; where the numerator shows the power to which the quantity is to be raised, and the denominator its root.

PROBLEM I.

To reduce a Rational Quantity to the Form of a Surd.

Raise the given quantity to the power denoted by the index of the surd; then over or above the new quantity set the radical sign, and it will be of the form required.

EXAMPLES.

1. To reduce 4 to the form of the square root.
First, $4^2 = 4 \times 4 = 16$; then $\sqrt{16}$ is the answer.

2. To reduce $3a^2$ to the form of the cube root.
First $3a^2 \times 3a^2 = \times 3a^2 = (3a^2)^3 = 27a^6$;

then $\sqrt[3]{27a^6}$ or $(27a^6)^{\frac{1}{3}}$ is the answer.

3. Reduce 6 to the form of the cube root.

Ans. $(216)^{\frac{1}{3}}$ or $\sqrt[3]{216}$.

4. Reduce $\frac{1}{2}ab$ to the form of the square root.

Ans. $\sqrt{\frac{1}{4}a^2b^2}$.

5. Reduce 2 to the form of the 4th root. Ans. $(16)^{\frac{1}{4}}$.

6. Reduce $a^{\frac{1}{3}}$ to the form of the 5th root.

7. Reduce $a + x$ to the form of the square root.

8. Reduce $a - x$ to the form of the cube root.

PROBLEM II.

To reduce Quantities to a Common Index.

1. REDUCE the indices of the given quantities to a common denominator, and involve each of them to the power denoted by its numerator; then 1 set over the common denominator will form the common index. Or,

2. If the common index be given, divide the indices of the quantities by the given index, and the quotients will be the new indices for those quantities. Then over the said quantities, with their new indices, set the given index, and they will make the equivalent quantities sought.

EXAMPLES.

1. Reduce $3^{\frac{1}{2}}$ and $5^{\frac{1}{5}}$ to a common index.

Here $\frac{1}{2}$ and $\frac{1}{5} = \frac{5}{10}$ and $\frac{2}{10}$.

Therefore $3^{\frac{5}{10}}$ and $5^{\frac{2}{10}} = (3^5)^{\frac{1}{10}}$ and $(5^2)^{\frac{1}{10}} = \sqrt[10]{3^5}$ and $\sqrt[10]{5^2}$ $= \sqrt[10]{243}$ and $\sqrt[10]{25}$.

2. Reduce a^3 and $b^{\frac{1}{3}}$ to the same common index $\frac{1}{2}$.

Here, $\frac{3}{1} \div \frac{1}{2} = \frac{3}{1} \times \frac{2}{1} = \frac{6}{1}$ the 1st index,

and $\frac{1}{3} \div \frac{1}{2} = \frac{1}{3} \times \frac{2}{1} = \frac{2}{3}$ the 2d index.

Therefore $(a^6)^{\frac{1}{2}}$ and $(b^{\frac{2}{3}})^{\frac{1}{2}}$, or $\sqrt{a^6}$ and $\sqrt{b^{\frac{2}{3}}}$ are the quantities.

3. Reduce $4^{\frac{1}{3}}$ and $5^{\frac{1}{2}}$ to the common index $\frac{1}{4}$.

Ans. $(256^{\frac{1}{3}})^{\frac{1}{4}}$ and $25^{\frac{1}{4}}$.

4. Reduce $a^{\frac{1}{3}}$ and $x^{\frac{1}{4}}$ to the common index $\frac{1}{6}$.

Ans. $(a^2)^{\frac{1}{6}}$ and $(x^{\frac{3}{2}})^{\frac{1}{6}}$.

5. Reduce a^2 and x^3 to the same radical sign.

Ans. $\sqrt{a^4}$ and $\sqrt{x^6}$.

6. Reduce $(a + x)^{\frac{1}{2}}$ and $(a - x)^{\frac{1}{3}}$ to a common index.

7. Reduce $(a + b)^{\frac{1}{2}}$ and $(a - b)^{\frac{1}{4}}$ to a common index.

PROBLEM III.

To reduce Surds to more Simple Terms.

Divide the surd, if possible, into two factors, one of which is a power of the kind that accords with the root sought; as a complete square, if it be a square root, a complete cube, if it be a cube root; and so on. Set the root of this complete power before the surd expression which indicates the root of the other factor; and the quantity is reduced, as required.

If the surd be a fraction, the reduction is effected by multiplying both its numerator and denominator by some number that will transform the denominator into a complete square, cube, &c. its root will be the denominator to a fraction that will stand before the remaining part, or surd. See Example 3, below.

EXAMPLES.

1. To reduce $\sqrt{32}$ to simpler terms.

Here $\sqrt{32} = \sqrt{(16 \times 2)} = \sqrt{16} \times \sqrt{2} = 4 \times \sqrt{2} = 4\sqrt{2}$.

2. To reduce $\sqrt[3]{320}$ to simpler terms.

$\sqrt[3]{320} = \sqrt[3]{(64 \times 5)} = \sqrt[3]{64} \times \sqrt[3]{5} = 4 \times \sqrt[3]{5} = 4\sqrt[3]{5}$.

3. Reduce $\sqrt{\frac{44}{75}}$ to simpler terms.

$$\sqrt{\frac{44}{75}} = \sqrt{\frac{44}{15.3}} = \sqrt{\left(\frac{44}{15.3}.\frac{5}{5}\right)} = \sqrt{\frac{4.11.5}{15.15}} = \sqrt{\frac{2^2.55}{15^2}} = \frac{2}{15}\sqrt{55}.$$

4. Reduce $\sqrt{75}$ to its simplest terms. Ans. $5\sqrt{3}$.
5. Reduce $\sqrt[3]{189}$ to its simplest terms. Ans. $3\sqrt[3]{7}$.
6. Reduce $\sqrt[3]{\frac{135}{32}}$ to its simplest terms. Ans. $\frac{3}{4}\sqrt[3]{10}$.
7. Reduce $\sqrt{75a^2b}$ to its simplest terms. Ans. $5a\sqrt{3b}$.

Note. There are other cases of reducing algebraic surds to simpler forms, that are practised on several occasions; one of which, on account of its simplicity and usefulness, may be here noticed, viz. in fractional forms having compound surds in the denominator, multiply both numerator and denominator by the same terms of the denominator, but having one sign changed, from + to — or from — to +, which will reduce the fraction to a rational denominator.

Ex. To reduce $\frac{\sqrt{20}+\sqrt{12}}{\sqrt{5}-\sqrt{3}}$, multiply it by $\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}$, and it becomes $\frac{16+4\sqrt{15}}{2} = 8 + 2\sqrt{15}$.

Also, to reduce $\frac{3\sqrt{15}-4\sqrt{5}}{\sqrt{15}+\sqrt{5}}$; multiply it by $\frac{\sqrt{15}-\sqrt{5}}{\sqrt{15}-\sqrt{5}}$, and it becomes $\frac{65-7\sqrt{75}}{15-5}=\frac{65-35\sqrt{3}}{10}=\frac{13-7\sqrt{3}}{2}$.

And the same method may easily be applied to examples with three or more surds.

PROBLEM IV.

To add Surd Quantities together.

1. BRING all fractions to a common denominator, and reduce the quantities to their simplest terms, as in the last problem.—2. Reduce also such quantities as have unlike indices to other equivalent ones, having a common index.—3. Then if the surd part be the same in them all, annex it to the sum of the rational parts, with the sign of multiplication, and it will give the total sum required.

But if the surd part be not the same in all the quantities, they can only be added by the signs + and −.

EXAMPLES.

1. Required to add $\sqrt{18}$ and $\sqrt{32}$ together.

First, $\sqrt{18}=\sqrt{(9\times2)}=3\sqrt{2}$; and $\sqrt{32}=\sqrt{(16\times2)}=4\sqrt{2}$: Then, $3\sqrt{2}+4\sqrt{2}=(3+4)\sqrt{2}=7\sqrt{2}=$ sum required.

2. It is required to add $\sqrt[3]{375}$, and $\sqrt[3]{192}$ together.

First, $\sqrt[3]{375}=\sqrt[3]{(125\times3)}=5\sqrt[3]{3}$; and $\sqrt[3]{192}=\sqrt[3]{(64\times3)}=4\sqrt[3]{3}$: Then, $5\sqrt[3]{3}+4\sqrt[3]{3}=(5+4)\sqrt[3]{3}=9\sqrt[3]{3}=$ sum required.

3. Required the sum of $\sqrt{27}$ and $\sqrt{48}$. Ans. $7\sqrt{3}$.
4. Required the sum of $\sqrt{50}$ and $\sqrt{72}$. Ans. $11\sqrt{2}$.
5. Required the sum of $\sqrt{\frac{3}{5}}$ and $\sqrt{\frac{1}{15}}$. Ans. $\frac{4}{15}\sqrt{15}$.
6. Required the sum of $\sqrt[3]{56}$ and $\sqrt[3]{189}$. Ans. $5\sqrt[3]{7}$.
7. Required the sum of $\sqrt[3]{\frac{1}{4}}$ and $\sqrt[3]{\frac{1}{32}}$ Ans. $\frac{3}{4}\sqrt[3]{2}$.
8. Required the sum of $3\sqrt{a^2b}$ and $5\sqrt{16a^4b}$.

PROBLEM V.

To find the Difference of Surd Quantities.

PREPARE the quantities the same way as in the last rule; then subtract the rational parts, and to the remainder annex the common surd, for the difference of the surds required.

But if the quantities have no common surd, they can only be subtracted by means of the sign $-$.

EXAMPLES.

1. To find the difference between $\sqrt{320}$ and $\sqrt{80}$.

First, $\sqrt{320}=\sqrt{(64\times5)}=8\sqrt{5}$; and $\sqrt{80}=\sqrt{(16\times5)}=4\sqrt{5}$.
Then, $8\sqrt{5}-4\sqrt{5}=4\sqrt{5}$ the difference sought.

2. To find the difference between $\sqrt[3]{128}$ and $\sqrt[3]{54}$.

First, $\sqrt[3]{128}=\sqrt[3]{(64\times2)}=4\sqrt[3]{2}$; and $\sqrt[3]{54}=\sqrt[3]{(27\times2)}=3\sqrt[3]{2}$.
Then, $4\sqrt[3]{2}-3\sqrt[3]{2}=\sqrt[3]{2}$, the difference required.

3. Required the difference of $\sqrt{75}$ and $\sqrt{48}$. Ans. $\sqrt{3}$.
4. Required the difference of $\sqrt[3]{256}$ and $\sqrt[3]{32}$. Ans $2\sqrt[3]{4}$.
5. Required the difference of $\sqrt{\frac{3}{4}}$ and $\sqrt{\frac{1}{3}}$. Ans. $\frac{1}{6}\sqrt{3}$.
6. Find the difference of $\sqrt{\frac{2}{3}}$ and $\sqrt{\frac{3}{8}}$. Ans. $\frac{1}{12}\sqrt{6}$.
7. Required the difference of $\sqrt[3]{\frac{3}{5}}$ and $\sqrt[3]{\frac{25}{9}}$. Ans. $\frac{2}{15}\sqrt[3]{75}$.
8. Find the difference of $\sqrt{24a^2b^2}$ and $\sqrt{54b^4}$.
Ans. $\sqrt{(3b^2-2ab)}\sqrt{6}$.

PROBLEM VI.

To multiply Surd Quantities together.

Reduce the surds to the same index, if necessary; next multiply the rational quantities together, and the surds together; then annex the one product to the other for the whole product required; which may be reduced to more simple terms if necessary.

EXAMPLES.

1. Required to find the product of $4\sqrt{12}$ and $3\sqrt{2}$.

Here, $4\times3\times\sqrt{12}\times\sqrt{2}=12\sqrt{(12\times2)}=12\sqrt{24}=12\sqrt{(4\times6)}$
$=12\times2\times\sqrt{6}=24\sqrt{6}$, the product required.

2. Required to multiply $\frac{1}{4}\sqrt[3]{\frac{3}{4}}$ by $\frac{1}{3}\sqrt[3]{\frac{3}{8}}$.

Here $\frac{1}{4}\times\frac{1}{3}\sqrt[3]{\frac{3}{4}}\times\sqrt[3]{\frac{3}{8}}=\frac{1}{12}\times\sqrt[3]{\frac{9}{32}}=\frac{1}{12}\times\sqrt[3]{\frac{18}{64}}=\frac{1}{12}\times\frac{1}{4}\times\sqrt[3]{18}$
$=\frac{1}{48}\sqrt[3]{18}$, the product required.

3. Required the product of $3\sqrt{2}$ and $2\sqrt{8}$. Ans. 24.
4. Required the product of $\frac{1}{3}\sqrt[3]{4}$ and $\frac{3}{4}\sqrt[3]{12}$. Ans. $\frac{1}{2}\sqrt[3]{6}$.
5. To find the product of $\frac{5}{9}\sqrt{\frac{3}{8}}$ and $\frac{2}{15}\sqrt{\frac{4}{5}}$. Ans. $\frac{3}{56}\sqrt{15}$.
6. Required the product of $2\sqrt[3]{14}$ and $3\sqrt[3]{4}$. Ans. $12\sqrt[3]{7}$.
7. Required the product of $2a^{\frac{2}{3}}$ and $a^{\frac{4}{3}}$. Ans. $2a^2$.
8. Required the product of $(a+b)^{\frac{1}{2}}$ and $(a+b)^{\frac{3}{4}}$.

9. Required the product of $2x+\sqrt{b}$ and $2x-\sqrt{b}$.

10. Required the product of $(a+2\sqrt{b})^{\frac{1}{3}}$, and $(a-2\sqrt{b})^{\frac{1}{3}}$.

11. Required the product of $2x^{\frac{1}{n}}$ and $3x^{\frac{1}{n}}$.

12. Required the product of $4x^{\frac{1}{n}}$ and $2y^{\frac{1}{n}}$.

PROBLEM VII.

To divide one Surd Quantity by another.

Reduce the surds to the same index, if necessary; then take the quotient of the rational quantities, and annex it to the quotient of the surds, and it will give the whole quotient required; which may be reduced to more simple terms if requisite.

EXAMPLES.

1. Required to divide $6\sqrt{96}$ by $3\sqrt{8}$.
Here $6 \div 3 . \sqrt{(96 \div 8)} = 2\sqrt{12} = 2\sqrt{(4\times3)} = 2\times2\sqrt{3}$
$= 4\sqrt{3}$, the quotient required.

2. Required to divide $12\sqrt[3]{280}$ by $3\sqrt[3]{5}$.
Here $12 \div 3 = 4$, and $280 \div 5 = 56 = 8\times7 = 2^3 . 7$;
Therefore $4 \times 2 \times\sqrt[3]{7} = 8\sqrt[3]{7}$, is the quotient required.

3. Let $4\sqrt{50}$ be divided by $2\sqrt{5}$. Ans. $2\sqrt{10}$.

4. Let $6\sqrt[3]{100}$ be divided $3\sqrt[3]{5}$. Ans: $2\sqrt[3]{20}$.

5 Let $\frac{4}{5}\sqrt{\frac{1}{36}}$ be divided by $\frac{3}{4}\sqrt{\frac{2}{5}}$. Ans. $\frac{1}{3}\sqrt{5}$.

6. Let $\frac{3}{4}\sqrt[3]{\frac{3}{16}}$ be divided by $\frac{2}{3}\sqrt[3]{\frac{2}{5}}$. Ans. $\frac{4}{15}\sqrt[3]{30}$.

7. Let $\frac{4}{5}\sqrt{a}$, or $\frac{4}{5}a^{\frac{1}{2}}$, be divided by $\frac{2}{3}a^{\frac{1}{3}}$. Ans. $\frac{6}{5}a^{\frac{1}{6}}$.

8. Let $a^{\frac{4}{5}}$ be divided by $a^{\frac{2}{3}}$.

9. To divide $3a^{\frac{1}{n}}$ by $4a^{\frac{1}{m}}$.

PROBLEM VIII.

To involve or raise Surd Quantities to any Power.

Raise both the rational part and the surd part. Or multiply the index of the quantity by the index of the power to which it is to be raised, and to the result annex the power of the rational parts, which will give the power required.

EXAMPLES.

1. Required to find the square of $\frac{3}{4}a^{\frac{1}{2}}$.

First, $(\frac{3}{4})^2 = \frac{3}{4} \times \frac{3}{4} = \frac{9}{16}$, and $(a^{\frac{1}{2}})^2 = a^{\frac{1}{2} \times 2} = a^{\frac{2}{2}} = a$.

Therefore, $(\frac{3}{4}a^{\frac{1}{2}})^2 = \frac{9}{16}a$, is the square required.

2. Required to find the square of $\frac{1}{2}a^{\frac{2}{3}}$.

First, $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$, and $(a^{\frac{2}{3}})^2 = a^{\frac{4}{3}} = a\sqrt[3]{a}$;

Therefore $(\frac{1}{2}a^{\frac{2}{3}})^2 = \frac{1}{4}a\sqrt[3]{a}$ is the square required.

3. Required to find the cube of $\frac{2}{3}\sqrt{6}$ or $\frac{2}{3} \times 6^{\frac{1}{2}}$.

First, $(\frac{2}{3})^3 = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{8}{27}$, and $(6^{\frac{1}{2}})^3 = 6^{\frac{3}{2}} = 6\sqrt{6}$;
Theref. $(\frac{2}{3}\sqrt{6})^3 = \frac{8}{27} \times 6\sqrt{6} = \frac{16}{9}\sqrt{6}$, the cube required.

4. Required the square of $2\sqrt[3]{2}$. Ans. $4\sqrt[3]{4}$.

5. Required the cube of $3^{\frac{1}{2}}$, or $\sqrt{3}$. Ans. $3\sqrt{3}$.

6. Required the 3d power of $\frac{1}{3}\sqrt{3}$. Ans. $\frac{1}{9}\sqrt{3}$.

7. Required to find the 4th power of $\frac{1}{2}\sqrt{2}$. Ans. $\frac{1}{4}$.

8. Required to find the mth power of $a^{\frac{1}{n}}$.

9. Required to find the square of $2 + \sqrt{3}$.

PROBLEM IX.

To evolve or extract the Roots of Surd Quantities.*

EXTRACT both the rational part and the surd part. Or divide the index of the given quantity by the index of the

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* The square root of a binomial or residual surd, $a + b$, or $a - b$, may be found thus: Take $\sqrt{a^2 - b^2} = c$;

then $\sqrt{a+b} = \sqrt{\frac{a+c}{2}} + \sqrt{\frac{a+c}{2}}$;

and $\sqrt{a-b} = \sqrt{\frac{a+c}{2}} - \sqrt{\frac{a-c}{2}}$.

Thus, the square root of $4 + 2\sqrt{3} = 1 + \sqrt{3}$;
and the square root of $6 - 2\sqrt{5} = \sqrt{5} - 1$.
But for the cube, or any higher root, no general rule is known.

For more on the subject of Surds, see *Bonnycastle's Algebra*, the 8vo. edition, and the *Elementary Treatise of Algebra*, by Mr. *J. R. Young*.

root to be extracted; then to the result annex the root of the rational part, which will give the root required.

EXAMPLES.

1. Required to find the square root of $16\sqrt{6}$.

First, $\sqrt{16} = 4$, and $(6^{\frac{1}{2}})^{\frac{1}{2}} = 6^{\frac{1}{2} \div 2} = 6^{\frac{1}{4}}$;

theref. $(16\sqrt{6})^{\frac{1}{2}} = 4 \cdot 6^{\frac{1}{4}} = 4\sqrt[4]{6}$, is the sq. root required.

2. Required to find the cube root of $\frac{1}{27}\sqrt{3}$.

First, $\sqrt[3]{\frac{1}{27}} = \frac{1}{3}$, and $(\sqrt{3})^{\frac{1}{3}} = 3^{\frac{1}{2} \div 3} = 3^{\frac{1}{6}}$;

theref. $(\frac{1}{27}\sqrt{3})^{\frac{1}{3}} = \frac{1}{3} \cdot 3^{\frac{1}{6}} = \frac{1}{3}\sqrt[6]{3}$, is the cube root required.

3. Required the square root of 6^3. Ans. $6\sqrt{6}$.

4. Required the cube root of $\frac{1}{8}a^3b$. Ans. $\frac{1}{2}a\sqrt[3]{b}$.

5. Required the 4th root of $16a^2$. Ans. $2\sqrt{a}$.

6. Required to find the nth root of $x^{\frac{1}{n}}$.

7. Required the square root of $a^2 - 6a\sqrt{b} + 9b$.

ARITHMETICAL PROPORTION AND PROGRESSION.

Arithmetical Proportion is the relation which two quantities, of the same kind, bear to each other, in respect to their difference.

Four quantities are said to be in Arithmetical Proportion, when the difference between the first and second is equal to the difference between the third and fourth.

Thus, 3, 7, 12, 16, and a, $a + b$, c, $c + b$, are arithmetically proportional.

Arithmetical Progression is when a series of quantities either increase or decrease by the same common difference.

Thus, 1, 3, 5, 7, 9, 11, &c. and a, $a + b$, $a + 2b$, $a + 3b$, $a + 4b$, $a + 5b$, &c. are series in arithmetical progression, whose common differences are 2 and b.

The most useful part of arithmetical proportion and progression has been exhibited in the Arithmetic. The same may be given algebraically, thus:

Let a denote the least term,
z the greatest term,
d the common difference,
n the number of the terms,
and s the sum of the series;
then the principal properties are expressed by these equations, viz.

1. $z = a + d.(n-1)$
2. $a = z - d.(n-1)$
3. $s = (a+z)\frac{1}{2}n,$
4. $s = (z - \frac{1}{2}d . \overline{n-1})n,$
5. $s = (a + \frac{1}{2}d . \overline{n-1})n.$

Moreover, when the first term a is 0 or nothing, the theorems become $z = d\,(n-1)$
and $s = \frac{1}{2}zn.$

EXAMPLES FOR PRACTICE.

1. The first term of an increasing arithmetical series is 1, the common difference 2, and the number of terms 21; required the sum of the series?

First, $1 + 2 \times 20 = 1 + 40 = 41$, is the last term.

Then $\frac{1+41}{2} \times 20 = 21 \times 20 = 420$, the sum required.

2. The first term of a decreasing arithmetical series is 199, the common difference 3, and the number of terms 67; required the sum of the series?

First, $199 - 3 . 66 = 199 - 198 = 1$, is the last term.

Then $\frac{199+1}{2} \times 67 = 100 \times 67 = 6700$, the sum required.

3. To find the sum of 100 terms of the natural numbers 1, 2, 3, 4, 5, 6, &c. And. 5050.

4. * Required the sum of 99 terms of the odd numbers 1, 3, 5, 7, 9, &c. Ans. 9801.

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* The sum of any number (n) of terms of the arithmetical series of odd numbers 1, 3, 5, 7, 9, &c. is equal to the square (n^2) of that number. That is,

If 1, 3, 5, 7, 9, &c. be the numbers, then will
$1^2, 2^2, 3^2, 4^2, 5^2,$ be the sums of 1, 2 3, &c. terms,
Thus, $0 + 1 = 1$ or 1^2, the sum of 1 term,
$1 + 3 = 4$ or 2^2, the sum of 2 terms,
$4 + 5 = 9$ or 3^2, the sum of 3 terms,
$9 + 7 = 16$ or 4^2, the sum of 4 terms, &c.

5. The first term of a decreasing arithmetical series is 10, the common difference $\frac{1}{3}$, and the number of terms 21; required the sum of the series? Ans. 140.

6. One hundred stones being placed on the ground, in a straight line, at the distance of 2 yards from each other; how far will a person travel, who shall bring them one by one to a basket, which is placed 2 yards from the first stone? Ans. 11 miles and 840 yards.

APPLICATION OF ARITHMETICAL PROGRESSION.

Qu. I. A Triangular Battalion * consists of thirty ranks, in which the first rank is formed of one man only, the second of 3; the 3d of 5; and so on: What is the strength of such a triangular battalion? Answer, 900 men.

Qu. II. A detachment having 12 successive days to march, with orders to advance the first day only 2 leagues, the second $3\frac{1}{3}$, and so on, increasing $1\frac{1}{3}$ league each day's march: What is the length of the whole march, and what is the last day's march?

Answer, the last day's march is $18\frac{2}{3}$ leagues, and 128 leagues is the length of the whole march.

Qu. III. A brigade of sappers†, having carried on 15 yards of sap the first night, the second only 13 yards, and

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For, by the 1st theorem, $1 + 2(n - 1) = 1 + 2n - 2 = 2n - 1$ is the last term, when the number of terms is n; to this last term $2n - 1$, add the first term 1, gives $2n$ the sum of the extremes, or n half the sum of the extremes; then, by the 3d theorem, $n \times n = n^2$ is the sum of all the terms. Hence it appears, in general, that half the sum of the extremes is always the same as the number of the terms, n; and that the sum of all the terms is the same as the square of the same number, n^2.

See more on Arithmetical Proportion in the Arithmetic.

* By triangular battalion, is to be understood, a body of troops ranged in the form of a triangle, in which the ranks exceed each other by an equal number of men: if the first rank consist of one man only, and the difference between the ranks be also 1, then its form is that of an equilateral triangle; and when the difference between the ranks is more than 1, its form may then be an isosceles or scalene triangle. The practice of forming troops in this order, which is now laid aside, was formerly held in greater esteem than forming them in a solid square, as admitting of a greater front, especially when the troops were to make simply a stand on all sides.

† A brigade of sappers consists generally of 8 men, divided equally into two parties. While one of these parties is advancing the sap, the other is furnishing the gabions, fascines, and other necessary implements:

so on, decreasing 2 yards every night, till at last they carried on in one night only 3 yards: What is the number of nights they were employed; and what is the whole length of the sap.

Answer, they were employed 7 nights, and the length of the whole sap was 63 yards.

Qu. iv. A number of gabions* being given to be placed in six ranks, one above the other, in such a manner as that each rank exceeding one another equally, the first may consist of 4 gabions, and the last of 9: What is the number of gabions in the six ranks; and what is the difference between each rank?

Answer, the difference between the ranks will be 1, and the number of gabions in the six ranks will be 39.

Qu. v. Two detachments, distant from each other 37 leagues, and both designing to occupy an advantageous post equi-distant from each other's camp, set out at different times; the first detachment increasing every day's march 1 league and a half, and the second detachment increasing each day's march 2 leagues: both the detachments arrive at the same time; the first after 5 days' march, and the second after 4 days' march: What is the number of leagues marched by each detachment each day?

The progression $\frac{7}{10}$, $2\frac{2}{10}$, $3\frac{7}{10}$, $5\frac{2}{10}$, $6\frac{7}{10}$, answers the conditions of the first detachment: and the progression $1\frac{5}{8}$, $3\frac{5}{8}$, $5\frac{5}{8}$, $7\frac{5}{8}$, answers the condition of the second detachment.

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and when the first party is tired, the second takes its place, and so on, till each man in turn has been at the head of the sap. A sap is a small ditch, between 3 and 4 feet in breadth and depth; and is distinguished from the trench by its breadth only, the trench having between 10 and 15 feet breadth. As an encouragement to sappers, the pay for all the work carried on by the whole brigade is given to the survivors.

* Gabions are baskets, open at both ends, made of ozier twigs, and of a cylindrical form; those made use of at the trenches are 2 feet wide, and about 3 feet high; which, being filled with earth, serve as a shelter from the enemy's fire: and those made use of to construct batteries, are generally higher and broader. There is another sort of gabion, made use of to raise a low parapet: its height is from 1 to 2 feet, and 1 foot wide at top, but somewhat less at bottom, to give room for placing the muzzle of a firelock between them: these gabions serve instead of sand bags. A sand bag is generally made to contain about a cubic foot of earth.

OF COMPUTING SHOT OR SHELLS IN A FINISHED PILE.

SHOT and Shells are generally piled in three different forms, called triangular, square, or oblong piles, according as their base is either a triangle, a square, or a rectangle.

Fig. 1.

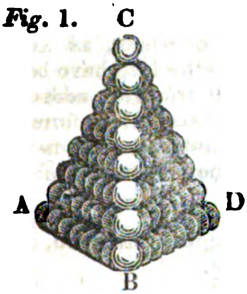
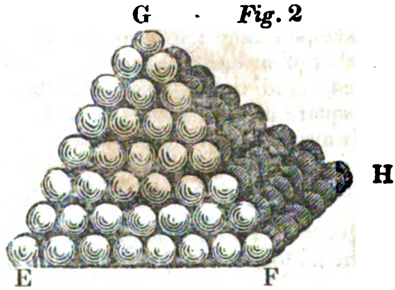
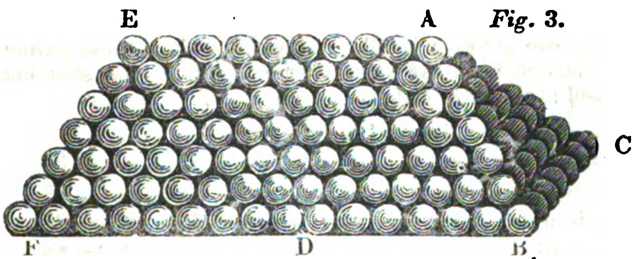


Fig. 2



ABCD, fig. 1, is a triangular pile.
EFGH, fig. 2. is a square pile.

Fig. 3.



ABCDEF, fig. 3, is an oblong pile.

A triangular pile is formed by the continual laying of triangular horizontal courses of shot one above another, in such a manner, as that the sides of these courses, called rows, decrease by unity from the bottom row to the top row, which ends always in 1 shot.

A square pile is formed by the continual laying of square horizontal courses of shot one above another, in such a manner, as that the sides of these courses decrease by unity from the bottom to the top row, which ends also in 1 shot.

In the triangular and the square piles, the sides or faces being equilateral triangles, the shot contained in those faces form an arithmetical progression, having for first term unity, and for last term and number of terms, the shot contained in the bottom row ; for the number of horizontal rows, or the number counted on one of the angles from the bottom to the top, is always equal to those counted on one side in the bottom : the sides or faces in either the triangular or square piles, are called arithmetical triangles ; and the numbers contained in these, are called triangular numbers : ABC, fig. 1, EFG, fig. 2, are arithmetical triangles.

The oblong pile may be conceived as formed from the square pile ABCD ; to one side or face of which, as AD, a number of arithmetical triangles equal to the face have been added : and the number of arithmetical triangles added to the square pile, by means of which the oblong pile is formed, is always one less than the shot in the top row ; or which is the same, equal to the difference between the bottom row of the greater side and that of the lesser.

Qu. VI. To find the shot in the triangular pile ABCD, fig. 1, the bottom row AB consisting of 8 shot.

Solution. The proposed pile consisting of 8 horizontal courses, each of which forms an equilateral triangle ; that is, the shot contained in these being in an arithmetical progression, of which the first and last term, as also the number of terms, are known ; it follows, that the sum of these particular courses, or of the 8 progressions, will be the shot contained in the proposed pile ; then

The shot of the first or lower triangular course will be	$(8+1) \times 4$	$= 36$
the second - - - -	$(7+1) \times 3\frac{1}{2}$	$= 28$
the third - - - -	$(6+1) \times 3$	$= 21$
the fourth - - - -	$(5+1) \times 2\frac{1}{2}$	$= 15$
the fifth - - - -	$(4+1) \times 2$	$= 10$
the sixth - - - -	$(3+1) \times 1\frac{1}{2}$	$= 6$
the seventh - - - -	$(2+1) \times 1$	$= 3$
the eighth - - - -	$(1+1) \times \frac{1}{2}$	$= 1$
	Total -	120 shot

in the pile proposed.

Qu. VII. To find the shot of the square pile EFGH, fig. 2, the bottom row EF consisting of 8 shot.

Solution. The bottom row containing 8 shot, and the second only 7 ; that is, the rows forming the progression, 8, 7, 6, 5, 4, 3, 2, 1, in which each of the terms being the square root of the shot contained in each separate square course employed in forming the square pile; it follows, that the sum of the squares of these roots will be the shot required; and the sum of the squares divided by 8, 7, 6, 5, 4, 3, 2, 1, being 204, expresses the shot in the proposed pile.

Qu. VIII. To find the shot of the oblong pile ABCDEF, fig. 3; in which BF = 16, and BC = 7.

Solution. The oblong pile proposed, consisting of the square pile ABCD, whose bottom row is 7 shot; besides 9 arithmetical triangles or progressions, in which the first and last term, as also the number of terms, are known; it follows, that,

if to the contents of the square pile - -	140
we add the sum of the 9th progression -	252
their total gives the contents required - -	392 shot.

REMARK I.

The shot in the triangular and the square piles, as also the shot in each horizontal course, may at once be ascertained by the following table: the vertical column A contains the shot in the bottom row, from 1 to 40 inclusive; the column B contains the triangular numbers, or number of each course; the column C contains the sum of the triangular numbers, that is, the shot contained in a triangular pile, commonly called pyramidal numbers; the column D contains the square of the numbers of the column A, that is, the shot contained in each square horizontal course; and the column E contains the sum of these squares or shot in a square pile.

C Pyramidal numbers	B Triangular numbers.	A Natural numbers.	D Square of the natural numbers.	E Sum of these square numbers.
1	1	1	1	1
4	3	2	4	5
10	6	3	9	14
20	10	4	16	30
35	15	5	25	55
56	21	6	36	91
84	28	7	49	140
120	36	8	64	204
165	45	9	81	285
220	55	10	100	385
286	66	11	121	506
364	78	12	144	650
455	91	13	169	819
560	105	14	196	1015
680	120	15	225	1240
816	136	16	256	1496
969	153	17	289	1785
1140	171	18	324	2109
1330	190	19	361	2470
1540	210	20	400	2870
1771	231	21	441	3311
2024	253	22	484	3795
2300	276	23	529	4324
2600	300	24	576	4900
2925	325	25	625	5525
3276	351	26	676	6201
3654	378	27	729	6930
4060	406	28	784	7714
4495	435	29	841	8555
4960	465	30	900	9455
5456	496	31	961	10416
5984	528	32	1024	11440
6545	561	33	1089	12529
7140	595	34	1156	13685
7770	630	35	1225	14910
8436	666	36	1296	16206
9139	703	37	1369	17575
9880	741	38	1444	19019
10660	780	39	1521	20540
11480	820	40	1600	22140

Thus, the bottom row in a triangular pile, consisting of 19 shot, the contents will be 1330 ; and when of 19 in the square pile, 2470.—In the same manner, the contents either of a square or triangular pile being given, the shot in the bottom row may be easily ascertained.

The contents of any oblong pile by the preceding table may be also with little trouble ascertained, the less side not exceeding 40 shot, nor the difference between the less and the greater side 40. Thus, to find the shot in an oblong pile, the less side being 15, and the greater 35, we are first to find the contents of the square pile, by means of which the oblong pile may be conceived to be formed; that is, we are to find the contents of a square pile, whose bottom row is 15 shot: which being 1240, we are, secondly, to add these 1240 to the product 2400 of the triangular number 120, answering to 15, the number expressing the bottom row of the arithmetical triangle, multiplied by 20, the number of those triangles; and their sum, being 3640, expresses the number of shot in the proposed oblong pile.

REMARK II.

The following algebraical expressions, deduced from the investigations of the sums of the powers of numbers in arithmetical progression, which are seen upon many gunners' callipers*, serve to compute with ease and expedition the shot or shells in any pile.

That serving to compute any triangular pile, is represented by $\dfrac{(n+2)\times(n+1)\times n}{6}$

That serving to compute any square pile, is represented by $\dfrac{(n+1)\times(2n+1)\times n}{6}$

In each of these, the letter n represents the number in the bottom row: hence, in a triangular pile, the number in the bottom row being 30; then this pile will be $(30+2)\times(30+1)\times\frac{30}{6}=4960$ shot or shells. In a square pile, the number in the bottom row being also 30; then this pile will be $(30+1)\times(60+1)\times\frac{30}{6}=9455$ shot or shells.

* Callipers are large compasses, with bowed shanks, serving to take the diameters of convex and concave bodies. The gunners' callipers consist of two thin rules or plates, which are moveable quite round a joint, by the plates folding one over the other: the length of each rule or plate is 6 inches, the breadth about 1 inch. It is usual to represent, on the plates, a variety of scales, tables, proportions, &c. such as are esteemed useful to be known by persons employed about artillery; but, except the measuring of the caliber of shot and cannon, and the measuring of saliant and re-entering angles, none of the articles, with which the callipers are usually filled, are essential to that instrument.

That serving to compute any oblong pile, is represented by $\frac{(2n+1+3m) \times (n+1) \times n}{6}$, in which the letter n denotes the number of courses, and the letter m the number of shot, less one, in the top row; hence, in an oblong pile the number of courses being 30, and the top row 31; this pile will be $\overline{60+1+90} \times \overline{30+1} \times \frac{30}{6} = 23405$ shot or shells.

REMARK III.

One practical rule, of easy recollection, will include the three cases of the triangular, square, and rectangular, complete piles.

Thus, recurring to the diagrams 1, 2, and 3, we shall have, balls in

$(\text{BD} + \text{A} + \text{C}) \times \frac{1}{3}\text{BDC} =$ triangular pile.
$(\text{EF} + \text{EF} + \text{G}) \times \frac{1}{3}\text{GFH} =$ square pile.
$(\text{BF} + \text{BF} + \text{AE}) \times \frac{1}{3}\text{ABC} =$ rectangular pile.

Hence, for a general rule: add to the number of balls or shells in one side of the base, the numbers in its two parallels at bottom and top (whether row or ball), the sum being multiplied by a third of the slant end or face, gives the number in the pile.

GEOMETRICAL PROPORTION, AND PROGRESSION.

Geometrical Proportion contemplates the relation of quantities considered as to what part or what multiple one is of another, or how often one contains, or is contained in, another.—Of two quantities compared together, the first is called the Antecedent, and the second the Consequent, Their ratio is the quotient which arises from dividing the one by the other.

Four Quantities are proportional, when the two couplets have equal ratios, or when the first is the same part or multiple of the second, as the third is of the fourth. Thus, 3, 6, 4, 8, and a, ar, b, br, are geometrical proportionals. For $\frac{6}{3} = \frac{8}{4} = 2$, and $\frac{ar}{a} = \frac{br}{b} = r$. And they are stated thus, 3 : 6 : : 4 : 8, &c. See the Arithmetic.

Geometrical Progression is one in which the terms have all successively the same ratio ; as 1, 2, 4, 8, 16, &c. where the common ratio is 2.

The general and common property of a geometrical progression is, that the product of any two terms, or the square of any one single term, is equal to the product of every other two terms that are taken at an equal distance on both sides from the former. So of these terms,

1, 2, 4, 8, 16, 32, 64, &c.
$1 \times 64 = 2 \times 32 = 4 \times 16 = 8 \times 8 = 64.$

In any geometrical progression, if

a denote the least term,
z the greatest term,
r the common ratio,
n the number of the terms,
s the sum of the series, or all the terms ;

then any of these quantities may be found from the others, by means of these general values or equations, viz.

$$1.\ r = \left(\frac{z}{a}\right)^{\frac{1}{n-1}}.$$

$$2.\ z = a \times r^{n-1}.$$

$$3.\ a = \frac{z}{r^{n-1}}.$$

$$4.\ n = \frac{\log. \frac{rz}{a}}{\log. r} = \frac{\log. r + \log. z - \log. a}{\log. r}.$$

$$5.\ s = \frac{r^n - 1}{r - 1} \times a = \frac{r^n - 1}{r - 1} \times \frac{z}{r^{n-1}} = \frac{rz - a}{r - 1}.$$

When the series is infinite, then the least term a is nothing, and the sum $s = \frac{rz}{r-1}$.

In any increasing geometrical progression, or series beginning with 1, the 3d, 5th, 7th, &c. terms will be squares ; the 4th, 7th, 10th, &c. cubes ; and the 7th will be both a square and a cube. Thus, in the series 1, r, r^2, r^3, r^4, r^5, r^6, r^7, r^8, r^9, &c. r^2, r^4, r^6, r^8, are squares ; r^3, r^6, r^9, cubes ; and r^6 both a square and a cube.

In a decreasing geometrical progression, the ratio, r, is a fraction, and then $s = \frac{1 - r^n}{1 - r} a$. If n be infinite, this becomes $s = \frac{a}{1-r}$; a being the first term.

When four quantities, a, ar, b, br, or 2, 6, 4, 12, are proportional; then any of the following forms of those quantities are also proportional, viz.

1. Directly, $a : ar :: b : br$; or $2 : 6 :: 4 : 12$.
2. Inversely, $ar : a :: br : b$; or $6 : 2 :: 12 : 4$.
3. Alternately, $a : b :: ar : br$; or $2 : 4 :: 6 : 12$.
4. Compoundedly, $a : a+ar :: b : b+br$; or $2 : 8 :: 4 : 16$.
5. Dividedly, $a : ar-a :: b : br-b$; or $2 : 4 :: 4 : 8$.
6. Mixed, $ar+a : ar-a :: br+b : br-b$; or $8 : 4 :: 16 : 8$.
7. Multiplication, $ac : arc :: bc : brc$; or $2.3 : 6.3 :: 4 : 12$,
8. Division, $\frac{a}{c} : \frac{ar}{c} :: b : br$; or $1 : 3 :: 4 : 12$.

9. The numbers a, b, c, d, are in harmonical proportion, when $a : d :: a \backsim b : c \backsim d$; or when their reciprocals $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}$, are in arithmetical proportion.

EXAMPLES.

1. Given the first term of a geometrical series 1, the ratio 2, and the number of terms 12; to find the sum of the series?
First, $1 \times 2^{11} = 1 \times 2048$, is the last term.

Then $\frac{2048 \times 2-1}{2-1} = \frac{4096-1}{1} = 4095$, the sum required.

2. Given the first term of a geometric series $\frac{1}{3}$, the ratio $\frac{1}{2}$, and the number of terms 8; to find the sum of the series?
First, $\frac{1}{3} \times (\frac{1}{2})^7 = \frac{1}{3} \times \frac{1}{128} = \frac{1}{384}$, is the last term.
Then $(\frac{1}{3} - \frac{1}{384} \times \frac{1}{2}) \div (1-\frac{1}{2}) = (\frac{1}{3} - \frac{1}{768}) \div \frac{1}{2} = \frac{255}{768} \times \frac{2}{1}$
$= \frac{255}{384}$, the sum required.

3. Required the sum of 12 terms of the series 1, 3, 9, 27, 81, &c. Ans. 265720.

4. Required the sum of 12 terms of the series, 1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, $\frac{1}{81}$, &c. Ans. $\frac{265720}{177147}$.

5. Required the sum of 100 terms of the series, 1, 2, 4, 8, 16, 32, &c. Ans. 1267650600228229401496703205375.

See more of Geometrical Proportion in the Arithmetic.

INFINITE SERIES.

An Infinite Series is formed either from division, dividing by a compound divisor, or by extracting the root of a compound surd quantity, or by other general processes; and is such as, being continued, would run on infinitely, in the manner of a continued decimal fraction*.

But, by obtaining a few of the first terms, the law of the progression will be manifest; so that the series may thence be continued, without actually performing the whole operation.

PROBLEM I.

To reduce Fractional Quantities into Infinite Series by Division.

Divide the numerator by the denominator, as in common division; then the operation, continued as far as may be thought necessary, will give the infinite series required.

EXAMPLE.

1. To change $\frac{2ab}{a+b}$ into an infinite series.

$$a+b)\ 2ab\ ..\ (2b - \frac{2b^2}{a} + \frac{2b^3}{a^2} - \frac{2b^4}{a^3} + \&c.$$

$$2ab + 2b^2$$

$$-2b^2$$

$$-2b^2 - \frac{2b^3}{a}$$

$$\frac{2b^3}{a}$$

$$\frac{2b^3}{a} + \frac{2b^4}{a^2}$$

$$-\frac{2b^4}{a^2}$$

$$-\frac{2b^4}{a^2} - \frac{2b^5}{a^3}$$

$$\frac{2b^5}{a^3},\ \&c.$$

* The doctrine of infinite series was commenced by Dr. Wallis; who, in his arithmetical works published in 1657, first reduced the fraction $\frac{a}{1-r}$ by a perpetual division into the infinite series $a + ar + ar^2 + ar^3 + ar^4 + \&c.$

2. Let $\frac{1}{1-a}$ be changed into an infinite series.

$$
\begin{array}{l}
1-a)\ 1\ \ldots\ (1+a+a^2+a^3+a^4+\&c. \\
\underline{1-a} \\
\qquad a \\
\qquad \underline{a-a^2} \\
\qquad\qquad a^2 \\
\qquad\qquad \underline{a^2-a^3} \\
\qquad\qquad\qquad a^3 \\
\qquad\qquad\qquad \underline{a^3-a^4} \\
\qquad\qquad\qquad\qquad a^4
\end{array}
$$

3. Expand $\frac{b}{a+c}$ into an infinite series.

Ans. $\frac{b}{a} \times (1 - \frac{c}{a} + \frac{c^2}{a^2} - \frac{c^3}{a^3} + \&c.)$

4. Expand $\frac{a}{a-b}$ into an infinite series.

Ans. $1 + \frac{b}{a} + \frac{b^2+b^3}{a^2+a^3} + \&c.$

5. Expand $\frac{1-x}{1+x}$ into an infinite series.

Ans. $1 - 2x + 2x^2 - 2x^3 + 2x^4$, &c.

6. Expand $\frac{a^2}{(a+b)^2}$ into an infinite series.

Ans. $1 - \frac{2b}{a} + \frac{3b^2}{a^2} - \frac{4b^3}{a^3}$, &c.

7. Expand $\frac{1}{1+1} = \frac{1}{2}$, into an infinite series.

PROBLEM II.

To reduce a Compound Surd into an Infinite Series.

Extract the root as in common arithmetic; then the operation, continued as far as may be thought necessary, will give the series required. But this method is chiefly of use in extracting the square root, the operation being too tedious for the higher powers.

EXAMPLES.

1. Extract the root of a^2-x^2 in an infinite series.

$$a^2-x^2\left(a-\frac{x^2}{2a}-\frac{x^4}{8a^3}-\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}\text{ \&c.}\right.$$

$$a^2$$

$$2a-\frac{x^2}{2a})\quad -x^2$$

$$-x^2+\frac{x^4}{4a^2}$$

$$2a-\frac{x^2}{a}-\frac{x^4}{8a^3})-\frac{x^4}{4a^2}$$

$$-\frac{x^4}{4a^2}+\frac{x^6}{8a^4}+\frac{x^8}{64a^6}$$

$$2a-\frac{x^2}{a}-\frac{x^4}{4a^3}\text{ \&c.})-\frac{x^6}{8a^4}-\frac{x^8}{64a^6}$$

$$-\frac{x^6}{8a^4}+\frac{x^8}{16a^6}\text{ \&c.}$$

$$-\frac{5x^8}{64a^6}\text{ \&c.}$$

2. Expand $\sqrt{(1+1)}=\sqrt{2}$, into an infinite series.
Ans. $1+\frac{1}{2}-\frac{1}{8}+\frac{1}{16}-\frac{5}{128}$ &c.
3. Expand $\sqrt{(1-1)}$ into an infinite series.
Ans. $1-\frac{1}{2}-\frac{1}{8}-\frac{1}{16}-\frac{5}{128}$ &c.
4. Expand $\sqrt{(a^2+x)}$ into an infinite series.
5. Expand $\sqrt{(a^2-2bx-x^2)}$ to an infinite series.

PROBLEM III.

To extract any Root of a Binomial: or to reduce a Binomial Surd into an infinite Series.

This will be done by substituting the particular letters of the binomial, with their proper signs, in the following general theorem or formula, viz.

$$(P+PQ)^{\frac{m}{n}}=P^{\frac{m}{n}}+\frac{m}{n}AQ+\frac{m-n}{2n}BQ+\frac{m-2n}{3n}CQ+\text{\&c.}$$

and it will give the root required: observing that P denotes the first term, Q the second term divided by the first, $\frac{m}{n}$ the index of the power or root; and A, B, C, D, &c. denote the several foregoing terms with their proper signs.

EXAMPLES.

1. To extract the sq. root of $a^2 + b^2$, in an infinite series.

Here $P = a^2$, $Q = \frac{b^2}{a^2}$, and $\frac{m}{n} = \frac{1}{2}$: therefore

$P^{\frac{m}{n}} = (a^2)^{\frac{m}{n}} = (a^2)^{\frac{1}{2}} = a =$ A, the 1st term of the series.

$\frac{m}{n}AQ = \frac{1}{2} \times a \times \frac{b^2}{a^2} = \frac{b}{2a} =$ B, the 2d term.

$\frac{m-n}{2n}BQ = \frac{1-2}{4} \times \frac{b^2}{2a} \times \frac{b^2}{a^2} = -\frac{b^4}{2.4a^3} =$ C, the 3d term.

$\frac{m-2n}{3n}CQ = \frac{1-4}{6} \times -\frac{b^4}{2.4a^3} \times \frac{b^2}{a^2} = \frac{3b^6}{2.4.6a^5} =$ D, the 4th.

Hence $a + \frac{b}{2a} - \frac{b^4}{2.4a^3} + \frac{3.b^6}{2.4.6a^5} -$ &c. or

$a + \frac{b^2}{2a} - \frac{b^4}{8a^3} + \frac{b^6}{16a^5} - \frac{5b^8}{128a^7}$ &c. is the series required.

2. To find the value of $\frac{1}{(a-x)^2}$, or its equal $(a-x)^{-2}$ in an infinite series*.

Here $P=a$, $Q=\frac{-x}{a}=-a^{-1}x$, and $\frac{m}{n}=\frac{-2}{1}=-2$; theref.

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* *Note.* To facilitate the application of the rule to fractional examples, it is proper to observe, that any surd may be taken from the denominator of a fraction and placed in the numerator, and vice versa, by only changing the sign of its index. Thus,

$\frac{1}{x^2} = 1 \times x^{-2}$ or only x^{-2}; and $\frac{1}{(a+b)^2} = 1 \times (a+b)^{-2}$ or $(a+b)^{-2}$;

and $\frac{a^2}{(a+x)^2} = a^2(a+x)^{-2}$; and $\frac{x^{\frac{1}{3}}}{x^{\frac{1}{2}}} = x^{\frac{1}{3}} \times x^{-\frac{1}{2}}$; also $\frac{(a^2+x^2)^{\frac{1}{2}}}{(a^2-x^2)^{\frac{1}{2}}} =$

$(a^2+x^2)^{\frac{1}{2}} \times (a^2-x^2)^{-\frac{1}{2}}$; &c.

The theorem above given is only the Binomial Theorem so expressed as to facilitate its application to roots and series.

$P^n = (a)^{-2} = a^{-2} = \frac{1}{a^2} = $ A, the first term of the series.

$\frac{m}{n}AQ = -2 \times \frac{1}{a^2} \times \frac{-x}{a} = \frac{2x}{a^3} = 2a^{-3}x = $ B, the 2d term.

$\frac{m-n}{2n}BQ = -\frac{3}{2} \times \frac{2x}{a^3} \times \frac{-x}{a} \times = \frac{3x^2}{a^4} = 3a^{-4}x^2 = $ C, the 3d.

$\frac{m-2n}{3n}CQ = -\frac{4}{3} \times \frac{3x^2}{a^4} \times \frac{-x}{a} = \frac{4x^3}{a^5} = 4a^{-5}x^3 = $ D.

Hence $a^{-2} + 2a^{-3}x + 3a^{-4}x^2 + 4a^{-5}x^3 +$ &c. or

$\frac{1}{a^2} + \frac{2x}{a^3} + \frac{3x^2}{a^4} + \frac{4x^3}{a^5} + \frac{5x^4}{a^6}$ &c. is the series required.

3. To find the value of $\frac{a^2}{a-x}$, in an infinite series.

Ans. $a + x + \frac{x^2}{a} + \frac{x^3}{a^2} + \frac{x^4}{a^3}$ &c.

4. To expand $\sqrt{\frac{1}{(a^2+x^2)}}$ or $\frac{1}{(a^2+x^2)^{\frac{1}{2}}}$ in a series.

Ans. $\frac{1}{a} - \frac{x^2}{2a^3} + \frac{3x^4}{8a^5} - \frac{5x^6}{16a^7}$ &c.

5. To expand $\frac{a^2}{(a-b)^2}$ in an infinite series.

Ans. $1 + \frac{2b}{a} + \frac{3b^2}{a^2} + \frac{4b^3}{a^3} + \frac{5b^4}{a^4}$ &c.

6. To expand $\sqrt{a^2-x^2}$ or $(a^2-x^2)^{\frac{1}{2}}$ in a series.

Ans. $a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5} - \frac{5x^8}{128a^7}$ &c.

7. To find the value of $\sqrt[3]{(a^3-b^3)}$ or $(a^3-b^3)^{\frac{1}{3}}$ in a series.

Ans. $a - \frac{b^3}{3a^2} - \frac{b^6}{9a^5} - \frac{5b^9}{81a^8}$ &c.

8. To find the value of $\sqrt[5]{(a^5+x^5)}$ or $(a^5+x^5)^{\frac{1}{5}}$ in a series.

Ans. $a + \frac{x^5}{5a^4} + \frac{2x^{10}}{25a^9} + \frac{6x^{15}}{125^{14}}$ &c.

9. To find the square root of $\frac{a-b}{a+b}$ in an infinite series.

Ans. $1 - \frac{b}{a} + \frac{b^2}{2a^2} - \frac{b^3}{2a^3}$ &c.

10. Find the cube root of $\frac{a^3}{a^3+b^3}$ in a series.

Ans. $1 - \frac{b^3}{3a^3} + \frac{2b^6}{9a^6} - \frac{14b^9}{81a^9}$ &c.

INFINITE SERIES: PART THE SECOND.

PROBLEM I*.

A SERIES being given, to find the several orders of differences of the successive terms.

RULE I. Subtract the first term from the second, the second from the third, the third from the fourth, and so on; the several remainders will constitute a new series, called *the first order of differences*.

II. In this new series, take the first term from the second, the second from the third, &c. as before, and the remainders will form another new series, called *the second order of differences*.

III. Proceed in the same manner for the *third, fourth, fifth, &c. orders*, until either the differences become 0, or the work will be carried as far as is thought necessary†.

EXAMPLES.

1. Given the series 1, 4, 8, 13, 19, 26, &c. to find the several orders of differences.

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* The study of this second part of Infinite Series may be conveniently postponed till Simple and Quadratic Equations have been learnt.

† Let a, b, c, d, e, &c. be the terms of a given series, then if D = the first term of the nth order of differences, the following theorem will exhibit the value of D: viz. $\pm a \mp nb \pm n \cdot \frac{n-1}{2} \cdot c \mp n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot d \pm n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot e \mp$, &c. (to $n+1$ terms) = D, where the upper signs must be taken when n is an even number, and the lower signs when n is odd.

If the differences be very great, the logarithms of the quantities may be used, the differences of which will be much smaller than those of the quantities themselves; and at the close of the operation the natural number answering to the logarithmical result will be the answer. See *Emerson's Differential Method*, prop. 1.

Thus	1, 4, 8, 13, 19, 26,	&c. the given series.
Then,	3, 4, 5, 6, 7,	&c. the first differences.
And	1, 1, 1, 1,	&c. the second differences.
Also	0, 0, 0,	&c. the third differences.

where the work evidently must terminate.

2. Given the series 1, 4, 8, 16, 32, 64, 128, &c. to find the several orders of differences.

Here	1, 4, 8, 16, 32, 64, 128,	&c. given series.
And	3, 4, 8, 16, 32, 64,	&c. 1st diff.
	1, 4, 8, 16, 32,	&c. 2nd diff.
	3, 4, 8, 16,	&c. 3rd diff.
	1, 4, 8,	&c. 4th diff.
	3, 4,	&c. 5th diff.
	1,	&c. 6th diff.

3. Find the several orders of differences in the series 1, 2, 3, 4, &c.

Ans. First diff. 1, 1, 1, 1, &c. Second diff. 0, 0, 0, &c.

4. To find the several orders of differences in the series 1, 4, 9, 16, 25, &c. of squares.

Ans. First differences 3, 5, 7, 9, &c. Second, 2, 2, 2, &c. Third, 0, 0, &c.

5. Required the orders of differences in the series 1, 8, 27, 64, 125, &c. being cubes.

6. Given 1, 6, 20, 50, 105, &c. to find the several orders of differences.

PROBLEM. II.

To Find any term of a given series.

RULE I. Let a, b, c, d, e, &c. be the given series; d', d'', d''', d^{iv}, &c. respectively, the first term of the first, second, third, fourth, &c. order of differences, as found by the preceding article; n = the number denoting the place of the term required.

II. Then will $a + \frac{n-1}{1}. d' + \frac{n-1}{1} . \frac{n-2}{2} . d'' + \frac{n-1}{1} . \frac{n-2}{2} . \frac{n-3}{3} . d''' + \frac{n-1}{1} . \frac{n-2}{2} . \frac{n-3}{3} . \frac{n-4}{4} . d^{iv} +$ &c. = to the nth term required.

EXAMPLES.

1. To find the 10th term of the series 2, 5, 9, 14, 20, &c.

Here 2, 5, 9, 14, 20, &c. series.
3, 4, 5, 6, &c. 1st diff.
1, 1, 1, &c. 2nd diff.
0, 0, &c. 3rd diff.

Where $d' = 3$, $d'' = 1$, $d''' = 0$, also $a = 2$, $n = 10$; wherefore $a + \frac{n-1}{1} . d' + \frac{n-1}{1} . \frac{n-2}{2} . d'' = (2 + \frac{10-1}{1} \times 3 + \frac{10-1}{1} \times \frac{10-2}{2} \times 1 =) 2 + 27 + 36 = 65 =$ the 10th term required.

2. To find the 20th term of the series 2, 6, 12, 20, 30, &c.

Here $a = 2$, $n = 20$; and Art. 12.
2, 6, 12, 20, 30, &c. series.
4, 6, 8, 10, &c. 1st diff.
2, 2, 2, &c. 2nd diff. or $d' = 4$, $d'' = 2$;

whence $a + \frac{n-1}{1} . d' + \frac{n-1}{1} . \frac{n-2}{2} . d'' = (2 + \frac{19}{1} \times 4 + \frac{19}{1} \times \frac{18}{2} \times 2 =) 2 + 76 + 342 = 420 =$ the 20th term required.

3. Required the 5th term of the series, 1, 3, 6, 10, &c. Ans. 15.

4. To find the 10th term of the series, 1, 4, 8, 13, 19, &c. Ans. 64.

5. Required the 20th term of the series, 1, 8, 27, 64, 125, &c. Ans. 8000.

PROBLEM III.

If the succeeding terms of a given series be at an unit's distance from each other, to find any intermediate term by interpolation.

RULE 1. Let y be the term to be interpolated, x its distance from the beginning of the series, d', d'', d''', d^{iv}, &c. the first terms of the several orders of differences.

2. Then will $a + xd' + x \cdot \frac{x-1}{2} \cdot d'' + x \cdot \frac{x-1}{2} \cdot \frac{x-2}{3} \cdot d''' +$ $x \cdot \frac{x-1}{2} \cdot \frac{x-2}{3} \cdot \frac{x-3}{4} \cdot d^{\text{iv}} +$ &c. $= y$, the term required.

EXAMPLES.

1. Given the logarithmic sines of 3° 4′, 3° 5′, 3° 6′, 3° 7′, and 3° 8′, to find the sine of 3° 6′ 15″.

Series.	*Logarithms.*	*1st diff.*	*2nd diff.*	*3rd diff.*
3° 4′......	8·7283366	23516		
			—126	
3 5	8·7306882	23390		1
			—127	
3 6	8·7330272	23263		—4
			—123	
3 7	8·7353535	23140		
3 8	8·7376675			

Here $x = $ (3° 6′ 15″ — 3° 4′ = 2′ 15″ =) $\frac{9}{4}$ = the distance of the term y, to be interpolated; $a = 8{\cdot}7283366$, $d' = 23516$, $d'' = -126$, $d''' = 1$, and $y = a + xd' + x \cdot \frac{x-1}{2} \cdot d'' + x \frac{x-1}{2} \cdot \frac{x-2}{3} \cdot d''' = (a + \frac{9}{4}d' + \frac{45}{32}d'' + \frac{15}{128}d''' =)$ $8{\cdot}7283366 + {\cdot}0052911 - {\cdot}00001771875 + {\cdot}0000000117 = 8{\cdot}733609999296$, the log. sine required.

2. Given the series $\frac{1}{50}$, $\frac{1}{51}$, $\frac{1}{52}$, $\frac{1}{53}$, $\frac{1}{54}$, to find the term which stands in the middle, between $\frac{1}{52}$ and $\frac{1}{53}$. Ans. $\frac{1}{105}$.

3. Given the logarithmic sines of 1° 0′, 1° 1′, 1° 2′, and 1° 3′, to find the logarithmic sine of 1° 1′ 40″. Ans. 8·2537533.

PROBLEM IV.

To find any intermediate Term by Interpolation, when the first Differences of a Series of equidifferent Terms are small.

Rule 1. Let a, b, c, d, e, &c. represent the given series, and n = the number of terms given.

2. Then will $a - nb + n \cdot \frac{n-1}{2} \cdot c - n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot d + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot e +$ &c. $= 0$, from whence by transportation, &c. any required term may be obtained*.

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* For the investigation of these rules, see *Emerson's Differential Method.*

EXAMPLES.

1. Given the square root of 10, 11, 12, 13, and 15, to find the square root of 14.

Here $n = 5$, and e is the term required.

$$a = (\sqrt{10} =)\ 3\cdot1622776$$
$$b = (\sqrt{11} =)\ 3\cdot3166248$$
$$c = (\sqrt{12} =)\ 3\cdot4641016$$
$$d = (\sqrt{13} =)\ 3\cdot6055512$$
$$f = (\sqrt{15} =)\ 3\cdot8729833$$

And since $n = 5$, the series must be continued to 6 terms.

Therefore $a - nb + n \cdot \frac{n-1}{2} \cdot c - n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot d + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot e - n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot \frac{n-4}{5} \cdot f = 0.$

Whence, by transposition, in order to find e we shall have $n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot e = -a + nb - n \cdot \frac{n-1}{2} \cdot c + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot d + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot \frac{n-4}{5} \cdot f$; this in numbers becomes $5c = -3\cdot1622776 + (5 \times 3\cdot3166248) - (10 \times 3\cdot4641016) + (10 \times 3\cdot6055512) + 3\cdot8729833 = 56\cdot5116193 - 37\cdot8032936 = 18\cdot7083257$, and $e = \frac{18\cdot7083257}{5} = 3\cdot74166514$ = the root, nearly.

2. Given the square roots of 37, 38, 39, 41, and 42, to find the square root of 40. Ans. 6·32455532.

3. Given the cube roots of 45, 46, 47, 48, and 49, to find the cube root of 50. Ans. 3·684033.

PROBLEM V.

To revert a given Series.

When the powers of an unknown quantity are contained in the terms of a series, the finding the value of the unknown quantity in another series, which involves the powers of the quantity to which the given series is equal, and known quantities only, is called reverting the series*.

Rule 1. Assume a series for the value of the unknown quantity, of the same form with the series which is required to be reverted.

2. Substitute this series and its powers, for the unknown quantity and its powers, in the given series.

3. Make the resulting terms equal to the corresponding terms of the given series, whence the values of the assumed co-efficients will be obtained.

EXAMPLES.

1. Let $ax + bx^2 + cx^3 + dx^4 +$ &c. $= z$ be given, to find the value of x in terms of z and known quantities.

Let $z^n = x$, then it is plain that if z^n and its powers be substituted in the given series for x and its powers, the indices of z will be n, $2n$, $3n$, $4n$, &c. and 1; whence $n = 1$, and the differences of these indices are 0, 1, 2, 3, 4, &c. Wherefore the indices of the series to be assumed, must have the same differences; let therefore this series be $Az + Bz^2 + Cz^3 + Dz^4 +$ &c. $= x$. And if this series be involved, and substituted for the several powers of x, in the given series, it will become

$$\left.\begin{array}{l} aAz + aBz^2 + aCz^3 + aDz^4 + \&c. \\ \quad * + bA^2z^2 + 2bABz^3 + 2bACz^4 + \&c. \\ \quad * \quad * \quad * + bB^2z^4 + \&c. \\ \quad * \quad * + cA^3z^3 + 3cA^2Bz^4 + \&c. \\ \quad * \quad * \quad * + dA^4z^4. + \&c. \end{array}\right\} = z.$$

Whence, by equating the terms which contain like powers of z, we obtain ($aAz = z$, or) $A = \frac{1}{a}$; ($aBz^2 + bA^2z^2 = 0$, whence) $B = (-\frac{bA^2}{a}, =) - \frac{b}{a^3}$, ($aCz^3 + 2bABz^3 + cA^3z^3 = 0$, whence) $C = (-\frac{2bAB + cA^3}{a} =) \frac{2b^2 - ac}{a^5}$; $D = (-\frac{2bAC + bB^2 + 3cA^2B + dA^4}{a} =) \frac{5abc - 5b^3 - a^2d}{a^7}$, &c. and conse-

* Other methods of reversion are given by different mathematicians. The above is selected for its simplicity.

quently $x=(Az+Bz^2+Cz^3+ \&c.=)\frac{z}{a}-\frac{bz^2}{a^3}+\frac{2b^2-ac}{a^5}.z^3-\frac{5b^3-5abc+a^2d}{a^7}.z^4+$ &c. the series required.

This conclusion forms a general theorem for every similar series, involving the like powers of the unknown quantity.

2. Let the series $x-x^2+x^3-x^4+\&c.=z$, be proposed for reversion.

Here $a=1$, $b=-1$, $c=1$, $d=-1$, &c. these values being substituted in the theorem derived from the preceding example, we thence obtain $x=z+z^2+z^3+z^4+$ &c. the answer required.

3. Let $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\&c.=y$, be given for reversion.

Substituting as before, we have $a=1$, $b=-\frac{1}{2}$, $c=\frac{1}{3}$, and $d=-\frac{1}{4}$, &c. These values being substituted, we shall have $x=y+\frac{y^2}{2}+\frac{y^3}{6}+\frac{y^4}{24}+$ &c. from which if y be given, and sufficiently small for the series to approximate, the value of x will be known.

PROBLEM VI.

To find the Sum of n Terms of an Infinite Series.

RULE 1. Let a, b, c, d, e, &c. be the given series, $s=$ the sum of n terms, and d', d'', d''', d^{iv}, &c. respectively the first terms of the several orders of differences, found by prob. 1.

2. Then will $na+n.\frac{n-1}{2}.d'+n.\frac{n-1}{2}.\frac{n-2}{3}.d''+n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}.d'''+n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}.\frac{n-4}{5}.d^{iv}+\&c.=s$, the sum of n terms of the series, as was required.

CASE 1. To find the sum of n terms of the series 1, 2, 3, 4, 5, &c.

First, 1, 2, 3, 4, 5, &c. the given series.
1, 1, 1, 1, &c. first differences.
0, 0, 0, &c. second differences.

Here $a=1$, $d'=1$, $d''=0$; then will $na+n.\frac{n-1}{2}.d'=$

$(\frac{2na + \overline{n^2 - n} \cdot d'}{2}$, which, (since a and d' each $= 1$) $=$

$\frac{2n + n^2 - n}{2} =) \frac{n \cdot \overline{n+1}}{2} = s$, the sum required.

EXAMPLES.

1. Let the sum of 20 terms of the above series be required.

Here $n = 20$, and $s = \frac{\overline{n.n+1}}{2} = \frac{20 \times 21}{2} = 210$, the ans.

2. Let the sum of 1000 terms be required. Ans. 500500.

3. Let the sum of 12345 terms be required.

CASE 2. To find the sum of n terms of the series, 1, 3, 5, 7, 9, &c.

Here 1, 3, 5, 7, 9, &c. the given series.
2, 2, 2, 2, &c. . . . first difference.
0, 0, 0, &c. . . . second difference.

Wherefore $a = 1$, $d' = 2$, $d'' = 0$, and $na + n \cdot \frac{n-1}{2} \cdot d'$

$= (na + \frac{n^2 - n}{2} \cdot d' =$ (since $a = 1$ and $d = 2$) $n + n^2 - n =) \; n^2 = s$, the sum required.

EXAMPLE.

To find the sum of 10 terms of the above series.

Here $n = 10$, and $s = (n^2 =)$ 100, the answer.

CASE 3. To find the sum of n terms of the series of squares 1, 4, 9, 16, 25, &c.

Here 1, 4, 9, 16, 25, &c. the series.
3, 5, 7, 9, &c. . . . 1st diff.
2, 2, 2, &c. . . . 2nd diff.
0, 0, &c. . . . 3rd diff.

Whence $a = 1$, $d' = 3$, $d'' = 2$, $d''' = 0$, and $na + n \cdot \frac{n-1}{2} \cdot d' + \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot d'' = (n + 3n \cdot \cdot \frac{n-1}{2} \cdot + 2n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} = \frac{3n^2 - n}{2} + \frac{n^3 - 3n^2 + 2n}{3} =) \; \frac{n.\overline{n+1}.\overline{2n+1}}{6} = s$, the sum required.

EXAMPLE.

Let the sum of 30 terms of the above series be required.

Here $n = 30$; wherefore $\frac{n(n+1)(2n+1)}{6} = \frac{30 \times 31 \times 61}{6} =$ 9455, the answer. See the table, pa. 210.

PROBLEM VII.

To find the Sums of Series, by the Method of Subtraction.

This method will be rendered evident by two or three simple examples.

EXAMPLE 1.

$$\text{Let } 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \&c. \textit{ in inf.} = s$$
$$\text{then } \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \&c. \textit{ in inf.} = s - 1.$$
$$\text{by sub. } \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \&c. \textit{ in inf.} = 1.$$

EXAMPLE 2.

$$\text{Let } 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \&c. = s$$
$$\text{then } \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \&c. = s = \tfrac{3}{2}.$$
$$\text{by sub. } \frac{2}{1.3} + \frac{2}{2.4} + \frac{2}{3.5} + \frac{2}{4.6} + \&c. = \tfrac{3}{2},$$
$$\text{or} \div 2, \frac{1}{1.3} + \frac{1}{2.4} + \frac{1}{3.5} + \frac{1}{4.6} + \&c. = \tfrac{3}{4}.$$

EXAMPLE 3.

$$\text{Let } \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \&c. = s$$
$$\text{then } \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \&c. = s - \tfrac{1}{2}$$
$$\text{by sub. } \frac{2}{1.2.3} + \frac{2}{2.3.4} + \frac{2}{3.4.5} + \&c. = \tfrac{1}{2}$$
$$\therefore \frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \&c. = \tfrac{1}{4}.$$

EXAMPLE 4.

Find the sum of the series $\frac{1}{2.4.6} + \frac{1}{4.6.8} + \frac{1}{6.8.10} +$ &c. *in infinitum*.

Take away the last factor out of each denominator, and assume $\frac{1}{2.4} + \frac{1}{4.6} + \frac{1}{6.8} + \&c. = s.$

then $\frac{1}{4.6} + \frac{1}{6.8} + \frac{1}{8.10} + \&c. = s - \frac{1}{8}$

by sub. $\frac{4}{2.4.6} + \frac{4}{4.6.8} + \frac{4}{6.8.10} + \&c. = \frac{1}{8}$

$$\therefore \frac{1}{2.4.6} + \frac{1}{4.6.8} + \frac{1}{6.8.10} + \&c. = \frac{1}{8} \times \frac{1}{4} = \frac{1}{32}.$$

EXAMPLE 4.

Find the sum of the infinite series

$$\frac{1}{2.4.6.8} + \frac{1}{4.6.8.10} + \frac{1}{6.8.10.12} + \frac{1}{8.10.12.14} + \&c.$$

Ans. $\frac{1}{288}$.

EXAMPLE 5.

Find the sum of the infinite series

$$\frac{1}{3.5.8.11} + \frac{1}{5.8.11.14} + \frac{1}{8.11.14.17} + \frac{1}{11.14.17.20} + \&c.$$

Ans. $\frac{1}{720}$.

PROBLEM VIII.

To sum an infinite series by supposing it to arise from the expansion of some fractional expression.

RULE. Assume the series equal to a fraction, whose denominator is such, that when the series is multiplied by it, the product may be finite; this product being equal to the numerator of the assumed fraction, determines its value.

EXAMPLES.

1. Required the sum of the infinite series $x + x^2 + x^3$ + &c.

Assume the series $= \dfrac{x}{1-x}$

$$\begin{array}{rl} \text{then} & x + x^2 + x^3 + \&c. \\ \text{into} & 1 - x \\ \hline & x + x^2 + x^3 + \&c. \\ & \quad\; - x^2 - x^3 - \&c. \\ \hline \end{array}$$

$$x = x$$

$$\therefore x + x^2 + x^3 + \&c. = \frac{x}{1-x}.$$

Thus, if $x = \frac{1}{2}$, then $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \&c. = \frac{1}{2} \div \frac{1}{2} = 1$;
if $x = \frac{1}{3}$, then $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \&c. = \frac{1}{3} \div \frac{2}{3} = \frac{1}{2}$.

2. Required the sum of the infinite series $x + 2x^2 + 3x^3 + \&c.$

Assume the series $= \dfrac{x}{(1-x)^2} = \dfrac{x}{1-2x+x^2}$;

$$\begin{array}{rl} \text{then} & x + 2x^2 + 3x^3 + \&c. \\ \text{into} & 1 - 2x + x^2 \\ \hline & x + 2x^2 + 3x^3 + \&c. \\ & \quad\; - 2x^2 - 4x^3 - \&c. \\ & \qquad\qquad + x^3 + \&c. \\ \hline \end{array}$$

$$x = x$$

$$\therefore x + 2x^2 + 3x^3 + \&c. = \frac{x}{(1-x)^2}.$$

If $x = \frac{1}{2}$, then $\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \&c. = \frac{1}{2} \div \frac{1}{4} = 2$.
If $x = \frac{1}{3}$, then $\frac{1}{3} + \frac{2}{9} + \frac{3}{27} + \frac{4}{81} + \&c. = \frac{1}{3} \div \frac{4}{9} = \frac{3}{4}$.
And so on, in other cases *.

3. Find the sum of the infinite series $x + 4x^2 + 9x^3 + 16x^4 + \&c.$

Ans. $\dfrac{x(1+x)}{(1-x)^3}$.

* The preceding is only a sketch of an inexhaustible subject. For the algebraical investigation of infinite series, consult *Dodson's Mathematical Repository*, and Mr. *J. R. Young's Algebra*. The subject, however, is much more extensively treated by means of the fluxional analysis.

SIMPLE EQUATIONS.

An Equation is the expression of two equal quantities with the sign of equality (=) placed between them. Thus $10 - 4 = 6$ is an equation, denoting the equality of the quantities $10 - 4$ and 6.

Equations are either simple or compound. A Simple Equation, is that which contains only one power of the unknown quantity, without including different powers. Thus, $x - a = b + c$, or $ax^2 = b$, is a simple equation, containing only one power of the unknown quantity x. But $x^2 - 2ax = b^2$ is a compound one.

GENERAL RULE.

Reduction of Equations, is the finding the value of the unknown quantity. And this consists in disengaging that quantity from the known ones; or in ordering the equation so, that the unknown letter or quantity may stand alone on one side of the equation, or of the mark of equality, without a co-efficient; and all the rest, or the known quanties, on the other side.—*In general, the unknown quantity is disengaged from the known ones, by performing always the reverse operations.* So, if the known quantities are connected with it by + or addition, they must be subtracted; if by minus (—), or subtraction, they must be added; if by multiplication, we must divide by them; if by division, we must multiply; when it is in any power, we must extract the root; and when in any radical, we must raise it to the power. As in the following particular rules; which are founded on the general principle, that when equal operations are performed on equal quantities, the results must still be equal; whether by equal additions, or subtractions, or multiplications, or divisions, or roots, or powers.

PARTICULAR RULE I.

When known quantities are connected with the unknown by + or —; transpose them to the other side of the equation, and change their signs. Which is only adding or subtracting the same quantities on both sides, in order to get all the unknown terms on one side of the question, and all the known ones on the other side*.

Thus, if $x+5=8$; then transposing 5 gives $x=8-5=3$.

And if $x-3+7=9$; then transposing the 3 and 7, gives $x=9+3-7=5$.

Also, if $x-a+b=cd$, then by transposing a and b, it is $x=a-b+cd$.

In like manner, if $5x-6=4x+10$, then by transposing 6 and $4x$, it is $5x-4x=10+6$, or $x=16$.

RULE II.

When the unknown term is multiplied by any quantity; divide all the terms of the equation by it.

Thus, if $ax=ab-4a$; then dividing by a, gives $x=b-4$.

And, if $3x+5=20$; then first transposing 5 gives $3x=15$; and then by dividing by 3, it is $x=5$.

In like manner, if $ax+3ab=4c^2$; then by dividing by a, it is $x+3b=\frac{4c^2}{a}$; and then transposing $3b$, gives $x=\frac{4c^2}{a}-3b$.

RULE III.

When the unknown term is divided by any quantity; we must then multiply all the terms of the equation by that divisor; which takes it away.

Thus, if $\frac{x}{4}=3+2$: then mult. by 4, gives $x=12+8=20$.

And, if $\frac{x}{a}=3b+2c-d$:

then mult. by a, it gives $x=3ab+2ac-ad$.

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* Here it is earnestly recommended that the pupil be accustomed, at every line or step in the reduction of the equations, to name the particular operation to be performed in the equation in the last line, in order to produce the next form or state of the equation, in applying each of these rules, according as the particular form of the equation may require; applying them according to the order in which they are here placed: and beginning every line with the words *Then by*, as in the following specimens of Examples; which two words will always bring to his recollection, that he is to pronounce what particular operation he is to perform on the last line, in order to give the next; allotting always a single line for each operation, and ranging the equations neatly just under each other, in the several lines, as they are successively produced.

Also, if $\frac{3x}{5} - 3 = 5 + 2$:

Then by transposing 3, it is $\frac{3}{5}x = 10$.
And multiplying by 5, it is $3x = 50$.
Lastly, dividing by 3, gives $x = 16\frac{2}{3}$.

RULE IV.

When the unknown quantity is included in any root or surd: transpose the rest of the terms, if there be any, by Rule 1; then raise each side to such a power as is denoted by the index of the surd; viz. square each side when it is the square root; cube each side when it is the cube root; &c. which clears that radical.

Thus, if $\sqrt{x} - 3 = 4$; then transposing 3, gives $\sqrt{x} = 7$;
And squaring both sides gives $x = 49$.
And, if $\sqrt{(2x + 10)} = 8$:
Then by squaring it, it becomes $2x + 10 = 64$;
And by transposing 10, it is $2x = 54$;
Lastly, dividing by 2, gives $x = 27$.
Also, if $\sqrt[3]{(3x + 4)} + 3 = 6$;
Then by transposing 3, it is $\sqrt[3]{(3x + 4)} = 3$;
And by cubing, it is $3x + 4 = 27$;
Also, by transposing 4, it is $3x = 23$;
Lastly, dividing by 3, gives $x = 7\frac{2}{3}$.

RULE V.

When that side of the equation which contains the unknown quantity is a complete power, or can easily be reduced to one, by rule 1, 2, or 3; then extract the root of the said power on both sides of the equation; that is, extract the square root when it is a square power, or the cube root when it is a cube, &c.

Thus, if $x^2 + 8x + 16 = 36$, or $(x + 4)^2 = 36$;
Then by extracting the root, it is $x + 4 = 6$;
And by transposing 4, it is $x = 6 - 4 = 2$.
And if $3x^2 - 19 = 21 + 35$.
Then, by transposing 19, it is $3x^2 = 75$;
And dividing by 3, gives $x^2 = 25$;
And extracting the root, gives $x = 5$.
Also, if $\frac{3}{4}x^2 - 6 = 24$.
Then transposing 6, gives $\frac{3}{4}x^2 = 30$;

And multiplying by 4, gives $3x^2 = 120$;
Then dividing by 3, gives $x^2 = 40$;
Lastly, extracting the root, gives $x = \sqrt{40} = 6{\cdot}324555$.

RULE VI.

When there is any analogy or proportion, it is to be changed into an equation, by multiplying the two extreme terms together, and the two means together, and making the one product equal to the other.

Thus, if $2x : 9 :: 3 : 5$.
Then mult. the extremes and means, gives $10x = 27$;
And dividing by 10, gives $x = 2\frac{7}{10}$.

And if $\frac{3}{4}x : a :: 5b : 2c$.
Then mult. extremes and means, gives $\frac{3}{2}cx = 5ab$;
And multiplying by 2, gives $3cx = 10ab$;
Lastly, dividing by $3c$, gives $x = \dfrac{10ab}{3c}$.

Also, if $10 - x : \frac{2}{3}x :: 3 : 1$.
Then mult. extremes and means, gives $10 - x = 2x$;
And transposing x, gives $10 = 3x$;
Lastly, dividing by 3, gives $3\frac{1}{3} = x$.

RULE VII.

When the same quantity is found on both sides of an equation, with the same sign, either plus or minus, it may be left out of both: and when every term in an equation is either multiplied or divided by the same quantity, it may be struck out of them all.

Thus, if $3x + 2a = 2a + b$:
Then, by taking away $2a$, it is $3x = b$.
And, dividing by 3, it is $x = \frac{1}{3}b$.

Also, if there be $4ax + 6ab = 7ac$.
Then striking out or dividing by a, gives $4x + 6b = 7c$.
Then by transposing $6b$, it becomes $4x = 7c - 6b$;
And then dividing by 4, gives $x = \frac{7}{4}c - \frac{3}{2}b$.

Again, if $\frac{2}{3}x - \frac{7}{3} = \frac{10}{3} - \frac{7}{3}$.
Then, taking away the $\frac{7}{3}$, it becomes $\frac{2}{3}x = \frac{10}{3}$;
And taking away the 3's, it is $2x = 10$;
Lastly, dividing by 2, gives $x = 5$.

MISCELLANEOUS EXAMPLES.

1. Given $7x - 18 = 4x + 6$; to find the value of x.
First, transposing 18 and $4x$ gives $3x = 24$;
Then dividing by 3, gives $x = 8$.

2. Given $20 - 4x - 12 = 92 - 10x$; to find x.
First, transposing 20 and 12 and $10x$, gives $6x = 84$;
Then dividing by 6, gives $x = 14$.

3. Let $4ax - 5b = 3dx + 2c$ be given: to find x.
First, by trans. $5b$ and $3dx$, it is $4ax - 3dx = 5b + 2c$:
Then dividing by $4a - 3d$, gives $x = \frac{5b + 2c}{4a - 3d}$.

4. Let $5x^2 - 12x = 9x + 2x^2$ be given; to find x.
First, by dividing by x, it is $5x - 12 = 9 + 2x$;
Then transposing 12 and $2x$, gives $3x = 21$;
Lastly, dividing by 3, gives $x = 7$.

5. Given $9ax^3 - 15abx^2 = 6ax^3 + 12ax^2$; to find x.
First, dividing by $3ax^2$, gives $3x - 5b = 2x + 4$;
Then transposing $5b$ and $2x$, gives $x = 5b + 4$.

6. Let $\frac{x}{3} - \frac{x}{4} + \frac{x}{5} = 2$ be given, to find x.

First, multiplying by 3, gives $x - \frac{3}{4}x + \frac{3}{5}x = 6$;
Then, multiplying by 4, gives $x + \frac{12}{5}x = 24$.
Also multiplying by 5, gives $17x = 120$:
Lastly, dividing by 17, gives $x = 7\frac{1}{17}$.

7. Given $\frac{x-5}{3} + \frac{x}{2} = 12 - \frac{x-10}{3}$; to find x.

First, mult. by 3, gives $x - 5 + \frac{3}{2}x = 36 - x + 10$;
Then transposing 5 and x, gives $2x + \frac{3}{2}x = 51$;
And multiplying by 2, gives $7x = 102$;
Lastly, dividing by 7, gives $x = 14\frac{4}{7}$.

8. Let $\sqrt{\frac{3x}{4}} + 7 = 10$, be given; to find x.

First, transposing 7, gives $\sqrt{\frac{3}{4}x} = 3$;
Then squaring the equation, gives $\frac{3}{4}x = 9$;

Then dividing by 3, gives $\frac{1}{4}x = 3$;
Lastly, multiplying by 4, gives $x = 12$.

9. Let $2x + 2\sqrt{(a^2 + x^2)} = \frac{5a^2}{\sqrt{(a^2+x^2)}}$, be given, to find x.

First, mult. by $\sqrt{(a^2+x^2)}$, gives $2x\sqrt{(a^2+x^2)}+2a^2+2x^2 = 5a^2$.
Then transp. $2a^2$ and $2x^2$, gives $2x\sqrt{(a^2+x^2)}=3a^2-2x^2$;
Then by squaring, it is $4x^2 \times (a^2 + x^2) = (3a^2 - 2x^2)^2$;
That is, $4a^2x^2 + 4x^4 = 9a^4 - 12a^2x^2 + 4x^4$;
By taking $4x^4$ from both sides, it is $4a^2x^2=9a^4 - 12a^2x^2$;
Then transposing $12a^2x^2$, gives $16a^2x^2=9a^4$;
Dividing by a^2 gives $16x^2 = 9a^2$;
And dividing by 16, gives $x^2=\frac{9}{16}a^2$;
Lastly, extracting the root, gives $x=\frac{3}{4}a$.

EXAMPLES FOR PRACTICE.

1. Given $2x - 5 + 16 = 21$; to find x. Ans. $x=5$.
2. Given $6x - 15 = x + 6$; to find x. Ans. $x=4\frac{1}{5}$.
5. Given $8-3x+12=30-5x+4$; to find x. Ans. $x=7$.
4. Given $x + \frac{1}{3}x + \frac{1}{4}x=13$; to find x. Ans. $x=12$.
5. Given $3x+\frac{1}{2}x+2=5x-4$; to find x. Ans. $x=4$.
6. Given $4ax+\frac{1}{3}a-2=ax-bx$; to find x. Ans. $x = \frac{6-a}{9a+3b}$.
7. Given $\frac{1}{3}x - \frac{1}{4}x + \frac{1}{5}x = \frac{1}{2}$; to find x. Ans. $x = \frac{30}{17}$.
8. Given $\sqrt{(4+x)}=4-\sqrt{x}$; to find x. Ans. $x = 2\frac{1}{4}$.
9. Given $4a + x = \frac{x^2}{4a+x}$; to deter. x. Ans. $x = -2a$.
10. Given $\sqrt{(4a^2+ x^2)}= \sqrt[4]{(4b^4 + x^4)}$; to find x. Ans. $x = \sqrt{\frac{b^4-4a^4}{2a^2}}$.
11. Given $\sqrt{x} + \sqrt{(2a+x)} = \frac{4a}{\sqrt{(2a+x)}}$; to find x. Ans. $x = \frac{2}{3}a$.
12. Given $\frac{a}{1+2x} + \frac{a}{1-2x} = 2b$; to find x. Ans. $x=\frac{1}{2}\sqrt{\frac{b-a}{b}}$.
13. Given $a+x= \sqrt{(a^2+x\sqrt{(4b^2+x^2)})}$; to find x. Ans. $x = \frac{b^2}{a} - a$.

OF REDUCING DOUBLE, TRIPLE, &C. EQUATIONS, CONTAINING TWO, THREE, OR MORE UNKNOWN QUANTITIES.

PROBLEM I.

To exterminate two Unknown Quantities; Or, to reduce the two Simple Equations containing them, to a Single one.

RULE I.

Find the value of one of the unknown letters, in terms of the other quantities, in each of the equations, by the methods already explained. Then put those two values equal to each other for a new equation, with only one unknown quantity in it, whose value is to be found as before.

Note. It is evident that we must first begin to find the values of that letter which is easiest to be found in the two proposed equations.

EXAMPLES.

1. Given $\left\{\begin{matrix} 2x + 3y = 17 \\ 5x - 2y = 14 \end{matrix}\right\}$; to find x and y.

In the 1st equat. trans. $3y$ and div. by 2, gives $x = \frac{17-3y}{2}$;

In the 2d trans. $2y$ and div. by 5, gives $x = \frac{14+2y}{5}$;

Putting these two values equal, gives $\frac{14+2y}{5} = \frac{17-3y}{2}$;

Then mult. by 2 and 5, gives $28 + 4y = 85 - 15y$;
Transposing 28 and $15y$, gives $19y = 57$;
And dividing by 19, gives $y = 3$.
And hence $x = 4$.

Or, effect the same by finding two values of y, thus

In the 1st equat. tr. $2x$ and div. by 3, gives $y = \frac{17-2x}{2}$;

In the 2d tr. $2y$ and 14, and div. by 2, gives $y = \frac{5x-14}{2}$;

Putting these two values equal, gives $\frac{5x-14}{2} = \frac{17-2x}{3}$,

Mult. by 2 and by 3, gives $15x - 42 = 34 - 4x$;
Transp. 42 and $4x$, gives $19x = 76$;
Dividing by 19, gives $x = 4$.
Hence $y = 3$, as before.

2. Given $\left\{ \begin{matrix} \frac{1}{2}x + 2y = a \\ \frac{1}{2}x - 2y = b \end{matrix} \right\}$; to find x and y.

Ans. $x = a + b$, and $y = \frac{1}{4}a - \frac{1}{4}b$.

3. Given $3x + y = 22$, and $3y + x = 18$; to find x and y.

Ans. $x = 6$, and $y = 4$.

4. Given $\left\{ \begin{matrix} \frac{1}{2}x + \frac{1}{3}y = 4 \\ \frac{1}{3}x + \frac{1}{2}y = 3\frac{1}{2} \end{matrix} \right\}$; to find x and y.

Ans. $x = 6$, and $y = 3$.

5. Given $\frac{2x}{3} + \frac{3y}{5} = \frac{22}{5}$ and $\frac{3x}{5} + \frac{2y}{3} = \frac{67}{15}$; to find x and y.

Ans. $x = 3$, and $y = 4$.

6. Given $x + 2y = s$, and $x^2 - 4y^2 = d^2$; to find x and y.

Ans. $x = \frac{s^2 + d^2}{2s}$, and $y = \frac{s^2 - d^2}{4s}$.

7. Given $x - 2y = d$, and $x : y :: a : b$; to find x and y.

Ans. $x = \frac{ad}{a - 2b}$, and $y = \frac{bd}{a - 2b}$.

RULE II.

Find the value of one of the unknown letters, in only one of the equations, as in the former rule; and substitute this value instead of that unknown quantity in the other equation, and there will arise a new equation, with only one unknown quantity, whose value is to be found as before.

Note. It is evident that it is best to begin first with that letter whose value is easiest found in the given equations.

EXAMPLES.

1. Given $\left\{ \begin{matrix} 2x + 3y = 17 \\ 5x - 2y = 14 \end{matrix} \right\}$; to find x and y.

This will admit of four ways of solution; thus; First, in the 1st eq. trans. $3y$ and div. by 2, gives $x = \frac{17 - 3y}{2}$.

This val. subs. for x in the 2d, gives $\frac{85 - 15y}{2} - 2y = 14$;

Mult. by 2, this becomes $85 - 15y - 4y = 28$;

Transp. $15y$ and $4y$ and 28, gives $57 = 19y$;
And dividing by 19, gives $3 = y$.

Then $x = \frac{17 - 3y}{2} = 4$.

2dly, in the 2d trans. $2y$ and div. by 5, gives $x = \frac{14 + 2y}{5}$;

This subst. for x in the 1st, gives $\frac{28 + 4y}{5} + 3y = 17$;

Mult. by 5, gives $28 + 4y + 15y = 85$;
Transpo. 28, gives $19y = 57$;
And dividing by 19, gives $y = 3$.

Then $x = \frac{14 + 2y}{5} = 4$, as before.

3dly, in the 1st trans. $2x$ and div. by 3, gives $y = \frac{17 - 2x}{3}$;

This subst. for y in the 2d, gives $5x - \frac{34 - 4x}{3} = 14$;

Multiplying by 3, gives $15x - 34 + 4x = 42$;
Transposing 34, gives $19x = 76$;
And dividing by 19, gives $x = 4$.

Hence $y = \frac{17 - 2x}{3} = 3$, as before.

4thly, in the 2d tr. $2y$ and 14 and div. by 2, gives $y = \frac{5x - 14}{2}$;

This substituted in the 1st, gives $2x + \frac{15x - 42}{2} = 17$;

Multiplying by 2, gives $19x - 42 = 34$;
Transposing 42, gives $19x = 76$;
And dividing by 19, gives $x = 4$;

Hence $y = \frac{5x - 14}{2} = 3$, as before.

2. Given $2x + 3y = 29$, and $3x - 2y = 11$: to find x and y. Ans. $x = 7$, and $y = 5$.

3. Given $\left\{ \begin{array}{l} x + y = 14 \\ x - y = 2 \end{array} \right\}$; to find x and y.

Ans. $x = 8$, and $y = 6$.

4. Given $\left\{\begin{array}{l} x : y :: 3 : 2 \\ x^2 - y^2 = 20 \end{array}\right\}$; to find x and y.
Ans. $x = 6$, and $y = 4$.

5. Given $\frac{x}{3} + 3y = 21$, and $\frac{y}{3} + 3x = 29$; to find x and y.
Ans. $x = 9$, and $y = 6$.

6. Given $10 - \frac{x}{2} = \frac{y}{3} + 4$, and $\frac{x-y}{2} + \frac{x}{4} - 2 = \frac{3y - x}{5} - 1$; to find x and y.
Ans. $x = 8$, and $y = 6$.

7. Given $x : y :: 4 : 3$, and $x^3 - y^3 = 37$; to find x and y.
Ans. $x = 4$, and $y = 3$.

RULE III.

Let the given equations be so multiplied, or divided, &c. and by such numbers or quantities, as will make the terms which contain one of the unknown quantities the same in both equations; if they are not the same when first proposed.

Then by adding or subtracting the equations, according as the signs may require, there will result a new equation, with only one unknown quantity, as before. That is, add the two equations when the signs are unlike, but subtract them when the signs are alike, to cancel that common term.

Note. To make two unequal terms become equal, as above, multiply each term by the co-efficient of the other.

EXAMPLES.

Given $\left\{\begin{array}{l} 5x - 3y = 9 \\ 2x + 5y = 16 \end{array}\right\}$; to find x and y.

Here we may either make the two first terms, containing x, equal, or the two 2d terms, containing y, equal. To make the two first terms equal, we must multiply the 1st equation by 2, and the 2d by 5; but to make the two 2d terms equal, we must multiply the 1st equation by 5, and the 2d by 3; as follows.

1. By making the two first terms equal:

Mult. the 1st equ. by 2, gives	$10x - 6y = 18$;
And mult. the 2d by 5, gives	$10x + 25y = 80$;
Subtr. the upper from the under, gives	$31y = 62$;
And dividing by 31, gives	$y = 2$.

Hence, from the 1st given equ. $x = \frac{9+3y}{5} = 3$.

2. By making the two 2d terms equal:

Mult. the 1st equat. by 5, gives	$25x - 15y = 45$;
And mult. the 2d by 3, gives	$6x + 15y = 48$;
Adding these two, gives	$31x = 93$;
And dividing by 31, gives	$x = 3$.

Hence, from the 1st equ. $y = \frac{5x-9}{3} = 2$.

MISCELLANEOUS EXAMPLES.

1. Given $\frac{x+8}{4} + 6y = 21$, and $\frac{y+6}{3} + 5x = 23$; to find x and y. Ans. $x = 4$, and $y = 3$.

2. Given $\frac{3x-y}{4} + 10 = 13$, and $\frac{3y+x}{2} + 5 = 12$; to find x and y. Ans. $x = 5$, and $y = 3$.

3. Given $\frac{3x+4y}{5} + \frac{x}{4} = 10$, and $\frac{6x-2y}{3} + \frac{y}{6} = 14$; to find x and y. Ans. $x = 8$, and $y = 4$.

4. Given $3x+4y = 38$, and $4x - 3y = 9$; to find x and y. Ans. $x = 6$, and $y = 5$.

PROBLEM III.

To exterminate three or more Unknown Quantities; Or, to reduce the simple Equations, containing them, to a Single one.

RULE.

This may be done by any of the three methods in the last problem: viz.

1. After the manner of the first rule in the last problem, find the value of one of the unknown letters in each of the given equations; next put two of these values equal to each other, and then one of these and a third value equal, and so on for all the values of it; which gives a new set of equations,

with which the same process is to be repeated, and so on till there is only one equation, to be reduced by the rules for a single equation.

2. Or, as in the 2d rule of the same problem, find the value of one of the unknown quantities in one of the equations only; then substitute this value instead of it in the other equations; which gives a new set of equations to be resolved as before, by repeating the operation.

3. Or, as in the 3d rule, reduce the equations, by multiplying or dividing them, so as to make some of the terms to agree: then, by adding or subtracting them, as the signs may require, one of the letters may be exterminated, &c. as before.

EXAMPLES.

1. Given $\left\{\begin{array}{l} x+y+z=9 \\ x+2y+3z=16 \\ x+3y+4z=21 \end{array}\right\}$; to find x, y, and z.

1. By the 1st method:

Transp. the terms containing y and z in each equa. gives

$$x = 9 - y - z,$$
$$x = 16 - 2y - 3z,$$
$$x = 21 - 3y - 4z;$$

Then putting the 1st and 2d values equal, and the 2d and 3d values equal, give

$$9 - y - z = 16 - 2y - 3z,$$
$$16 - 2y - 3z = 21 - 3y - 4z;$$

In the 1st trans. 9, z, and $2y$, gives $y = 7 - 2z$;
In the 2d trans. 16, $3z$, and $3y$, gives $y = 5 - z$;
Putting these two equal, gives $5 - z = 7 - 2z$.
Trans. 5 and $2z$, gives $z = 2$.
Hence $y = 5 - z = 3$, and $x = 9 - y - z = 4$.

2dly. By the 2d method:

From the 1st equa. $x = 9 - y - z$;
This value of x substit. in the 2d and 3d, gives

$$9 + y + 2z = 16,$$
$$9 + 2y + 3z = 21;$$

In the 1st trans. 9 and $2z$, gives $y = 7 - 2z$;
This substit. in the last, gives $23 - z = 21$;
Trans. z and 21, gives $2 = z$.
Hence again $y = 7 - 2z = 3$, and $x = 9 - y - z = 4$.

3dly. By the 3d method: subtracting the 1st equ. from the 2d, and the 2d from the 3d, gives

$$y + 2z = 7,$$
$$y + z = 5;$$

Subtr. the latter from the former, gives $z = 2$.
Hence $y = 5 - z = 3$, and $x = 9 - y - z = 4$.

2. Given $\begin{cases} x + y + z = 18 \\ x + 3y + 2z = 38 \\ x + \frac{1}{2}y + \frac{1}{3}z = 10 \end{cases}$; to find x, y, and z.
Ans. $x = 4, y = 6, z = 8$.

3. Given $\begin{cases} x + \frac{1}{2}y + \frac{1}{3}z = 27 \\ x + \frac{1}{3}y + \frac{1}{4}z = 20 \\ x + \frac{1}{4}y + \frac{1}{5}z = 16 \end{cases}$; to find x, y, and z.
Ans. $x = 1, y = 12, z = 60$.

4. Given $x - y = 2, x - z = 3$, and $y + z = 9$; to find x, y, and z. Ans. $x = 7, y = 5, z = 4$.

5. Given $\begin{cases} 2x + 3y + 4z = 34 \\ 3x + 4y + 5z = 46 \\ 2x + 6y + 8z = 58 \end{cases}$; to find x, y, and z.

6. Given $\begin{cases} x(x + y + z) = 45 \\ y(x + y + z) = 70 \\ z(x + y + z) = 105 \end{cases}$; to find x, y, and z.

A COLLECTION OF QUESTIONS PRODUCING SIMPLE EQUATIONS.

QUEST. 1. To find two numbers, such, that their sum shall be 10, and their difference 6.

Let x denote the greater number, and y the less*.
Then, by the 1st condition $x + y = 10$,
And by the 2d . . . $x - y = 6$,
Transp. y in each, gives $x = 10 - y$,
and $x = 6 + y$;
Put these two values equal, gives $6 + y = 10 - y$;
Transpos. 6 and $-y$, gives . $2y = 4$;
Dividing by 2, gives . . $y = 2$.
And hence . . . . . $x = 6 + y = 8$.

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* In these solutions, as many unknown letters are always used as there are unknown numbers to be found, purposely for exercise in the modes of reducing the equations: avoiding the short ways of notation, which, though they may give neater solutions, afford less exercise in practising the several rules in reducing equations.

QUEST. 2. Divide 100*l* among A, B, C, so that A may have 20*l* more than B, and B 10*l* more than C.

Let $x =$ A's share, $y =$ B's, and $z =$ C's.
Then $x + y + z = 100$,
$x = y + 20$,
$y = z + 10$.
In the 1st substit. $y + 20$ for x, gives $2y + z + 20 = 100$;
In this substituting $z + 10$ for y, gives $3z + 40 = 100$;
By transposing 40, gives . . $3z = 60$;
And dividing by 3, gives . . $z = 20$.
Hence $y = z + 10 = 30$, and $x = y + 20 = 50$.

QUEST. 3. A prize of 500*l* is to be divided between two persons, so as their shares may be in proportion as 7 to 8; required the share of each.

Put x and y for the two shares; then by the question,
$7 : 8 :: x : y$, or mult. the extremes.
and the means, $7y = 8x$,
and $x + y = 500$;
Transposing y, gives $x = 500 - y$;
This substituted in the 1st, gives $7y = 4000 - 8y$;
By transposing $8y$, it is $15y = 4000$;
By dividing by 15, it gives $y = 266\frac{2}{3}$;
And hence $x = 500 - y = 233\frac{1}{3}$.

QUEST. 4. What fraction is that, to the numerator of which if 1 be added, the value will be $\frac{1}{2}$; but if 1 be added to the denominator, its value will be $\frac{1}{3}$?

Let $\frac{x}{y}$ denote the fraction.

Then by the quest. $\frac{x + 1}{y} = \frac{1}{2}$, and $\frac{x}{y + 1} = \frac{1}{3}$.

The 1st mult. by 2 and y, gives $2x + 2 = y$;
The 2d mult. by 3 and $y + 1$, is $3x = y + 1$;
The upper taken from the under leaves $x - 2 = 1$;
By transpos. 2, it gives $x = 3$.
And hence $y = 2x + 2 = 8$; and the fraction is $\frac{3}{8}$.

QUEST. 5. A labourer engaged to serve for 30 days on these conditions: that for every day he worked, he was to receive 20*d*, but for every day he played, or was absent, he was to forfeit 10*d*. Now at the end of the time he had to receive just 20 shillings, or 240 pence. It is required to find how many days he worked, and how many he was idle?

Let x be the days worked, and y the days idled.
Then $20x$ is the pence earned, and $10y$ the forfeits;
Hence, by the question - $x + y = 30$,
and $20x - 10y = 240$;
The 1st mult. by 10, gives $10x + 10y = 300$;
These two added, give - $30x = 540$;
This div. by 30, gives - $x = 18$, the days worked;
Hence - $y = 30 - x = 12$, the days idle.

QUEST. 6. Out of a cask of wine which had leaked away $\frac{1}{4}$, 30 gallons were drawn; and then, being guaged, it appeared to be half full; how much did it hold?

Let it be supposed to have held x gallons,
Then it would have leaked $\frac{1}{4}x$ gallons,
Conseq. there had been taken away $\frac{1}{4}x + 30$ gallons.
Hence $\frac{1}{2}x = \frac{1}{4}x + 30$ by the question.
Then mult. by 4, gives $2x = x + 120$;
And transposing x, gives $x = 120$ the gallons it held.

QUEST. 7. To divide 20 into two such parts, that 3 times the one part added to 5 times the other may make 76.

Let x and y denote the two parts.
Then by the question - - $x + y = 20$,
and $3x + 5y = 76$.
Mult. the 1st by 3, gives - $3x + 3y = 60$;
Subtr. the latter from the former, gives $2y = 16$;
And dividing by 2, gives - - $y = 8$.
Hence, from the 1st, - $x = 20 - y = 12$.

QUEST. 8. A market woman bought in a certain number of eggs at 2 a penny, and as many more at 3 a penny, and sold them all out again at the rate of 5 for two-pence, and by so doing, contrary to expectation, found she lost 3*d*; what number of eggs had she?

Let x = number of eggs of each sort,
Then will $\frac{1}{2}x$ = cost of the first sort,
And $\frac{1}{3}x$ = cost of the second sort;
But $5 : 2 :: 2x$ (the whole number of eggs) : $\frac{4}{5}x$;
Hence $\frac{4}{5}x$ = price of both sorts, at 5 for 2 pence;
Then by the question $\frac{1}{2}x + \frac{1}{3}x - \frac{4}{5}x = 3$;
Mult. by 2, gives - $x + \frac{2}{3}x - \frac{8}{5}x = 6$;
And mult. by 3, gives $5x - \frac{24}{5}x = 18$;
Also mult. by 5, gives $x = 90$, the number of eggs of each sort.

QUEST. 9. Two persons, A and B, engage at play. Before they begin, A has 80 guineas, and B has 60. After a certain number of games won and lost between them, A rises with three times as many guineas as B. Query, how many guineas did A win of B?

Let x denote the number of guineas A won.
Then A rises with $80 + x$,
And B rises with $60 - x$;
Theref. by the quest. $80 + x = 180 - 3x$;
Transp. 80 and $3x$, gives $4x = 100$;
And dividing by 4, gives $x = 25$, the guineas won.

QUESTIONS FOR PRACTICE.

1. To determine two numbers such, that their difference may be 4, and the difference of their squares 64.
Ans. 6 and 10.

2. To find two numbers with these conditions, viz. that half the first with a third part of the second may make 9, and that a 4th part of the first with a 5th part of the second may make 5. Ans. 8 and 15.

3. To divide the number 20 into two such parts, that a 3d of the one part added to a 5th of the other, may make 6.
Ans. 15 and 5.

4. To find three numbers such, that the sum of the 1st and 2d shall be 7, the sum of the 1st and 3d 8, and the sum of the 2d and 3d 9. Ans. 3, 4, 5.

5. A father, dying, bequeathed his fortune, which was 2800*l*, to his son and daughter, in this manner; that for every half crown the son might have, the daughter was to have a shilling. What then were their two shares?
Ans. The son 2000*l* and the daughter 800*l*.

6. Three persons, A, B, C, make a joint contribution, which in the whole amounts to 400*l*: of which sum B contributes twice as much as A and 20*l* more; and C as much as A and B together. What sum did each contribute?
Ans. A 60*l*, B 140*l*, and C 200*l*.

7. A person paid a bill of 100*l* with half guineas and crowns, using in all 202 pieces; how many pieces were there of each sort? Ans. 180 half guineas, and 22 crowns.

8. Says A to B, if you give me 10 guineas of your money, I shall then have twice as much as you will have left: but says B to A, give me 10 of your guineas, and then I shall have 3 times as many as you. How many had each?

Ans. A 22, B 26.

9. A person goes to a tavern with a certain quantity of money in his pocket, where he spends 2 shillings; he then borrows as much money as he had left, and going to another tavern, he there spends 2 shillings also; then borrowing again as much money as was left, he went to a third tavern, where likewise he spent 2 shillings; and thus repeating the same at a fourth tavern, he then had nothing remaining. What sum had he at first? Ans. 3*s*. 9*d*.

10. A man with his wife and child dine together at an inn. The landlord charged 1 shilling for the child; and for the woman he charged as much as for the child and $\frac{1}{4}$ as much as for the man; and for the man he charged as much as for the woman and child together. How much was that for each? Ans. The woman 20*d* and the man 32*d*.

11. A cask, which held 60 gallons, was filled with a mixture of brandy, wine, and cyder, in this manner, viz. the cyder was 6 gallons more than the brandy, and the wine was as much as the cyder and $\frac{1}{5}$ of the brandy. How much was there of each?

Ans. Brandy 15, cyder 21, wine 24.

12. A general, disposing his army into a square form, finds that he has 284 men more than a perfect square; but increasing the side by 1 man, he then wants 25 men to be a complete square. How many men had he under his command? Ans. 24000.

13. What number is that, to which if 3, 5, and 8, be severally added, the three sums shall be in geometrical progression? Ans. 1.

14. The stock of three traders amounted to 760*l*: the shares of the first and second exceeded that of the third by 240: and the sum of the 2d and 3d exceeded the first by 360. What was the share of each?

Ans. The 1st 200, the 2d 300, the 3d 260.

15. What two numbers are those, which, being in the ratio of 3 to 4, their product is equal to 12 times their sum?

Ans. 21 and 28.

16. A certain company at a tavern, when they came to settle their reckoning, found that had there been 4 more in company, they might have paid a shilling each *less* than they did; but that if there had been 3 fewer in company, they must have paid a shilling each *more* than they did. What then was the number of persons in company, what each paid, and what was the whole reckoning?

Ans. 24 persons, each paid 7*s*, and the whole reckoning 8 guineas.

17. A jockey has two horses: and also two saddles, the one valued at 18*l*. the other at 3*l*. Now when he sets the better saddle on the 1st horse, and the worse on the 2d, it makes the first horse worth double the 2d; but when he places the better saddle on the 2d horse, and the worse on the first, it makes the 2d horse worth three times the 1st. What then were the values of the two horses?

Ans. The 1st 6*l*, and the 2d 9*l*.

18. What two numbers are as 2 to 3, to each of which if 6 be added, the sums will be as 4 to 5? Ans. 6 and 9.

19. What are those two numbers, of which the greater is to the less as their sum is to 20, and as their difference is to 10? Ans. 15 and 45.

20. What two numbers are those, whose difference, sum, and product, are to each other, as the three numbers 2, 3, 5? Ans. 2 and 10.

21. To find three numbers in arithmetical progression, of which the first is to the third as 5 to 9, and the sum of all three is 63. Ans. 15, 21, 27.

22. It is required to divide the number 24 into two such parts, that the quotient of the greater part divided by the less, may be to the quotient of the less part divided by the greater, as 4 to 1. Ans. 16 and 8.

23. A gentleman being asked the age of his two sons, answered, that if to the sum of their ages 18 be added, the result will be double the age of the elder; but if 6 be taken from the difference of their ages, the remainder will be equal to the age of the younger. What then were their ages?

Ans. 30 and 12.

24. To find four numbers such, that the sum of the 1st, 2d, and 3d shall be 13; the sum of the 1st, 2d, and 4th, 15; the sum of the 1st, 3d, and 4th, 18; and lastly, the sum of the 2d, 3d, and 4th, 20. Ans. 2, 4, 7, 9.

25. To divide 48 into 4 such parts, that the first increased by 3, the second diminished by 3, the third multiplied by 3, and the 4th divided by 3, may be all equal to each other.

Ans. 6, 12, 3, 27.

QUADRATIC EQUATIONS.

Quadratic Equations are either simple or compound.

A simple quadratic equation, is that which involves the square only of the unknown quantity. As $ax^2 = b$. The solution of such quadratics has been already given in simple equations.

A compound quadratic equation, is that which contains the square of the unknown quantity in one term, and the first power in another term. As $ax^2 + bx = c$.

All compound quadratic equations, after being properly reduced, fall under the three following forms, to which they must always be reduced by preparing them for solution.

1. $x^2 + ax = b$
2. $x^2 - ax = b$
3. $x^2 - ax = -b$

The general method of solving quadratic equations, is by what is called completing the square, which is as follows :

1. Reduce the proposed equation to a proper simple form, as usual, such as the forms above ; namely, by transposing all the terms which contain the unknown quantity to one side of the equation, and the known terms to the other ; placing the square term first, and the single power second ; dividing the equation by the co-efficient of the square or first term, if it has one, and changing the signs of all the terms, when that term happens to be negative, as that term must always be made positive before the solution. Then the proper solution is by completing the square as follows, viz.

2. Complete the unknown side to a square, in this manner, viz. Take half the co-efficient of the second term, and square it ; which square add to both sides of the equation, then that side which contains the unknown quantity will be a complete square.

3. Then extract the square root on both sides of the equation*, and the value of the unknown quantity will be determined, making the root of the known side either + or −, which will give two roots of the equation, or two values of the unknown quantity.

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* As the square root of any quantity may be either + or —, therefore all quadratic equations admit of two solutions. Thus, the square root of $+n^2$ is either $+n$ or $-n$; for $+n \times +n$ and $-n \times -n$ are each equal to $+n^2$. But the square root of $-n^2$, or $\sqrt{-n^2}$ is imaginary or impossible, as neither $+n$ nor $-n$, when squared, gives $-n^2$.

So, in the first form, $x^2+ax=b$, where $x+\frac{1}{2}a$ is found $=\sqrt{(b+\frac{1}{4}a^2)}$, the root may be either $+\sqrt{(b+\frac{1}{4}a^2)}$, or $-\sqrt{(b+\frac{1}{4}a^2)}$, since either of them being multiplied by itself produces $b+\frac{1}{4}a^2$. And this ambiguity is expressed by writing the uncertain or double sign $\pm$ before $\sqrt{(b+\frac{1}{4}a^2)}$; thus $x=\pm\sqrt{(b+\frac{1}{4}a^2)}-\frac{1}{2}a$.

In this form, where $x=\pm\sqrt{(b+\frac{1}{4}a^2)}-\frac{1}{2}a$, the first value of x, viz. $x=+\sqrt{(b+\frac{1}{4}a^2)}-\frac{1}{2}$, is always affirmative; for since $\frac{1}{4}a^2+b$ is greater than $\frac{1}{4}a^2$, the greater square must necessarily have the greater root; therefore $\sqrt{(b+\frac{1}{4}a^2)}$, will always be greater than $\sqrt{\frac{1}{4}a^2}$, or its equal $\frac{1}{2}a$; and consequently $+\sqrt{(b+\frac{1}{4}a^2)}-\frac{1}{2}a$ will always be affirmative.

The second value, viz. $x=-\sqrt{(b+\frac{1}{4}a^2)}-\frac{1}{2}a$ will always be negative, because it is composed of two negative terms. Therefore when $x^2+ax=b$, we shall have $x=+\sqrt{(b+\frac{1}{4}a^2)}-\frac{1}{2}a$ for the affirmative value of x, and $x=-\sqrt{(b+\frac{1}{4}a^2)}-\frac{1}{2}a$ for the negative value of x.

In the second form, where $x=\pm\sqrt{(b+\frac{1}{4}a^2)}+\frac{1}{2}a$ the first value, viz. $x=+\sqrt{(b+\frac{1}{4}a^2)}+\frac{1}{2}a$ is always affirmative, since it is composed of two affirmative terms. But the second value, viz. $x=-\sqrt{(b+\frac{1}{4}a^2)}+\frac{1}{2}a$, will always be negative; for since $b+\frac{1}{4}a^2$ is greater than $\frac{1}{4}a^2$, therefore $\sqrt{(b+\frac{1}{4}a^2)}$ will be greater than $\sqrt{\frac{1}{4}a^2}$, or its equal $\frac{1}{2}a$; and consequently $-\sqrt{(b+\frac{1}{4}a^2)}+\frac{1}{2}a$ is always a negative quantity.

Therefore, when $x^2-ax=b$, we shall have $x=+\sqrt{(b+\frac{1}{4}a^2)}+\frac{1}{2}a$ for the affirmative value of x; and $x=-\sqrt{(b+\frac{1}{4}a^2)}+\frac{1}{2}a$ for the negative value of x; so that in both the first and second forms, the unknown quantity has always two values, one of which is positive, and the other negative.

But, in the third form, where $x=\pm\sqrt{(\frac{1}{4}a^2-b)}+\frac{1}{2}a$, both the values of x will be positive, when $\frac{1}{4}a^2$ is greater than b. For the first value, viz. $x=+\sqrt{(\frac{1}{4}a^2-b)}+\frac{1}{2}a$ will then be affirmative, being composed of two affirmative terms.

The second value, viz. $x=-\sqrt{(\frac{1}{4}a^2-b)}+\frac{1}{2}a$ is affirmative also; for since $\frac{1}{4}a^2$ is greater than $\frac{1}{4}a^2-b$, therefore $\sqrt{\frac{1}{4}a^2}$ or $\frac{1}{2}a$ is greater than $\sqrt{(\frac{1}{4}a^2-b)}$; and consequently $-\sqrt{(\frac{1}{4}a^2-b)}+\frac{1}{2}a$ will always be an affirmative quantity. So that, when $x^2-ax=-b$, we shall have $x=+\sqrt{(\frac{1}{4}a^2-b)}+\frac{1}{2}a$. and also $x=-\sqrt{(\frac{1}{4}a^2-b)}+\frac{1}{2}a$, for the values of x, both positive.

But in this third form, if b be greater than $\frac{1}{4}a^2$, the solution of the proposed question will be impossible. For since the square of any quantity (whether that quantity be affirmative or negative) is always affirmative, the square root of a negative quantity is impossible, and cannot be assigned. But when b is greater than $\frac{1}{4}a^2$, then $\frac{1}{4}a^2-b$ is a negative quantity; and therefore its root $\sqrt{(\frac{1}{4}a^2-b)}$ is impossible, or imaginary; consequently, in that case, $x=\frac{1}{2}a\pm\sqrt{(\frac{1}{4}a^2-b)}$, or the two roots or values of x, are both impossible, or imaginary quantities.

Note, 1. The root of the first side of the equation, is always equal to the root of the first term, with half the coefficient of the second term joined to it, with its sign, whether $+$ or $-$.

2. All equations, in which there are two terms including the unknown quantity, and which have the index of the one just double that of the other, are resolved like quadratics, by completing the square, as above.

Thus, $x^4 + ax^2 = b$, or $x^{2n} + ax^n = b$, or $x + ax^{\frac{1}{2}} = b$, or $(x^2 \pm ax)^2 \pm m(x^2 \pm ax) = b$, are analagous to quadratics, and the value of the unknown quantity may be determined accordingly.

3. For the construction of Quadratics, see vol. ii.

EXAMPLES.

1. Given $x^2 + 4x = 60$; to find x.
   First, by completing the square, $x^2 + 4x + 4 = 64$;
   Then, by extracting the root, $x + 2 = \pm 8$;
   Then, transpos. 2, gives $x = 6$ or -10, the two roots.

2. Given $x^2 - 6x + 10 = 65$; to find x.
   First, trans. 10, gives $x^2 - 6x = 55$;
   Then by complet. the sq. it is $x^2 - 6x + 9 = 64$;
   And by extr. the root, gives $x - 3 = \pm 8$;
   Then trans. 3, gives $x = 11$ or -5.

3. Given $3x^2 - 3x + 9 = 8\frac{1}{3}$; to find x.
   First div. by 3, gives $x^2 - x + 3 = 2\frac{7}{9}$;
   Then transpos. 3, gives $x^2 - x = -\frac{2}{9}$;
   And compl. the sq. gives $x^2 - x + \frac{1}{4} = \frac{1}{36}$;
   Then extr. the root gives $x - \frac{1}{2} = \pm \frac{1}{6}$;
   And transp. $\frac{1}{2}$, gives $x = \frac{2}{3}$ or $\frac{1}{3}$.

4. Given $\frac{1}{2}x^2 - \frac{1}{3}x + 30\frac{1}{2} = 52\frac{2}{3}$; to find x.
   First by transpos. $30\frac{1}{2}$, it is $\frac{1}{2}x^2 - \frac{1}{3}x = 22\frac{1}{6}$;
   Then mult. by 2 gives $x^2 - \frac{2}{3}x = 44\frac{1}{3}$;
   And by compl. the sq. it is $x^2 - \frac{2}{3}x + \frac{1}{9} = 44\frac{4}{9}$.
   Then extr. the root gives $x - \frac{1}{3} = \pm 6\frac{2}{3}$;
   And trans. $\frac{1}{3}$, gives $x = 7$ or $-6\frac{1}{3}$;

5. Given $ax^2 - bx = c$; to find x.

   First by div. by a, it is $x^2 - \dfrac{b}{a}x = \dfrac{c}{a}$;

Then compl. the sq. gives $x^2 - \frac{b}{a}x + \frac{b}{4a^2} = \frac{c}{a} + \frac{b^2}{4a^2}$;

And extrac. the root, gives $x - \frac{b}{2a} = \pm\sqrt{\frac{4ac+b^2}{4a^2}}$;

Then transp. $\frac{b}{2a}$, gives $x = \pm\sqrt{\frac{4ac+b^2}{4a^2}} + \frac{b}{2a}$.

6. Given $x^4 - 2ax^2 = b$; to find x.

First by compl. the sq. gives $x^4 - 2ax^2 + a^2 = a^2 + b$;
And extract. the root, gives $x^2 - a = \pm\sqrt{(a^2-b)}$;
Then transpos. a, gives $x^2 = \pm\sqrt{(a^2+b)} + a$;
And extract. the root, gives $x = \pm\sqrt{[a \pm \sqrt{(a^2+b)}]}$.

EXAMPLES FOR PRACTICE*.

1. Given $x^2 - 6x - 7 = 33$; to find x. Ans. $x = 10$ or -4.

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* 1. Cubic equations, when occurring in pairs, may usually be reduced to quadratics, by extermination. Thus,

Suppose $4x^3 + 3x^2 + 5x = 150$
and $3x^3 + 2x^2 + 2x = 105$

Then mult. 1st equa. by 3, and 2d by 4,

$12x^3 + 9x^2 + 15x = 450$
$12x^3 + 8x^2 + 8x = 420$

By subtr. $x^2 + 7x = 30$
Compl. the sq. $x^2 + 7x + \frac{49}{4} = 30 + \frac{49}{4} = \frac{169}{4}$
Extr. the root $x + \frac{7}{2} = \pm\frac{13}{2}$
$x = \pm\frac{13}{2} - \frac{7}{2} = 3$ or -10.

2. Sometimes, when the unknown square has a co-efficient, the following method may be advantageously adopted: viz.

Having transposed the known terms to one side and the unknown terms to the other, multiply each side by 4 times the co-efficient of the unknown square.

Add the square of the co-efficient of the simple power of the unknown quantity, to both sides; the first side will then be a complete square.

Extract the root, and the value of the unknown quantity will be obtained.

Thus, if $5x^2 + 4x = 28$.

Then mult. by 4×5, $100x^2 + 80x = 560$
Add 4^2. . $100x^2 + 80x + 16 = 576$
Extr. the root, $10x + 4 = \pm 24$
Transposing . $10x = 20$ or -28
Dividing by 10, $x = 2$, or $-2{\cdot}8$.

The principal advantage of this method, which is due to the Indians, is that it does not introduce *fractions* into the operation. It will have the same advantage in cases where the square has no co-efficient, if that of the simple power be an *odd* number.

2. Given $x^2 - 5x - 10 = 14$; to find x. Ans. $x = 8$ or -3.

3. Given $5x^2 + 4x - 90 = 114$; to find x.
Ans. $x = 6$ or $-6\frac{4}{5}$.

4. Given $\frac{1}{2}x^2 - \frac{1}{4}x + 2 = 9$; to find x. Ans. $x = 4$ or $-3\frac{1}{2}$.

5. Given $3x^4 - 2x^2 = 40$; to find x. Ans. $x = 2$ or -2.

6. Given $\frac{1}{3}x - \frac{1}{2}\sqrt{x} = 1\frac{1}{2}$; to find x. Ans. $x = 9$ or $2\frac{1}{4}$.

7. Given $\frac{1}{2}x^2 + \frac{2}{3}x = \frac{3}{4}$; to find x.
Ans. $x = -\frac{2}{3} \pm \frac{1}{6}\sqrt{70} = \cdot7277668$ or $-2\cdot0611000$.

8. Given $x^6 + 4x^3 = 12$; to find x.
Ans. $x = \sqrt[3]{2} = 1\cdot259921$, or $\sqrt[3]{-6} = -1\cdot817121$.

9. Given $x^2 + 4x = a^2 + 2$; to find x.
Ans. $x = \sqrt{(a^2 + 6)} - 2$.

QUESTIONS PRODUCING QUADRATIC EQUATIONS.

1. To find two numbers whose difference is 2, and product 80.

Let x and y denote the two required numbers.
Then the first condition gives $x - y = 2$,
And the second gives $xy = 80$.
Then transp. y in the 1st gives $x = y + 2$;
This value of x substitut. in the 2d, is $y^2 + 2y = 80$;
Then comp. the square gives $y^2 + 2y + 1 = 81$;
And extrac. the root gives $y + 1 = 9$;
And transpos. 1 gives $y = 8$;
And therefore $x = y + 2 = 10$.

2. To divide the number 14 into two such parts, that their product may be 48.

Let x and y donote the two parts.
Then the 1st condition gives $x + y = 14$,
And the 2d gives $xy = 48$.
Then transp. y in the first gives $x = 14 - y$;
This value subst. for x in the 2d, is $14y - y^2 = 48$;
Changing all the signs, to make the square positive,
gives $y^2 - 14y = -48$;
Then compl. the square gives $y^2 - 14y + 49 = 1$;
And extrac. the root gives $y - 7 = \pm 1$;
Then transpos. 7, gives $y = 8$ or 6, the two parts.

3. What two numbers are those, whose sum, product, and difference of their squares, are all equal to each other?

Let x and y denote the two numbers.
Then the 1st and 2d expression give $x + y = xy$.
And the 1st and 3d give $x + y = x^2 - y^2$.

Then the last equa. div. by $x+y$, gives $1 = x - y$;
And transpos. y, gives $y + 1 = x$;
This val. substit. in the 1st gives $2y + 1 = y^2 + y$;
And transpos. $2y$, gives $1 = y^2 - y$;
Then complet. the sq. gives $\frac{5}{4} = y^2 - y + \frac{1}{4}$;
And extracting the root gives $\frac{1}{2}\sqrt{5} = y - \frac{1}{2}$;
And transposing $\frac{1}{2}$ gives $\frac{1}{2}\sqrt{5} + \frac{1}{2} = y$;
And therefore $x = y + 1 = \frac{1}{2}\sqrt{5} + \frac{3}{2}$.
And if these expressions be turned into numbers, by extracting the root of 5, &c. they give $x = 2{\cdot}6180 +$, and $y = 1{\cdot}6180 +$.

4. There are four numbers in arithmetical progression, of which the product of the two extremes is 22, and that of the means 40; what are the numbers?

Let $x =$ the less extreme,
and $y =$ the common difference;
Then x, $x+y$, $x+2y$, $x+3y$, will be the four numbers.
Hence, by the 1st condition $x^2 + 3xy = 22$,
And by the 2d $x^2 + 3xy + 2y^2 = 40$.
Then subtracting the first from the 2d gives $2y^2 = 18$;
And dividing by 2 gives $y^2 = 9$;
And extracting the root gives $y = 3$.
Then substit. 3 for y in the 1st, gives $x^2 + 9x = 22$;
And completing the square gives $x^2 + 9x + \frac{81}{4} = \frac{169}{4}$;
Then extracting the root gives $x + \frac{9}{2} = \frac{13}{2}$;
And transposing $\frac{9}{2}$ gives $x = 2$ the least number.
Hence the four numbers are 2, 5, 8, 11.

5. To find 3 numbers in geometrical progression, whose sum shall be 7, and the sum of their squares 21.

Let x, y, and z denote the three numbers sought.
Then by the 1st condition $xz = y^2$,
And by the 2d $x + y + z = 7$,
And by the 3d $x^2 + y^2 + z^2 = 21$.
Transposing y in the 2d gives $x + z = 7 - y$;
Sq. this equa. gives $x^2 + 2xz + z^2 = 49 - 14y + y^2$;
Substi. $2y^2$ for $2xz$, gives $x^2 + 2y^2 + z^2 = 49 - 14y + y^2$;
Subtr. y^2 from each side, leaves $x^2 + y^2 + z^2 = 49 - 14y$;
Putting the two values of $x^2 + y^2 + z^2$ equal to each other, gives $\Big\}\ 21 = 49 - 14y$;
Then transposing 21 and $14y$, gives $14y = 28$;
And dividing by 14, gives $y = 2$.
Then substit. 2 for y in the 1st equa. gives $xz = 4$,
And in the 4th, it gives $x + z = 5$;
Transposing z in the last, gives $x = 5 - z$;
This subst. in the next above, gives $5z - z^2 = 4$;

Changing all the signs, gives $z^2 - 5z = -4$;
Then completing the square, gives $z^2 - 5z + \frac{25}{4} = \frac{9}{4}$;
And extracting the root gives $z - \frac{5}{2} = \pm \frac{3}{2}$;
Then transposing $\frac{5}{2}$, gives z and $x = 4$ and 1, the two other numbers;
So that the three numbers are 1, 2, 4.

QUESTIONS FOR PRACTICE.

1. What number is that which added to its square makes 42? Ans. 6, or -7.

2. To find two numbers such, that the less may be to the greater as the greater is to 12, and that the sum of their squares may be 45. Ans. 3 and 6.

3. What two numbers are those, whose difference is 2, and the difference of their cubes 98? Ans. 3 and 5.

4. What two numbers are those, whose sum is 6, and the sum of their cubes 72? Ans. 2 and 4.

5. What two numbers are those, whose product is 20, and the difference of their cubes 61? Ans. 4 and 5.

6. To divide the number 11 into two such parts, that the product of their squares may be 784. Ans. 4 and 7.

7. To divide the number 5 into two such parts, that the sum of their alternate quotients may be $4\frac{1}{4}$, that is of the two quotients of each part divided by the other. Ans. 1 and 4.

8. To divide 12 into two such parts, that their product may be equal to 8 times their difference. Ans. 4 and 8.

9. To divide the number 10 into two such parts, that the square of 4 times the less part, may be 112 more than the square of 2 times the greater. Ans. 4 and 6.

10. To find two numbers such, that the sum of their squares may be 89, and their sum multiplied by the greater may produce 104. Ans. 5 and 8.

11. What number is that, which being divided by the product of its two digits, the quotient is $5\frac{1}{3}$; but when 9 is subtracted from it, there remains a number having the same digits inverted? Ans. 32.

12. To divide 20 into three parts such, that the continual product of all three may be 270, and that the difference of the first and second may be 2 less than the difference of the second and third. Ans. 5, 6, 9.

13. To find three numbers in arithmetical progression, such that the sum of their squares may be 56, and the sum

arising by adding together 3 times the first and 2 times the second and 3 times the third, may amount to 32.

Ans. 2, 4, 6.

14. To divide the number 13 into three such parts, that their squares may have equal differences, and that the sum of those squares may be 75. Ans. 1, 5, 7.

15. To find three numbers having equal differences, so that their sum may be 12, and the sum of their fourth powers 962. Ans. 3, 4, 5.

16. To find three numbers having equal differences, and such that the square of the least added to the product of the two greater may make 28, but the square of the greatest added to the product of the two less may make 44.

Ans. 2, 4, 6.

17. Three merchants, A, B, C, on comparing their gains find, that among them all they have gained 1444*l*; and that B's gained added to the square root of A's made 920*l*; but if added to the square root of C's it made 912*l*. What were their several gains? Ans. A 400, B 900, C 144.

18. To find three numbers in arithmetical progression, so that the sum of their squares shall be 93; also if the first be multiplied by 3, the second by 4, and the third by 5, the sum of the products may be 66. Ans. 2, 5, 8.

19. To find two numbers such, that their product added to their sum may make 47, and their sum taken from the sum of their squares may leave 62. Ans. 5 and 7.

RESOLUTION OF CUBIC AND HIGHER EQUATIONS.

A Cubic Equation, or Equation of the 3d degree or power, is one that contains the third power, of the unknown quantity. As $x^3 - ax^2 + bx = c$.

A *Biquadratic*, or Double Quadratic, is an equation that contains the 4th power of the unknown quantity:

$$\text{As } x^4 - ax^3 + bx^2 - cx = d.$$

An Equation of the 5th Power or Degree, is one that contains the 5th power of the unknown quantity.

$$\text{As } x^5 - ax^4 + bx^3 - cx^2 + dx = e.$$

And so on, for all other higher powers. Where it is to be noted, however, that all the powers, or terms, in the equation, are supposed to be freed from surds or fractional exponents.

There are many particular and prolix rules usually given for the solution of some of the above-mentioned powers or equations. But they may be all readily solved by the following easy rule of Double Position, sometimes called Trial-and-Error *.

RULE.

1. Find, by trial, two numbers, as near the true root as you can, and substitute them separately in the given equation, instead of the unknown quantity; and find how much the terms collected together, according to their signs + or −, differ from the absolute known term of the equation, marking whether these errors are in excess or defect.

2. Multiply the difference of the two numbers, found or taken by trial, by either of the errors, and divide the product by the difference of the errors, when they are alike, but by their sum when they are unlike. Or say, As the difference or sum of the errors, is to the difference of the two numbers, so is either error to the correction of its supposed number.

3. Add the quotient, last found, to the number belonging to that error, when its supposed number is too little, but subtract it when too great, and the result will give the true root *nearly*.

4. Take this root and the nearest of the two former, or any other that may be found nearer; and, by proceeding in like manner as above, a root will be had still nearer than before. And so on, to any degree of exactness required.

Note 1. It is best to employ always two assumed numbers that shall differ from each other only by unity in the last figure on the right hand; because then the difference, or multiplier, is only 1. It is also best to use always the least error in the above operation.

Note 2. It will be convenient also to begin with a single

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* See, farther, that portion of vol. ii. which relates to equations, their construction, &c.

A new and ingenious general method of solving equations has been recently discovered by Messrs. *H. Atkinson*, *Holdred*, and *Horner*, independently of each other. For the best pratical view of this new method and its applications, consult the *Elementary Treatise of Algebra*, by Mr. *J. R. Young*; a work which deserves our cordial recommendation.

figure at first, trying several single figures till there be found the two nearest the truth, the one too little, and the other too great; and in working with them, find only one more figure. Then substitute this corrected result in the equation, for the unknown letter, and if the result prove too little, substitute also the number next greater for the second supposition; but contrarywise, if the former prove too great, then take the next less number for the second supposition; and in working with the second pair of errors, continue the quotient only so far as to have the corrected number to four places of figures. Then repeat the same process again with this last corrected number, and the next greater or less, as the case may require, carrying the third corrected number to eight figures; because each new operation commonly doubles the number of true figures. And thus proceed to any extent that may be wanted.

EXAMPLES.

Ex. 1. To find the root of the cubic equation $x^3 + x^2 + x = 100$, or the value of x in it.

Here it is soon found that x lies between 4 and 5. Assume therefore these two numbers, and the operation will be as follows:

1st Sup.		2d Sup.
4	x	5
16	x^2	25
64	x^3	125
84	sums	155
100	but should be	100
−16	errors	+55

the sum of which is 71.
Then as 71 : 1 : : 16 : ·2
Hence $x = 4·2$ nearly.

Again, suppose 4·2 and 4·3, and repeat the work as follows:

1st Sup.		2d Sup.
4·2	x	4·3
17·64	x^2	18·49
74·088	x^3	79·507
95·928	sums	102·297
100		100
−4·072	errors	+2·297

the sum of which is 6·369.
As 6·369 : ·1 : : 2·297 : 0·036
This taken from 4·300
leaves x nearly = 4·264

Again, suppose 4·264, and 4·265, and work as follows:

4·264	x	4·265
18·181696	x^2	18·190225
77·526752	x^3	77·581310
99·972448	sums	100·036535
100		100
—0·027552	errors	+0·036535

the sum of which is ·064087.

Then as ·064087 : ·001 : : ·027552 : 0·0004299
To this adding - 4·264

gives x very nearly = 4·2644299

The work of the example above might have been much shortened, by the use of the Table of Powers in the Arithmetic, which would have given two or three figures by inspection. But the example has been worked out so particularly as it is, the better to show the method.

Ex. 2. To find the root of the equation $x^3 - 15x^2 + 63x = 50$, or the value of x in it.

Here it soon appears that x is very little above 1.

Suppose therefore 1·0 and 1·1, and work as follows:

1·0	x	1·1
63·0	$63x$	69·3
—15	$-15x^2$	—18·15
1	x^3	1·331
49	sums	52·481
50		50
—1	errors	+2·481

3·481 sum of the errors.

As 3·481 : 1 : : ·1 : ·03 correct.
1·00

Hence x = 1·03 nearly.

Again, suppose the two numbers 1·03 and 1·02, &c. as follows:

1·03	x	1·02
64·89	$63x$	64·26
—15·9135	$-15x^2$	—15·6060
1·092727	x^3	1·061208
50·069227	sums	49·715208
50		50
+·069227	errors	—·284792
·284792		

As ·354019 : ·01 : : ·069227 : ·0019555
This taken from 1·03

leaves x nearly = 1·02804

Note 3. Every equation has as many roots as it contains dimensions, or as there are units in the index of its highest power. That is, a simple equation has only one value of the root; but a quadratic equation has two values or roots, a cubic equation has three roots, a biquadratic equation has four roots, and so on.

When one of the roots of an equation has been found by approximation, as above, the rest may be found as follows. Take, for a dividend, the given equation, with the known term transposed, with its sign changed, to the unknown side of the equation; and, for a divisor, take x minus the root just found. Divide the said dividend by the divisor, and the quotient will be the equation depressed a degree lower than the given one.

Find a root of this new equation by approximation, as before, or otherwise, and it will be a second root of the original equation. Then, by means of this root, depress the second equation one degree lower, and from thence find a third root; and so on, till the equation be reduced to a quadratic; then the two roots of this being found, by the method of completing the square, they will make up the remainder of the roots. Thus, in the foregoing equation, having found one root to be 1·02804, connect it by minus with x for a divisor, and the equation for a dividend, &c. as follows:

$$x - 1{\cdot}02804\ )\ x^3 - 15x^2 + 63x - 50\ (\ x^2 - 13{\cdot}97196x + 48{\cdot}63627 = 0.$$

Then the two roots of this quadratic equation, or $x^2 - 13{\cdot}97196x = -48{\cdot}63627$, by completing the square, are 6·57653 and 7·39543, which are also the other two roots of the given cubic equation. So that all the three roots of that equation, viz. $x^3 - 15x^2 + 63x = 50$,

are 1·02804 and 6·57653 and 7·39543 sum 15·00000	and the sum of all the roots is found to be 15, being equal to the co-efficient, of the 2d term of the equation, which the sum of the roots always ought to be, when they are right.

Note 4. It is also a particular advantage of the foregoing rule, that it is not necessary to prepare the equation, as for other rules, by reducing it to the usual final form and state of equations. Because the rule may be applied at once to an unreduced equation, though it be ever so much embarrassed by surd and compound quantities. As in the following example:

Ex. 3. Let it be required to find the root x of the equation $\sqrt{(144x^2 - (x^2 + 20)^2)} + \sqrt{(196a^2 - (x^2 + 24)^2)} = 114$, or the value of x in it.

By a few trials it is soon found that the value of x is but little above 7. Suppose therefore first that x is $= 7$, and then $x = 8$.

First, when $x = 7$,		Second, when $x = 8$.
47·906	$\sqrt{[144x^2 - (x^2 + 20)^2]}$	46·476
65·384	$\sqrt{[196x^2 - (x^2 + 24)^2]}$	69·283
113·290	the sums of these	115·759
114·000	the true number	114·000
−0·710	the two errors	+1·759
+1·759		

As 2·469 : 1 :: 0·710 : 0·2 nearly.
7·0

Therefore $x = 7·2$ nearly.

Suppose again $x = 7·2$, and then, because it turns out too great, suppose x also $= 7·1$, &c. as follows:

Supp. $x = 7·2$,		Supp. $x = 7·1$.
47·990	$\sqrt{[144x^2 - (x^2 + 20)^2]}$	47·973
66·402	$\sqrt{[196x^2 - (x^2 + 24)^2]}$	65·904
114·392	the sums of these	113·877
114·000	the true number	114·000
+0·392	the two errors	−0·123
0·123		

As ·515 : ·1 :: ·123 : ·024 the correction,
7·100 add

Therefore $x = 7·124$ nearly the root required.

Note 5. The same rule also, among other more difficult forms of equations, succeeds very well in what are called exponential ones, or those which have an unknown quantity in the exponent of the power; as in the following example:

Ex. 4. To find the value of x in the exponential equation $x^x = 100$.

For more easily resolving such kinds of equations, it is convenient to take the logarithms of them, and then compute the terms by means of a table of logarithms. Thus, the logarithms of the two sides of the present equation are $x \times$ log. of $x = 2$, the log. of 100. Then, by a few trials, it is soon perceived that the value of x is somewhere between the two numbers 3 and 4, and indeed nearly in the middle between them, but rather nearer the latter than the former. Taking therefore first, $x = 3{\cdot}5$, and then $= 3{\cdot}6$, and working with the logarithms, the operation will be as follows:

First Supp. $x = 3{\cdot}5$.	Second Supp. $x = 3{\cdot}6$.
Log. of $3{\cdot}5 = 0{\cdot}544068$	Log. of $3{\cdot}6 = 0{\cdot}556303$
then $3{\cdot}5 \times \log. 3{\cdot}5 = 1{\cdot}904238$	then $3{\cdot}6 \times \log. 3{\cdot}6 = 2{\cdot}002689$
the true number $2{\cdot}000000$	the true number $2{\cdot}000000$
error, too little, $-{\cdot}095762$	error, too great, $+{\cdot}002689$
${\cdot}002689$	
${\cdot}098451$ sum of the errors. Then,	

As ${\cdot}098451 : {\cdot}1 :: {\cdot}002689 : 0{\cdot}00273$ the correction
taken from $3{\cdot}60000$

leaves - $3{\cdot}59727 = x$ nearly.

By repeating the operation with a larger table of logarithms, a nearer value of x may be found $3{\cdot}597285$.

This method, indeed, may be a little improved in practice: for since $x^x = a$, we have by logarithms $x \times \log. x = \log. a$; and again, $\log. x + \log. \overline{\log. x} = \log. \overline{\log. a}$. We have therefore only to find a number, which, added to its log. will will be equal to the log. of the log. of the given number; and the natural number answering to this number, is the value of x required.

In illustration of the above, take the 12th example:— $x^x = 123456789$. First, log. $123456789 = 8{\cdot}0915148$, and log. $8{\cdot}0915148 = {\cdot}9080298$. Searching in a table of logarithms, we find the nearest number ${\cdot}93651$; which added to its logarithm $- 1{\cdot}9715124 = {\cdot}9080224$. The next higher number ·93652 + its log. = ·9080371. Hence

·9080371	·9080298	
·9080224	·9080224	$74 \div 147 = {\cdot}503$
147	74	

Therefore, the number sought is ·93651503, the natural number answering to which is 8·640026 the value of x, which is true to the last figure, the value given by Dr. Hutton being 8·6400268.

The common logarithmic solution fails when a is less than unity, its log. being then negative. In this case, assume $x = 1 \div y$, and $a = 1 \div e$, which transforms the given equa. $x^x = a$, to $e^y = y$. Taking the logs. *twice*, we get y log. $e =$ log. y, and log. $y +$ log. of log. $e =$ log. of log. y; or, putting log. $y = v$, and log. of log. $e = s$, we have $v + s =$ log. v, an equation easy to solve.

Ex. 5. To find the value of x in the equation $x^3 + 10x^2 + 5x = 260$. Ans. $x = 4{\cdot}1179857$.

Ex. 6. To find the value of x in the equation $x^3 - 2x = 50$. Ans. 3·8648854.

Ex. 7. To find the value of x in the equation $x^3 + 2x^2 - 23x = 70$. Ans. $x = 5{\cdot}13457$.

Ex. 8. To find the value of x in the equation $x^3 - 17x^2 + 54x = 350$. Ans. $x = 14{\cdot}95407$.

Ex. 9. To find the value of x in the equation $x^4 - 3x^2 - 75x = 10000$. Ans. $x = 10{\cdot}2609$.

Ex. 10. To find the value of x in the equation $2x^4 - 16x^3 + 40x^2 - 30x = -1$. Ans. $x = 1{\cdot}284724$.

Ex. 11. To find the value of x in the equation $x^5 + 2x^4 + 3x^3 + 4x^2 + 5x = 54321$. Ans. $x = 8{\cdot}414455$.

Ex. 12. To find the value of x in the equation $x^x = 123456789$. Ans. $x = 8{\cdot}6400268$.

Ex. 13. Given $2x^4 - 7x^3 + 11x^2 - 3x = 11$, to find x.

Ex. 14. To find the value of x in the equation.

$$(3x^2 - 2\sqrt{x} + 1)^{\frac{2}{3}} - (x^2 - 4x\sqrt{x} + 3\sqrt{x})^{\frac{1}{2}} = 56.$$

Ans. $x = 18{\cdot}360877$.

To resolve Cubic Equations by Cardan's Rule.

Though the foregoing general method, by the application of Double Position, be the readiest way, in real practice, of finding the roots in numbers of cubic equations, as well as of all the higher equations universally, we may here add the particular method commonly called Cardan's Rule, for resolving cubic equations, in case any person should choose occasionally to employ that method; although it is only applicable when two of the roots are impossible.

The form that a cubic equation must necessarily have, to be resolved by this rule, is this, viz. $z^3 \pm az = b$, that is, wanting the second term, or the term of the 2d power z^2. Therefore, after any cubic equation has been reduced down to its final usual form, $x^3 + px^2 + qx = r$, freed from the co-efficient of its first term, it will then be necessary to take away the 2d term px^2; which is to be done in this manner; Take $\frac{1}{3}p$, or $\frac{1}{3}$ of the co-efficient of the second term, and annex it, with the contrary sign, to another unknown letter z, thus $z - \frac{1}{3}p$; then substitute this for x, the unknown letter in the original equation $x^3 + px^2 + qx = r$, and there will result this reduced equation $z^3 \pm az = b$, of the form proper for applying the following, or Cardan's rule. Or take $c = \frac{1}{3}a$, and $d = \frac{1}{2}b$, by which the reduced equation takes this form, $z^3 \pm 3cz = 2d$.

Then substitute the values of c and d in this

$$\left.\begin{array}{l} \text{form, } z = \sqrt[3]{[d + \sqrt{(d^2 + c^3)}]} + \sqrt[3]{[d - \sqrt{(d^2 + c^3)}]}, \\ \text{or } z = \sqrt[3]{[d + \sqrt{(d^2 + c^3)}]} - \dfrac{c}{\sqrt[3]{[d + \sqrt{(d^2 + c^3)}]}}, \end{array}\right\}$$

and the value of the root z, of the reduced equation $z^3 \pm az = b$, will be obtained. Lastly, take $x = z - \frac{1}{3}p$, which will give the value of x, the required root of the original equation $x^3 + px^2 + qx = r$, first proposed.

One root of this equation being thus obtained, then depressing the original equation one degree lower, after the manner described, p. 260, the other two roots of that equation will be obtained by means of the resulting quadratic equation.

Note. When the co-efficient a, or c, is negative, and c^3 is greater than d^2, this is called the irreducible case, because then the solution cannot be generally obtained by this rule*.

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* Suppose a root to consist of the two parts x and y, so that $(x + y) = z$; which sum substituted for z, in the given equation $z^3 + az = b$,

Ex. To find the roots of the equation $x^3 - 6x^2 + 10x = 8$.

First to take away the 2d term, its co-efficient being -6, its 3d part is -2; put therefore $x = z + 2$; then

$$\begin{array}{rcl} x^3 = & z^3 + 6z^2 + 12z + & 8 \\ -\ 6x^2 = & -6z^2 - 24z - & 24 \\ +10x = & +10z + & 20 \\ \hline \end{array}$$

$$\text{theref. the sum } z^3 \quad * \quad - 2z + 4 = 8$$
$$\text{or } z^3 \quad * \quad - 2z = 4$$

Here then $a = -2$, $b = 4$, $c = -\frac{2}{3}$, $d = 2$.

Theref. $\sqrt[3]{[d+\sqrt{(d^2+c^3)}]} = \sqrt[3]{[2+\sqrt{(4-\frac{8}{27})}]} = \sqrt[3]{(2+\sqrt{\frac{100}{27}})} = \sqrt[3]{(2+\frac{10}{9}\sqrt{3})} = 1\cdot57735$,

and $\sqrt[3]{[d-\sqrt{(d^2+c^3)}]} = \sqrt[3]{[2-\sqrt{(4-\frac{8}{27})}]} = \sqrt[3]{(2-\sqrt{\frac{100}{27}})} = \sqrt[3]{(2-\frac{10}{9}\sqrt{3})} = 0\cdot42265$;

then the sum of these two is the value of $z = 2$. Hence $x = z + 2 = 4$, one root of x in the eq. $x^3 - 6x^2 + 10x = 8$.

To find the two other roots, perform the division, &c. as in p. 261, thus:

$$\begin{array}{l} x - 4)\ x^3 - 6x^2 + 10x - 8\ (\ x^2 - 2x + 2 = 0 \\ \quad\quad\ \ x^3 - 4x^2 \\ \quad\quad\ \ \overline{\quad\quad\quad} \\ \quad\quad\quad\ -2x^2 + 10x \\ \quad\quad\quad\ -2x^2 + 8x \\ \quad\quad\quad\ \overline{\quad\quad\quad\quad} \\ \quad\quad\quad\quad\quad\quad\ 2x - 8 \\ \quad\quad\quad\quad\quad\quad\ 2x - 8 \\ \quad\quad\quad\quad\quad\quad\ \overline{\quad\quad} \end{array}$$

---

it becomes $x^3 + y^3 + 3xy(x + y) + a(x + y) = b$. Again, suppose $3xy = -a$; which substituted, the last equation becomes $x^3 + y^3 = b$. Now, from the square of this equation subtract four times the equation $xy = -\frac{1}{3}a$, and there results $x^6 - 2x^3y^3 + y^6 = b^2 + \frac{4}{27}a^3$, the square root of which is $x^3 - y^3 = \sqrt{(b^2 + \frac{4}{27}a^3)}$. This being added to and taken from the equation $x^3 + y^3 = b$, gives

$$\begin{cases} 2x^3 = b + \sqrt{(b^2 + \frac{4}{27}a^3)} = b + 2\sqrt{[(\frac{1}{2}b)^2 + (\frac{1}{3}a)^3]}, \\ 2y^3 = b - \sqrt{(b^2 - \frac{4}{27}a^3)} = b - 2\sqrt{[(\frac{1}{2}b)^2 + (\frac{1}{3}a)^3]}; \text{ or} \end{cases}$$

$$\begin{cases} 2x^3 = 2d + 2\sqrt{(d^2 + c^3)} \\ 2y^3 = 2d - 2\sqrt{(d^2 + c^3)} \end{cases}.$$ Hence, dividing by 2, and extracting the cube roots, we have $x = \sqrt[3]{d + \sqrt{(d^2 + c^3)}}$, and $y = \sqrt[3]{d - \sqrt{(d^2 + c^3)}}$; the sum of these two gives the first form of the root z above stated. And that the 2d form is equal to the first will be evident by reducing the two 2d quantities to the same denominator.

When c is negative, and c^3 greater than d^2, the root appears in an imaginary form.

Hence $x^2 - 2x = -2$, or $x^2 - 2x + 1 = -1$, and $x - 1 = \pm\sqrt{-1}$; $x = 1 + \sqrt{-1}$ or $= 1 - \sqrt{-1}$, the two other roots sought.

Ex. 2. Given $x^3 - 6x^2 + 36x = 44$, to find x.

Ans. $x \doteq 2{\cdot}32748$.

Ex. 3. To find the roots of $x^3 - 7x^2 + 14x = 20$.

Ans. $x = 5$, or $= 1 + \sqrt{-3}$, or $= 1 - \sqrt{-3}$.

Ex. 4. Find the three roots of $x^3 + 6x = 20$.

OF SIMPLE INTEREST.

As the interest of any sum, for any time, is directly proportional to the principal sum, and to the time; therefore the interest of 1 pound, for 1 year, being multiplied by any given principal sum, and by the time of its forbearance, in years and parts, will give its interest for that time. That is, if there be put

$r =$ the rate of interest of 1 pound per annum,
$p =$ any principal sum lent,
$t =$ the time it is lent for, and
$a =$ the amount or sum of principal and interest; then is $prt =$ the interest of the sum p, for the time t, and conseq. $p + prt$ or $p \times (1 + rt) = a$, the amount for that time.

From this expression, other theorems can easily be deduced, for finding any of the quantities above mentioned: which theorems, collected together, will be as follows:

1st, $a = p + prt$ the amount; 2d, $p = \dfrac{a}{1 + rt}$ the principal;

3d, $r = \dfrac{a - p}{pt}$ the rate; 4th, $t = \dfrac{a - p}{pr}$ the time.

For Example. Required to find in what time any principal sum will double itself, at any rate of simple interest.

In this case, we must use the first theorem, $a = p + prt$, in which the amount a must be made $= 2p$, or double the principal, that is, $p + prt = 2p$, or $prt = p$, or $rt = 1$; and hence $t = \dfrac{1}{r}$.

Hence r being the interest of $1l$ for 1 year, it follows, that the doubling at simple interest, is equal to the quotient of any sum divided by its interest for 1 year. So, if the rate of interest be 5 per cent. then $100 \div 5 = 20$, is the time of doubling at that rate. Or the 4th theorem gives at once $t = \frac{a-p}{pr} = \frac{2p-p}{pr} = \frac{2-1}{r} = \frac{1}{r}$, the same as before.

COMPOUND INTEREST.

Besides the quantities concerned in Simple Interest, namely,

p = the principal sum,
r = the rate of interest of $1l$ for 1 year,
a = the whole amount of the principal and interest,
t = the time,

there is another quantity employed in Compound Interest, viz. the ratio of the rate of interest, which is the amount of $1l$ for 1 time of payment, and which here let be denoted by R, viz.

$R = 1 + r$, the amount of $1l$ for 1 time.

Then the particular amounts for the several times may be thus computed, viz. As $1l$ is to its amount for any time, so is any proposed principal sum, to its amount for the same time; that is, as

$1l : R :: p : pR$, the 1st year's amount,
$1l : R :: pR : pR^2$, the 2d year's amount,
$1l : R :: pR^2 : pR^3$, the 3d year's amount,
and so on.

Therefore, in general, $pR^t = a$ is the amount for the t year, or t time of payment. Whence the following general theorems are deduced:

1st, $a = pR^t$ the amount; 2d, $p = \frac{a}{R^t}$ the principal;

3d, $R = \sqrt[t]{\frac{a}{p}}$ the ratio; 4th, $t = \frac{\text{log. of } a - \text{log. of } p}{\text{log. of } R}$ the time.

From which, any one of the quantities may be found, when the rest are given.

As to the whole interest, it is found by barely subtracting the principal p from the amount a.

Example. Suppose it be required to find, in how many years any principal sum will double itself, at any proposed rate of compound interest.

In this case the 4th theorem must be employed, making $a = 2p$; and then it is

$$t = \frac{\text{log. } a - \text{log. } p}{\text{log. } R} = \frac{\text{log. } 2p - \text{log. } p}{\text{log. } R} = \frac{\text{log. } 2}{\text{log. } R}.$$

So, if the rate of interest be 5 per cent. per annum; then $R = 1 + {\cdot}05 = 1{\cdot}05$; and hence

$$t = \frac{\text{log. } 2}{\text{log. } 1{\cdot}05} = \frac{{\cdot}301030}{{\cdot}021189} = 14{\cdot}2067 \text{ nearly};$$

that is, any sum doubles itself in $14\frac{1}{5}$ years nearly, at the rate of 5 per cent. per annum compound interest.

Hence, and from the like question in simple interest, above given, are deduced the times in which any sum doubles itself at several rates of interest, both simple and compound; viz.

At		At Simp. Int. (Years)	At Comp. Int. (Years)
2	per cent. per annum interest, 1*l.* or any other sum, will double itself in the following years.	in 50	in 35·0028
$2\frac{1}{2}$		40	28·0701
3		$33\frac{1}{3}$	23·4498
$3\frac{1}{2}$		$28\frac{4}{7}$	20·1488
4		25	17·6730
$4\frac{1}{2}$		$22\frac{2}{9}$	15·7473
5		20	14·2067
6		$16\frac{2}{3}$	11·8957
7		$14\frac{2}{7}$	10·2448
8		$12\frac{1}{2}$	9·0065
9		$11\frac{1}{9}$	8·0432
10		10	7·2725

The following Table will very much facilitate calculations of compound interest on any sum, for any number of years, at various rates of interest.

The Amounts of 1*l* in any Number of Years.

Yrs.	3	3½	4	4½	5	6
1	1·0300	1·0350	1·0400	1·0450	1·0500	1·0600
2	1·0609	1·0712	1·0816	1·0920	1·1025	1·1236
3	1·0927	1·1087	1·1249	1·1412	1·1576	1·1910
4	1·1255	1·1475	1·1699	1·1925	1·2155	1·2625
5	1·1593	1·1877	1·2167	1·2462	1·2763	1·3382
6	1·1948	1·2293	1·2653	1·3023	1·3401	1·4185
7	1·2299	1·2723	1·3159	1·3609	1·4071	1·5036
8	1·2668	1·3168	1·3686	1·4221	1·4775	1·5939
9	1·3048	1·3629	1·4233	1·4861	1·5513	1·6895
10	1·3439	1·4106	1·4802	1·5530	1·6289	1·7909
11	1·3842	1·4600	1·5895	1·6229	1·7103	1·8983
12	1·4258	1·5111	1·6010	1·6959	1.7959	2·0122
13	1·4685	1·5640	1·6651	1·7722	1·8856	2·1329
14	1·5126	1·6187	1·7317	1·8519	1·9799	2·2609
15	1·5580	1·6753	1·8009	1·9353	2·0789	2·3966
16	1·6047	1·7340	1·8730	2·0224	2·1829	2·5404
17	1·6528	1·7947	1·9479	2·1134	2·2920	2·6928
18	1·7024	1·8575	2·0258	2·2085	2·4066	2·8543
19	1·7535	1·9225	2·1068	2·3079	2·5270	3·0256
20	1·8061	1·9828	2·1911	2·4117	2·6533	3·2071

The use of this Table, which contains all the powers, R^t, to the 20th power, or the amounts of 1*l*, is chiefly to calculate the interest, or the amount of any principal sum, for any time, not more than 20 years.

For example, let it be required to find, to how much 523*l* will amount in 15 years, at the rate of 5 per cent. per annum compound interest.

In the table, on the line 15, and in the column 5 per cent.

is the amount of 1*l*, viz. - -	2·0789
this multiplied by the principal -	523
gives the amount - -	1087·2647
or - - - - -	1087*l* 5*s* 3¼*d*.
and therefore the interest	564*l* 5*s* 3¼*d*.

Note 1. When the rate of interest is to be determined to any other time than a year; as suppose to ½ a year, or ¼ a year, &c.: the rules are still the same; but then *t* will express that time, and R must be taken the amount for that time also.

Note 2. When the compound interest, or amount, of any sum, is required for the parts of a year; it may be determined in the following manner:

1*st*, For any time which is some aliquot part of a year:—Find the amount of 1*l* for 1 year, as before; then that root of it which is denoted by the aliquot part, will be the amount of 1*l*. This amount being multiplied by the principal sum, will produce the amount of the given sum as required.

2*d*, When the time is not an aliquot part of a year:—Reduce the time into days, and take the 365th root of the amount of 1*l* for 1 year, which will give the amount of the same for 1 day. Then raise this amount to that power whose index is equal to the number of days, and it will be the amount for that time. Which amount being multiplied by the principal sum, will produce the amount of that sum as before.—And in these calculations, the operation by logarithms will be very useful.

OF ANNUITIES.

Annuity is a term used for any periodical income, arising from money lent, or from houses, lands, salaries, pensions, &c. payable from time to time, but mostly by annual payments.

Annuities are divided into those that are in Possession, and those in Reversion: the former meaning such as have commenced; and the latter such as will not begin till some particular event has happened, or till after some certain time has elapsed.

When an annuity is forborn for some years, or the payments not made for that time, the annuity is said to be in Arrears.

An annuity may also be for a certain number of years; or it may be without any limit, and then it is called a Perpetuity.

The Amount of an annuity, forborn for any number of years, is the sum arising from the addition of all the annuities for that number of years, together with the interest due upon each after it becomes due.

The Present Worth or Value of an annuity, is the price or sum which ought to be given for it, supposing it to be bought off, or paid all at once.

Let a = the annuity, pension, or yearly rent ;
n = the number of years forborn, or lent for ;
R = the amount of 1*l* for 1 year ;
m = the amount of the annuity ;
v = its value, or its present worth.

Now, 1 being the present value of the sum R, by proportion the present value of any other sum a, is thus found : as $R : 1 :: a : \frac{a}{R}$ the present value of a due 1 year hence.

In like manner $\frac{a}{R^2}$ is the present value of a due 2 years hence; for $R : 1 :: \frac{a}{R} : \frac{a}{R^2}$. So also $\frac{a}{R^3}, \frac{a}{R^4}, \frac{a}{R^5}$, &c. will be the present values of a, due at the end of 3, 4, 5, &c. years respectively. Consequently the sum of all these, or $\frac{a}{R} + \frac{a}{R^2} + \frac{a}{R^3} + \frac{a}{R^4} +$ &c. $= (\frac{1}{R} + \frac{1}{R^2} + \frac{1}{R^3} + \frac{1}{R^4}$ &c.$) \times a$ continued to n terms, will be the present value of all the n years' annuities. And the value of the perpetuity, is the sum of the series to infinity.

But this series, it is evident, is a geometrical progression, having $\frac{1}{R}$ but for its first term and common ratio, and the number of its terms n ; therefore the sum v of all the terms, or the present value of all the annual payments, will be

$$v = \frac{\frac{1}{R} - \frac{1}{R} \times \frac{1}{R^n}}{1 - \frac{1}{R}} \times a, \text{ or } = \frac{R^n - 1}{R - 1} \times \frac{a}{R^n}.$$

When the annuity is a perpetuity ; n being infinite, R^n is also infinite, and therefore the quantity $\frac{1}{R^n}$ becomes $= 0$, therefore $\frac{a}{R-1} \times \frac{1}{R^n}$ also $= 0$; consequently the expression becomes barely $v = \frac{a}{R-1}$; that is, any annuity divided by the interest of 1*l* for 1 year, gives the value of the perpetuity. So, if the rate of interest be 5 per cent.

Then $100a \div 5 = 20a$ is the value of the perpetuity at 5 per cent.: Also $100a \div 4 = 25a$ is the value of the perpetuity at 4 per cent.: And $100a \div 3 = 33\frac{1}{3}a$ is the value of the perpetuity at 3 per cent.: and so on.

Again, because the amount of $1l$ in n years, is R^n, its increase in that time will be $R^n - 1$; but its interest for one single year, or the annuity answering to that increase, is $R - 1$; therefore, as $R - 1$ is to $R^n - 1$, so is a to m; that is, $m = \frac{R^n - 1}{R - 1} \times a$. Hence, the several cases relating to Annuities in Arrear, will be resolved by the following equations:

$$m = \frac{R^n - 1}{R - 1} \times a = vR^n;$$

$$v = \frac{R^n - 1}{R - 1} \times \frac{a}{R^n} = \frac{m}{R^n};$$

$$a = \frac{R - 1}{R^n - 1} \times m = \frac{R - 1}{R^n - 1} \times vR^n;$$

$$n = \frac{\log. m - \log. v}{\log. R} = \frac{\log. \frac{mR - m + a}{a}}{\log. R};$$

$$\text{Log. } R = \frac{\log. m - \log. v}{n};$$

$$r = \left(\frac{1}{R^p} - \frac{1}{R^n}\right) \times \frac{a}{R - 1}.$$

In this last theorem, r denotes the present value of an annuity in reversion, after p years, or not commencing till after the first p years, being found by taking the difference between the two values $\frac{R^n - 1}{R - 1} \times \frac{a}{R^n}$ and $\frac{R^p - 1}{R - 1} = \frac{a}{R^p}$, for n years and p years.

But the amount and present value of any annuity for any number of years, up to 21, will be most readily found by the two following tables.

TABLE I.

The Amount of an Annuity of 1*l* at Compound Interest.

Yrs.	at 3 per c.	3½ per c.	4 per c.	4½ per c.	5 per c.	6 per c.
1	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000
2	2·0300	2·0350	2·0400	2·0450	2·0500	2·0600
3	3·0909	3·1062	3·1216	3·1370	3·1525	3·1836
4	4·1836	4·2149	4·2465	4·2782	4·3101	4·3746
5	5·3091	5·3625	5·4163	5·4707	5·5256	5·6371
6	6·4684	6·5502	6·6330	6·7169	6·8019	6·9753
7	7·6625	7·7794	7·8983	8·0192	8·1420	8·3938
8	8·8923	9·0517	9·2142	9·3800	9·5491	9·8975
9	10·1591	10·3685	10·5828	10·8021	11·0266	11·4913
10	11·4639	11·7314	12·0061	12·2882	12·5779	13·1808
11	12·8078	13·1420	13·4864	13·8412	14·2068	14·9716
12	14·1920	14·6020	15·0258	15·4640	15·9171	16·8699
13	15·6178	16·1130	16·6268	17·1599	17·7130	18·8821
14	17·0863	17·6770	18·2919	18·9321	19·5986	21·0151
15	18·5989	19·2957	20·3236	20·7841	21·5786	23·2760
16	20·1569	20·9710	21·8245	22·7193	23·6575	25·6725
17	21·7616	22·7050	23·6975	24·7417	25·8404	28·2129
18	23·4144	24·4997	25·6454	26·8551	28·1324	30·9057
19	25·1169	26·3572	27·6712	29·0636	30·5390	33·7600
20	26·8704	28·2797	29·7781	31·3714	33·0660	36·7856
21	28·6765	30·2695	31·9692	33·7831	35·7193	39·9927

TABLE II. The Present Value of an Annuity of 1*l*.

Yrs.	at 3 per c.	3½ per c.	4 per c.	4½ per c.	5 per c.	6 per c.
1	0·9709	0·9662	0·9615	0·9569	0·9524	0·9524
2	1·9135	1·8997	1·8861	1·8727	1·8594	1·8334
3	2·8286	2·8016	2·7751	2·7490	2·7233	2·6730
4	3·7171	3·6731	3·6299	3·5875	3·5460	3·4651
5	4·5797	4·5151	4·4518	4·3900	4·3295	4·2124
6	5·4172	5·3286	5·2421	5·1579	5·0757	4·9173
7	6·2303	6·1145	6·0020	5·8927	5·7864	5·5824
8	7·0197	6·8740	7·7327	6·5959	6·4632	6·2098
9	7·7861	7·6077	7·4353	7·2688	7·1078	6·8017
10	8·5302	8·3166	8·1109	7·9127	7·7217	7·3601
11	9·5256	9·0016	8·7605	8·5289	8·3054	7·8869
12	9·9540	9·6633	9·3851	9·1186	8·8633	8·3838
13	10·6350	10·3027	9·9857	9·6829	9·3936	8·8527
14	11·2961	10·9205	10·5631	10·2228	9·8986	9·2950
15	11·9379	11·5174	11·1184	10·7396	10·3797	9·7123
16	12·5611	12·0941	11·6523	11·2340	10·8378	10·1059
17	13·1661	12·6513	12·1657	11·7072	11·2741	10·4773
18	13·7535	13·1897	12·6593	12·1600	11·6896	10·8276
19	14·3238	13·7098	13·1339	12·5933	12·0853	11·1581
20	14·8775	14·2124	13·5903	13·0079	12·4622	11·4699
21	15·4150	14·6980	14·0292	13·4047	12·8212	11·7641

To find the Amount of any Annuity forborn a certain number of years.

Take out the amount of 1*l* from the first table, for the proposed rate and time; then multiply it by the given annuity; and the product will be the amount, for the same number of years, and rate of interest. And the converse to find the rate of time.

Exam. To find how much an annuity of 50*l* will amount to in 20 years, at $3\frac{1}{2}$ per cent. compound interest.

On the line of 20 years, and in the column of $3\frac{1}{2}$ per cent. stands 28·2797, which is the amount of an annuity of 1*l* for the 20 years. Then $28{\cdot}2797 \times 50$, gives 1413·985*l* = 1413*l* 19*s* 8*d* for the answer required.

To find the Present Value of any Annuity for any number of years.—Proceed here by the 2d table, in the same manner as above for the 1st table, and the present worth required will be found.

Exam. 1. To find the present value of an annuity of 50*l*, which is to continue 20 years, at $3\frac{1}{2}$ per cent.—By the table, the present value of 1*l* for the given rate and time, is 14·2124; therefore $14{\cdot}2124 \times 50 = 710{\cdot}62$*l* or 710*l* 12*s* 4*d* is the present value required.

Exam 2. To find the present value of an annuity of 20*l*, to commence 10 years hence, and then to continue for 11 years longer, or to terminate 21 years hence, at 4 per cent. interest.—In such cases as this, we have to find the difference between the present values of two equal annuities, for the two given times; which therefore will be done by subtracting the tabular value of the one period from that of the other, and then multiplying by the given annuity. Thus,

tabular value for 21 years	14·0292
ditto for 10 years	8·1109
the difference	5·9183
multiplied by	20
gives -	118·366*l*
or - -	118*l* 7*s* $3\frac{1}{2}$*d* the answer.

END OF THE ALGEBRA.

GEOMETRY.

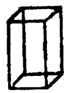
DEFINITIONS.

1. A Point is that which has position, but no magnitude, nor dimensions; neither length, breadth, nor thickness.

2. A Line is length, without breadth or thickness.

3. A Surface or Superficies, is an extension or a figure of two dimensions, length and breadth; but without thickness.

4. A Body or Solid, is a figure of three dimensions, namely, length, breadth, and depth, or thickness.



5. Lines are either Right, or Curved, or Mixed of these two.

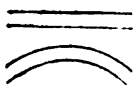
6. A Right Line, or Straight Line, lies all in the same direction, between its extremities; and is the shortest distance between two points.

When a Line is mentioned simply, it means a Right Line.

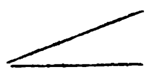
7. A Curve continually changes its direction between its extreme points.

8. Lines are either Parallel, Oblique, Perpendicular, or Tangential.

9. Parallel Lines are always at the same perpendicular distance; and they never meet, though ever so far produced.



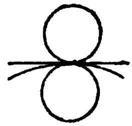
10. Oblique lines change their distance, and would meet, if produced on the side of the least distance.



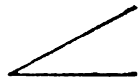
11. One line is Perpendicular to another, when it inclines not more on the one side

than the other, or when the angles on both sides of it are equal.

12. A line or circle is Tangential, or is a Tangent to a circle, or other curve, when it touches it, without cutting, when both are produced.



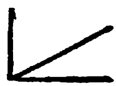
13. An Angle is the inclination or opening of two lines, having different directions, and meeting in a point.



14. Angles are Right or Oblique, Acute or Obtuse.

15. A Right Angle is that which is made by one line perpendicular to another. Or when the angles on each side are equal to one another, they are right angles.

16. An Oblique Angle is that which is made by two oblique lines; and is either less or greater than a right angle.



17. An Acute Angle is less than a right angle.

18. An Obtuse Angle is greater than a right angle.

19. Superfices are either Plane or Curved.

20. A Plane Superficies, or a Plane, is that with which a right line may, every way, coincide. Or, if the line touch the plane in two points, it will touch it in every point. But, if not, it is curved.

21. Plane Figures are bounded either by right lines or curves.

22. Plane figures that are bounded by right lines have names according to the number of their sides, or of their angles; for they have as many sides as angles; the least number being three.

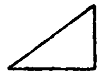
33. A figure of three sides and angles is called a Triangle. And it receives particular denominations from the relations of its sides and angles.

24. An Equilateral Triangle is that whose three sides are all equal.

25. An Isosceles Triangle is that which has two sides equal.

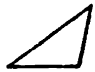
26. A Scalene Triangle is that whose three sides are all unequal.

27. A Right-angled Triangle is that which has one right angle.



28. Other triangles are Oblique-angled, and are either obtuse or acute.

29. An Obtuse-angled Triangle has one obtuse angle.



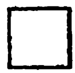
30. An Acute-angled Triangle has all its three angles acute.

31. A figure of Four sides and angles is called a Quadrangle, or a Quadrilateral.

32. A Parallelogram is a quadrilateral which has both its pairs of opposite sides parallel. And it takes the following particular names, viz. Rectangle, Square, Rhombus, Rhomboid.

33. A Rectangle is a parallelogram, having a right angle.

34. A Square is an equilateral rectangle; having its length and breadth equal.



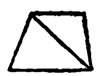
35. A Rhomboid is an oblique-angled parallelogram.

36. A Rhombus is an equilateral rhomboid; having all its sides equal, but its angles oblique.

37. A Trapezium is a quadrilateral which hath not its opposite sides parallel.

38. A Trapezöid has only one pair of opposite sides parallel.

39. A Diagonal is a line joining any two opposite angles of a quadrilateral.



40. Plane figures that have more than four sides are, in general, called Polygons: and they receive other particular names, according to the number of their sides or angles. Thus,

41. A Pentagon is a polygon of five sides; a Hexagon, of six sides; a Heptagon, seven; an Octagon, eight; a Nonagon, nine; a Decagon, ten; an Undecagon, eleven; and a Dodecagon, twelve sides.

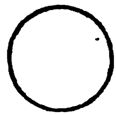
42. A Regular Polygon has all its sides and all its angles equal.—If they are not both equal, the polygon is Irregular.

43. An Equilateral Triangle is also a Regular Figure of three sides, and the Square is one of four: the former being also called a Trigon, and the latter a Tetragon.

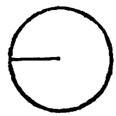
44. Any figure is equilateral, when all its sides are equal: and it is equiangular when all its angles are equal. When both these are equal, it is a regular figure.

45. A Circle is a plane figure bounded by a curve line, called the Circumference, which is every where equidistant from a certain point within, called its Centre.

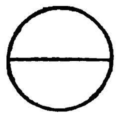
The circumference itself is often called a circle, and also the Periphery.



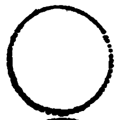
46. The Radius of a circle is a line drawn from the centre to the circumference.



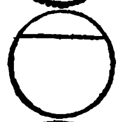
47. The Diameter of a cirle is a line drawn through the centre, and terminating at the circumference on both sides.



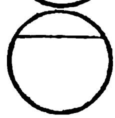
48. An Arc of a circle is any part of the circumference.



49. A Chord is a right line joining the extremities of an arc.

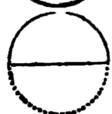


50. A Segment is any part of a circle bounded by an arc and its chord.

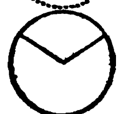


51. A Semicircle is half the circle, or a segment cut off by a diameter.

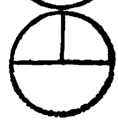
The half circumference is sometimes called the Semicircle.



52. A Sector is any part of a circle which is bounded by an arc, and two radii drawn to its extremities.



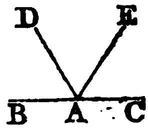
53. A Quadrant, or Quarter of a circle is a sector having a quarter of the circumference for its arc, and its two radii are perpendicular to each other. A quarter of the circumference is sometimes called a Quadrant.



54. The Height or Altitude of a figure is a perpendicular let fall from an angle, or its vertex, to the opposite side, called the base.

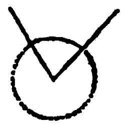
55. In a right-angled triangle, the side opposite the right angle is called the Hypothenuse; and the other two sides are called the Legs, and sometimes the Base and Perpendicular.

56. When an angle is denoted by three letters, of which one stands at the angular point, and the other two on the two sides, that which stands at the angular point is read in the middle.

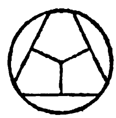


57. The circumference of every circle is supposed to be divided into 360 equal parts called degrees; and each degree into 60 Minutes, each Minute into 60 Seconds, and so on. Hence a semicircle contains 180 degrees, and a quadrant 90 degrees.

58. The Measure of an angle, is an arc of any circle contained between the two lines which form that angle, the angular point being the centre; and it is estimated by the number of degrees contained in that arc.

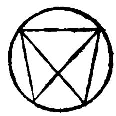


59. Lines, or chords, are said to be Equidistant from the centre of a circle, when perpendiculars drawn to them from the centre are equal.



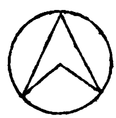
69. And the right line on which the Greater Perpendicular falls, is said to be farther from the centre.

61. An Angle In a Segment is that which is contained by two lines, drawn from any point in the arc of the segment, to the two extremities of that arc.

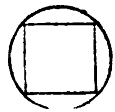


62. An Angle On a segment, or an arc, is that which is contained by two lines, drawn from any point in the opposite or supplemental part of the circumference, to the extremities of the arc, and containing the arc between them.

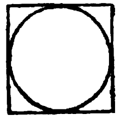
63. An Angle at the circumference, is that whose angular point or summit is any where in the circumference. And an angle at the centre, is that whose angular point is at the centre.



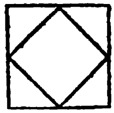
64. A right-lined figure is Inscribed in a circle, or the circle Circumscribes it, when all the angular points of the figure are in the circumference of the circle.



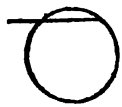
65. A right-lined figure Circumscribes a circle, or the circle is Inscribed in it, when all the sides of the figure touch the circumference of the circle.



66. One right-lined figure is Inscribed in another, or the latter circumscribes the former, when all the angular points of the former are placed in the sides of the latter.



67. A Secant is a line that cuts a circle, lying partly within, and partly without it.



66. Two triangles, or other right-lined figures, are said to be mutually equilateral, when all the sides of the one are equal to the corresponding sides of the other, each to each : and they are said to be mutually equiangular, when the angles of the one are respectively equal to those of the other.

69. Identical figures are such as are both mutually equilateral and equiangular ; or that have all the sides and all the angles of the one, respectively equal to all the sides and all the angles of the other, each to each ; so that if the one figure were applied to, or laid upon the other, all the sides of the one would exactly fall upon and cover all the sides of the other ; the two becoming as it were but one and the same figure.

70. Similar figures, are those that have all the angles of the one equal to all the angles of the other, each to each, and the sides about the equal angles proportional.

71. The Perimeter of a figure, is the sum of all its sides taken together.

72. A Proposition, is something which is either proposed to be done, or to be demonstrated, and is either a problem or a theorem.

73. A Problem, is something proposed to be done.

74. A Theorem, is something proposed to be demonstrated.

75. A Lemma, is something which is premised, or demontrated, in order to render what follows more easy.

76. A Corollary, is a consequent truth, gained immediately from some preceding truth, or demonstration.

77. A Scholium, is a remark or observation made upon something going before it.

AXIOMS.

1. Things which are equal to the same thing are equal to each other.

2. When equals are added to equals, the wholes are equal.

3. When equals are taken from equals, the remainders are equal.

4. When equals are added to unequals, the wholes are unequal.

5. When equals are taken from unequals, the remainders are unequal,

6. Things which are double of the same thing, or equal things, are equal to each other.

7. Things which are halves of the same thing, are equal.

8. Every whole is equal to all its parts taken together.

9. Things which coincide, or fill the same space, are identical, or mutually equal in all their parts.

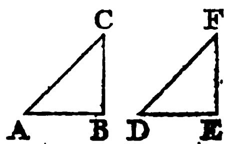
20. All right angles are equal to one another.

21. Angles that have equal measures, or arcs, are equal.

THEOREM I.

If two triangles have two sides and the included angle in the one, equal to two sides and the included angle in the other, the triangles will be identical, or equal in all respects.

In the two triangles ABC, DEF, if the side AC be equal to the side DF, and the side BC equal to the side EF, and the angle C equal to the angle F; then will the two triangles be identical, or equal in all respects.

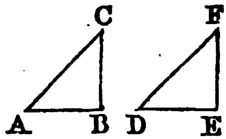


For conceive the triangle ABC to be applied to, or placed on, the triangle DEF, in such a manner that the point C may coincide with the point F, and the side AC with the side DF, which is equal to it.

Then, since the angle F is equal to the angle C (by hyp.), the side BC will fall on the side EF. Also, because AC is equal to DF, and BC equal to EF (by hyp.), the point A will coincide with the point D, and the point B with the point E; consequently the side AB will coincide with the side DE. Therefore the two triangles are identical, and have all their other corresponding parts equal (ax. 9), namely, the side AB equal to the side DE, the angle A to the angle D, and the angle B to the angle E. Q. E. D.

THEOREM II.

WHEN two triangles have two angles and the included side in the one, equal to two angles and the included side in the other, the triangles are identical, or have their other sides and angle equal.

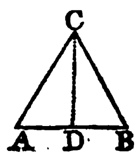


Let the two triangles ABC, DEF, have the angle A equal to the angle D, the angle B equal to the angle E, and the side AB equal to the side DE; then these two triangles will be identical.

For, conceive the triangle ABC to be placed on the triangle DEF, in such manner that the side AB may fall exactly on the equal side DE. Then, since the angle A is equal to the angle D (by hyp.), the side AC must fall on the side DF; and, in like manner, because the angle B is equal to the angle E, the side BC must fall on the side EF. Thus the three sides of the triangle ABC will be exactly placed on the three sides of the triangle DEF: consequently the two triangles are identical (ax. 9), having the other two sides AC, BC, equal to the two DF, EF, and the remaining angle C equal to the remaining angle F. Q. E. D.

THEOREM III.

IN an isosceles triangle, the angles at the base are equal. Or, if a triangle have two sides equal, their opposite angles will also be equal.



If the triangle ABC have the side AC equal to the side BC: then will the angle B be equal to the angle A.

For, conceive the angle C to be bisected, or divided into two equal parts, by the line CD, making the angle ACD equal to the angle BCD.

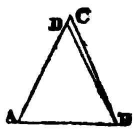
Then, the two triangles, ACD, BCD, have two sides and the contained angle of the one, equal to two sides and the contained angle of the other, viz. the side AC equal to BC, the angle ACD equal to BCD, and the side CD common; therefore these two triangles are identical, or equal in all respects (th. 1); and consequently the angle A equal to the angle B. Q. E. D.

Corol. 1. Hence the line which bisects the vertical angle of an isosceles triangle, bisects the base, and is also perpendicular to it.

Corol. 2. Hence too it appears, that every equilateral triangle, is also equiangular, or has all its angles equal.

THEOREM IV.

When a triangle has two of its angles equal, the sides opposite to them are also equal.



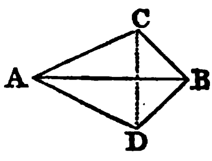
If the triangle ABC, have the angle CAB equal to the angle CBA, it will also have the side CA equal to the side CB.

For, if CA and CB be not equal, let CA be the greater of the two, and let DA be equal to CB, and join DB. Then, because DA, AB, are equal to CB, BA, each to each, and the angle DAB to CBA (hyp.), the triangles DAB, CBA, are equal in all respects (th. 1), a part to the whole, which is absurd; therefore CA is not greater than CB. In the same way it may be proved, that CB is not greater than CA. They are therefore equal. Q. E. D.

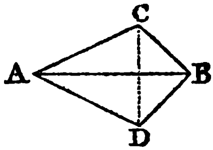
Corol. Hence every equiangular triangle is also equilateral.

THEOREM V.

When two triangles have all the three sides in the one, equal to all the three sides in the other, the triangles are identical, or have also their three triangles equal, each to each.



Let the two triangles ABC, ABD, have their three sides respectively, equal, viz. the side AB equal to AB, AC to AD, and BC to BD; then shall the two triangles be identical, or have their angles equal, viz. those angles that are opposite to the equal sides; namely, the angle BAC to the angle BAD, the angle ABC to the angle ABD, and the angle C to the angle D.



For, conceive the two triangles to be joined together by their longest equal sides, and draw the line CD.

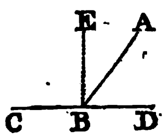
Then, in the triangle ACD, because the side AC is equal to AD (by hyp.), the angle ACD is equal to the angle ADC (th. 3). In like manner, in the triangle BCD, the angle BCD is equal to the angle BDC, because the side BC is equal to BD. Hence then, the angle ACD being equal to the angle ADC, and the angle BCD to the angle BDC, by equal additions the sum of the two angles ACD, BCD, is equal to the sum of the two ADC, BDC, (ax. 2), that is, the whole angle ACB equal to the whole angle ADB.

Since then, the two sides AC, CB, are equal to the two sides AD, DB, each to each, (by hyp.), and their contained angles ACB, ADB, also equal, the two triangles ABC, ABD, are identical (th. 1), and have the other angles equal, viz. the angle BAC to the angle BAD, and the angle ABC to the angle ABD. Q. E. D.

THEOREM VI.

When one line meets another, the angles which it makes on the same side of the other, are together equal to two right angles.

Let the line AB meet the line CD: then will the two angles ABC, ABD, taken together, be equal to two right angles.



For, first, when the two angles ABC, ABD, are equal to each other, they are both of them right angles (def. 15.)

But when the angles are unequal, suppose BE drawn perpendicular to CD. Then, since the two angles EBC, EBD, are right angles (def. 15), and the angle EBD is equal to the two angles EBA, ABD, together (ax. 8), the three angles, EBC, EBA, and ABD, are equal to two right angles.

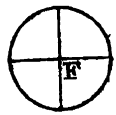
But the two angles EBC, EBA, are together equal to the angle ABC (ax. 8). Consequently the two angles ABC, ABD, are also equal to two right angles. Q. E. D.

Corol. 1. Hence also, conversely, if the two angles ABC, ABD, on both sides of the line AB, make up together two right angles, then CB and BD form one continued right line CD.

Corol. 2. Hence, all the angles which can be made, at any point E, by any number of lines, on the same side of the right line CD, are, when taken all together, equal to two right angles.

Corol. 3. And, as all the angles that can be made on the other side of the line CD are also equal to two right angles; therefore all the angles that can be made quite round a point B, by any number of lines, are equal to four right angles.

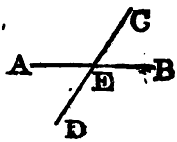
Corol. 4. Hence also the whole circumference of a circle, being the sum of the measures of all the angles that can be made about the centre F (def. 57), is the measure of four right angles. Consequently, a semicircle, or 180 degrees, is the measure of two right angles; and a quadrant, or 90 degrees, the measure of one right angle.



THEOREM VII.

When two lines intersect each other, the opposite angles are equal.

Let the two lines AB, CD, intersect in the point E; then will the angle AEC be equal to the angle BED, and the angle AED equal to the angle CEB.



For, since the line CE meets the line AB, the two angles AEC, BEC, taken together, are equal to two right angles (th. 6).

In like manner, the line BE, meeting the line CD, makes the two angles BEC, BED, equal to two right angles.

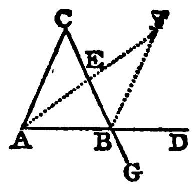
Therefore the sum of the two angles AEC, BEC, is equal to the sum of the two BEC, BED (ax. 1).

And if the angle BEC, which is common, be taken away from both these, the remaining angle AEC will be equal to the remaining angle BED (ax. 3).

And in like manner it may be shown, that the angle AED is equal to the opposite angle BEC.

THEOREM VIII.

When one side of a triangle is produced, the outward angle is greater than either of the two inward opposite angles.



Let ABC be a triangle, having the side AB produced to D; then will the outward angle CBD be greater than either of the inward opposite angles A or C.

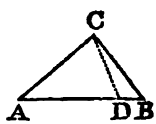
For, conceive the side BC to be bisected in the point E, and draw the line AE, producing it till EF be equal to AE; and join BF.

Then, since the two triangles AEC, BEF, have the side AE = the side EF, and the side CE = the side BE (by suppos.) and the included or opposite angles at E also equal (th. 7), therefore those two triangles are equal in all respects (th. 1), and have the angle C = the corresponding angle EBF. But the angle CBD is greater than the angle EBF; consequently the said outward angle CBD is also greater than the angle C.

In like manner, if CB be produced to G, and AB be bisected, it may be shown that the outward angle ABG, or its equal CBD, is greater than the other angle A.

THEOREM IX.

The greater side, of every triangle, is opposite to the greater angle; and the greater angle opposite to the greater side.



Let ABC be a triangle, having the side AB greater than the side AC; then will the angle ACB, opposite the greater side AB, be greater than the angle B, opposite the less side AC.

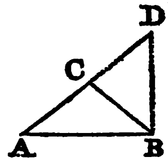
For, on the greater side AB, take the part AD equal to the less side AC, and join CD. Then, since BCD is a triangle, the outward angle ADC is greater than the inward opposite angle B (th. 8). But the angle ACD is equal to the said outward angle ADC, because AD is eqal to AC (th. 3). Consequently the angle ACD also is greater than the angle B. And since the angle ACD is only a part of ACB, much more must the whole angle ACB be greater than the angle B. Q. E. D.

Again, conversely, if the angle C be greater than the angle B, then will the side AB, opposite the former, be greater than the side AC, opposite the latter.

For, if AB be not greater than AC, it must be either equal to it, or less than it. But it cannot be equal, for then the angle C would be equal to the angle B (th. 3), which it is not, by the supposition. Neither can it be less, for then the angle C would be less than the angle B, by the former part of this ; which is also contrary to the supposition. The side AB, then, being neither equal to AC, nor less than it, must necessarily be greater. Q. E. D.

THEOREM X.

The sum of any two sides of a triangle is greater than the third side.



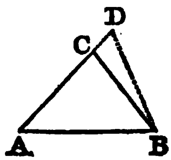
Let ABC be a triangle ; then will the sum of any two of its sides be greater than the third side, as for instance, AC + CB greater than AB.

For, produce AC till CD be equal to CB, or AD equal to the sum of the two AC + CB; and join BD :—Then, because CD is equal to CB (by constr.), the angle D is equal to the angle CBD (th. 3). But the angle ABD is greater than the angle CBD, consequently it must also be greater than the angle D. And, since the greater side of any triangle is opposite to the greater angle (th. 9), the side AD (of the triangle ABD) is greater than the side AB. But AD is equal to AC and CD, or AC and CB, taken together (by constr.) ; therefore AC + CB is also greater than AB. Q. E. D.

Corol. The shortest distance between two points, is a single right line drawn from the one point to the other.

THEOREM XI.

The difference of any two sides of a triangle, is less than the third side.



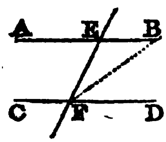
Let ABC be a triangle ; then will the difference of any two sides, as AB — AC, be less than the third side BC.

For, produce the less side AC to D, till AD be equal to the greater side AB, so that CD may be the difference of the two sides AB — AC ; and join BD. Then, because AD is equal to AB (by constr.), the opposite angels D and ABD are equal (th. 3). But the angle CBD is less than the angle ABD, and consequently also less than the equal angle D. And since the greater side of any triangle is opposite to the greater angle (th. 9), the side CD (of the triangle BCD) is less than the side BC. Q. E. D.

Otherwise. Set off upon AB a distance AI equal to AC. Then (th. 20) AC + CB is greater than AB, that is, greater than AI + IC. From these, take away the equal parts AC, AI, respectively; and there remains CB greater than IC. Consequently, IC is less than CB. Q. E. D.

THEOREM XII.

When a line intersects two parallel lines, it makes the alternate angles equal to each other.



Let the line EF cut the two parallel line AB, CD; then will the angle AEF be equal to the alternate angle EFD.

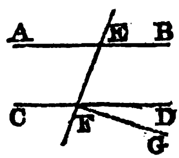
For if they are not equal, one of them must be greater than the other; let it be EFD for instance, which is the greater, if possible; and conceive the line FB to be drawn, cutting off the part or angle EFB equal to the angle AEF, and meeting the line AB in the point B.

Then, since the outward angle AEF, of the triangle BEF, is greater than the inward opposite angle EFB (th. 8); and since these two angles also are equal (by the constr.) it follows, that those angles are both equal and unequal at the same time: which is impossible. Therefore the angle EFD is not unequal to the alternate angle AEF, that is, they are equal to each other. Q. E. D.

Corol. Right lines which are perpendicular to one, of two parallel lines, are also perpendicular to the other.

THEOREM XIII.

When a line, cutting two other lines, makes the alternate angles equal to each other, those two lines are parallel.



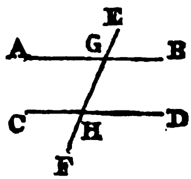
Let the line EF, cutting the two lines AB, CD, make the alternate angles AEF, DFE, equal to each other; then will AB be parallel to CD.

For if they be not parallel, let some other line, as FG, be parallel to AB. Then, because of these parallels, the angle AEF is equal to the alternate angle EFG (th. 12). But the angle AEF is equal to the angle EFD (by hyp.) Therefore the angle EFD is equal to the angle EFG (ax. 1); that is, a part is equal to the whole, which is impossible. Therefore no line but CD can be parallel to AB. Q. E. D.

Corol. Those lines which are perpendicular to the same lines, are parallel to each other.

THEOREM XIV.

When a line cuts two parallel lines, the outward angle is equal to the inward opposite one, on the same side; and the two inward angles, on the same side, equal to two right angles.



Let the line EF cut the two parallel lines AB, CD; then will the outward angle EGB be equal to the inward opposite angle GHD, on the same side of the line EF; and the two inward angles BGH, GHD, taken together, will be equal to two right angles.

For since the two lines AB, CD, are parallel, the angle AGH is equal to the alternate angle GHD, (th. 12.) But the angle AGH is equal to the opposite angle EGB (th. 7). Therefore the angle EGB is also equal to the angle GHD (ax. 1). Q. E. D.

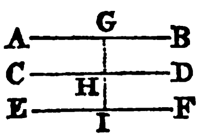
Again, because the two adjacent angles EGB, BGH, are together equal to two right angles (th. 6); of which the angle EGB has been shown to be equal to the angle GHD; therefore the two angles BGH, GHD, taken together, are also equal to two right angles.

Corol. 1. And, conversely, if one line meeting two other lines, make the angles on the same side of it equal, those two lines are parallels.

Corol. 2. If a line, cutting two other lines, make the sum of the two inward angles on the same side, less than two right angles, those two lines will not be parallel, but will meet each other when produced.

THEOREM XV.

Those lines which are parallel to the same line, are parallel to each other.

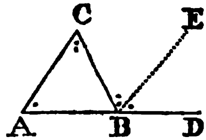


Let the lines AB, CD, be each of them parallel to the line EF; then shall the lines AB, CD, be parallel to each other.

For, let the line GI be perpendicular to EF. Then will this line be also perpendicular to both the lines AB, CD (corol. th. 12), and consequently the two lines AB, CD, are parallels (corol. th. 13). Q. E. D.

THEOREM XVI.

When one side of a triangle is produced, the outward angle is equal to both the inward opposite angles taken together.

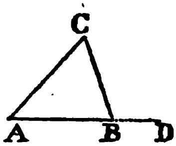


Let the side AB, of the triangle ABC, be produced to D; then will the outward angle CBD be equal to the sum of the two inward opposite angles A and C.

For, conceive BE to be drawn parallel to the side AC of the triangle. Then BC, meeting the two parallels AC, BE, makes the alternate angles C and CBE equal (th. 12). And AD, cutting the same two parallels AC, BE, makes the inward and outward angles on the same side, A and EBD, equal to each other (th. 14). Therefore, by equal additions, the sum of the two angles A and C, is equal to the sum of the two CBE and EBD, that is, to the whole angle CBD (by ax. 2). Q. E. D.

THEOREM XVII.

In any triangle, the sum of all the three angles is equal to two right angles.



Let ABC be any plane triangle; then the sum of the three angles A + B + C is equal to two right angles.

For, let the side AB be produced to D. Then the outward angle CBD is equal to the sum of the two inward opposite angles A + C (th. 16). To each of these equals add the inward angle B, then will the sum of the three inward angles A + B + C be equal to the sum of the two adjacent angles ABC + CBD (ax. 2). But the sum of these two last adjacent angles is equal to two right angles (th. 6). Therefore also the sum of the three angles of the triangle A + B + C is equal to two right angles (ax. 1). Q. E. D.

Corol. 1. If two angles in one triangle, be equal to two angles in another triangle, the third angles will also be equal (ax. 3), and the two triangles equiangular.

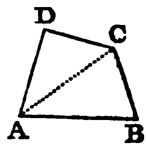
Corol. 2. If one angle in one triangle, be equal to one angle in another, the sums of the remaining angles will also be equal (ax. 3).

Corol. 3. If one angle of a triangle be right, the sum of the other two will also be equal to a right angle, and each of them singly will be acute, or less than a right angle.

Corol. 4. The two least angles of every triangle are acute, or each less than a right angle.

THEOREM XVIII.

In any quadrangle, the sum of all the four inward angles, is equal to four right angles.



Let ABCD be a quadrangle; then the sum of the four inward angles, A + B + C + D is equal to four right angles.

Let the diagonal AC be drawn, dividing the quadrangle into two triangles, ABC, ADC. Then, because the sum of the three angles of each of these triangles is equal to two right angles (th. 17); it follows, that the sum of all the angles of both triangles, which make up the four angles of the quadrangle, must be equal to four right angles (ax. 2).

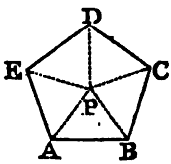
Q. E. D.

Corol. 1. Hence, if three of the angles be right ones, the fourth will also be a right angle.

Corol. 2. And if the sum of two of the four angles be equal to two right angles, the sum of the remaining two will also be equal to two right angles.

THEOREM XIX.

In any figure whatever, the sum of all the inward angles, taken together, is equal to twice as many right angles, wanting four, as the figure has sides.

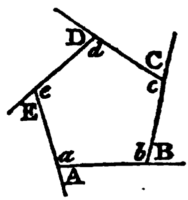


Let ABCDE be any figure; then the sum of all its inward angles, A + B + C + D + E, is equal to twice as many right angles, wanting four, as the figure has sides.

For, from any point P, within it, draw lines, PA, PB, PC, &c. to all the angles, dividing the polygon into as many triangles as it has sides. Now the sum of the three angles of each of these triangles, is equal to two right angles (th. 17); therefore the sum of the angles of all the triangles is equal to twice as many right angles as the figure has sides. But the sum of all the angles about the point P, which are so many of the angles of the triangles, but no part of the inward angles of the polygon, is equal to four right angles (corol. 3, th. 6), and must be deducted out of the former sum. Hence it follows that the sum of all the inward angles of the polygon alone, A + B + C + D + E, is equal to twice as many right angles as the figure has sides, wanting the said four right angles. Q. E. D.

THEOREM XX.

WHEN every side of any figure is produced out, the sum of all the outward angles thereby made, is equal to four right angles.

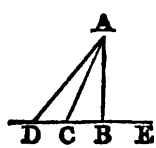


Let A, B, C, &c. be the outward angles of any polygon, made by producing all the sides; then will the sum A + B + C + D + E, of all those outward angles, be equal to four right angles.

For every one of these outward angles, together with its adjacent inward angle, make up two right angles, as A + *a* equal to two right angles, being the two angles made by one line meeting another (th. 6). And there being as many outward, or inward angles, as the figure has sides; therefore the sum of all the inward and outward angles, is equal to twice as many right angles as the figure has sides. But the sum of all the inward angles with four right angles, is equal to twice as many right angles as the figure has sides (th. 19). Therefore the sum of all the inward and all the outward angles, is equal to the sum of all the inward angles and four right angles (by ax. 1). From each of these take away all the inward angles, and there remains all the outward angles equal to four right angles (by ax. 3).

THEOREM XXI.

A PERPENDICULAR is the shortest line that can be drawn from a given point to an indefinite line. And, of any other lines drawn from the same point, those that are nearest the perpendicular are less than those more remote.



If AB, AC, AD, &c. be lines drawn from the given point A, to the indefinite line DE, of which AB is perpendicular; then shall the perpendicular AB be less than AC, and AC less than AD, &c.

For, the angle B being a right one, the

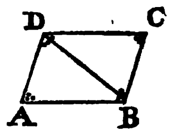
angle C is acute (by cor. 3, th. 17), and therefore less than the angle B. But the less angle of a triangle is subtended by the less side (th. 9). Therefore the side AB is less than the side AC.

Again, the angle ACB being acute, as before, the adjacent angle ACD will be obtuse (by th. 6); consequently the angle D is acute (corol. 3, th. 17), and therefore is less than the angle C. And since the less side is opposite to the less angle, therefore the side AC is less than the side AD. Q. E. D.

Corol. A perpendicular is the least distance of a given point from a line.

THEOREM XXII.

THE opposite sides and angles of any parallelogram are equal to each other; and the diagonal divides it into two equal triangles.



Let ABCD be a parallelogram, of which the diagonal is BD; then will its opposite sides and angles be equal to each other, and the diagonal BD will divide it into two equal parts, or triangles.

For, since the sides AB and DC are parallel, as also the sides AD and BC (defin. 32), and the line BD meets them; therefore the alternate angles are equal (th. 12), namely, the angle ABD to the angle CDB, and the angle ADB to the angle CBD. Hence the two triangles, having two angles in the one equal to two angles in the other, have also their third angles equal (cor. 1, th. 17), namely, the angle A equal to the angle C, which are two of the opposite angles of the parallelogram.

Also, if to the equal angles ABD, CDB, be added the equal angles CBD, ABD, the wholes will be equal (ax. 2), namely, the whole angle ABC to the whole ADC, which are the other two opposite angles of the parellelogram. Q. E. D.

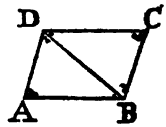
Again, since the two triangles are mutually equiangular and have a side in each equal, viz. the common side BD; therefore the two triangles are identical (th. 2), or equal in all respects, namely, the side AB equal to the opposite side DC, and AD equal to the opposite side BC, and the whole triangle ABD equal to the whole triangle BCD. Q. E. D.

Corol. 1. Hence, if one angle of a parallelogram be a right angle, all the other three will also be right angles, and the parallelogram a rectangle.

Corol. 2. Hence also, the sum of any two adjacent angles of a parallelogram is equal to two right angles.

THEOREM XXIII.

Every quadrilateral, whose opposite sides are equal, is a parallelogram, or has its opposite sides parallel.



Let ABCD be a quadrangle, having the opposite sides equal, namely, the side AB equal to DC, and AD equal to BC; then shall these equal sides be also parallel, and the figure a parallelogram.

For, let the diagonal BD be drawn. Then, the triangles, ABD, CBD, being mutually equilateral (by hyp.), they are also mutually equiangular (th. 5), or have their corresponding angles equal; consequently the opposite sides are parallel (th. 13); viz. the side AB parallel to DC, and AD parallel to BC, and the figure is a parallelogram. Q. E. D.

THEOREM XXIV.

Those lines which join the corresponding extremes of two equal and parallel lines, are themselves equal and parallel.

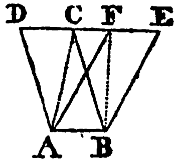
Let AB, DC, be two equal and parallel lines; then will the lines AD, BC, which join their extremes, be also equal and parallel. [See the fig. above.]

For, draw the diagonal BD. Then, because AB and DC are parallel (by hyp.), the angle ABD is equal to the alternate angle BDC (th. 12). Hence then, the two triangles having two sides and the contained angles equal, viz. the side AB equal to the side DC, and the side BD common, and the contained angle ABD equal to the contained angle BDC, they have the remaining sides and angles also respectively equal (th. 1); consequently AD is equal to BC, and also parallel to it (th. 12). Q. E. D.

THEOREM XXV.

Parallelograms, as also triangles, standing on the same base, and between the same parallels, are equal to each other.

Let ABCD, ABEF, be two parallelograms, and ABC, ABF, two triangles, standing on the same base AB, and between the same parallels AB, DE; then will the parallelogram ABCD be equal to the parallelogram ABEF, and the triangle ABC equal to the triangle ABF.



For, since the line DE cuts the two parallels AF, BE, and the two AD, BC, it makes the angle E equal to the angle AFD, and the angle D equal to the angle BCE (th. 14); the two triangles ADF, BCE, are therefore equiangular (cor. 1, th. 17); and having the two corresponding sides AD, BC, equal (th. 22), being opposite sides of a parallelogram, these two triangles are identical, or equal in all respects (th. 2). If each of these equal triangles then be taken from the whole space ABED, there will remain the parallelogram ABEF in the one case, equal to the parallelograms ABCD in the other (by ax. 3).

Also the triangles ABC, ABF, on the same base AB, and between the same parallels, are equal, being the halves of the said equal parallelograms (th. 22). Q. E. D.

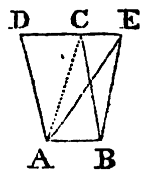
Corol. 1. Parallelograms, or triangles, having the same base and altitude, are equal. For the altitude is the same as the perpendicular or distance between the two parallels, which is every where equal, by the definition of parallels.

Corol. 2. Parallelograms, or triangles, having equal bases and altitudes, are equal. For, if the one figure be applied with its base on the other, the bases will coincide or be the same, because they are equal: and so the two figures, having the same base and altitude, are equal.

THEOREM XXVI.

If a parallelogram and a triangle, stand on the same base, and between the same parallels, the parallelogram will be double the triangle, or the triangle half the parallelogram.

Let ABCD be the parallelogram, and ABE a triangle, on the same base AB, and between the same parallels AB, DE; then will the parallelogram ABCD be double the triangle ABE, or the triangle half the parallelogram.



For, draw the diagonal AC of the parallelogram, dividing it into two equal parts (th. 22). Then because the triangles

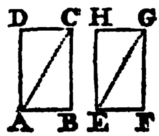
ABC, ABE, on the same base, and between the same parallels, are equal (th. 25); and because the one triangle ABC is half the parallelogram ABCD (th. 22), the other equal triangle ABE is also equal to half the same parallelogram ABCD. Q. E. D.

Corol. 1. A triangle is equal to half a parallelogram of the same base and altitude, because the altitude is the perpendicular distance between the parallels, which is every where equal, by the definition of parallels.

Corol. 2. If the base of a parallelogram be half that of a triangle, of the same altitude, or the base of the triangle be double that of the parallelogram, the two figures will be equal to each other.

THEOREM XXVII.

Rectangles that are contained by equal lines, are equal to each other.



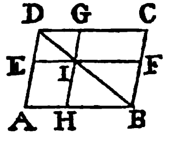
Let BD, FH, be two rectangles, having the sides AB, BC, equal to the sides EF, FG, each to each; then will the rectangle BD be equal to the rectangle FH.

For, draw the two diagonals AC, EG, dividing the two parallelograms each into two equal parts. Then the two triangles ABC, EFG, are equal to each other (th. 1), because they have the two sides AB, BC, and the contained angle B, equal to the two sides EF, FG, and the contained angle F (by hyp.). But these equal triangles are the halves of the respective rectangles. And because the halves, or the triangles, are equal, the wholes, or the rectangles DB, HF, are also equal (by ax. 6). Q. E. D.

Corol. The squares on equal lines are also equal; for every square is a species of rectangle.

THEOREM XXVIII.

The complements of the parallelograms, which are about the diagonal of any parallelogram, are equal to each other.



Let AC be a parallelogram, BD a diagonal, EIF parallel to AB or DC, and GIH parallel to AD or BC, making AI, IC, complements to the parallelograms EG, HF, which are about the diagonal DB: then will the complement AI be equal to the complement IC.

For, since the diagonal DB bisects the three parallelograms AC, EG, HF (th. 22); therefore, the whole triangle DAB being equal to the whole triangle DCB, and the parts DEI, IHB, respectively equal to the parts DGI, IFB, the remaining parts AI, IC, must also be equal (by ax. 3). Q. E. D.

THEOREM XXIX.

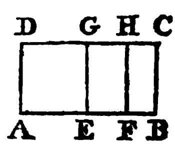
A TRAPEZOID, or trapezium having two sides parallel, is equal to half a parallelogram, whose base is the sum of those two sides, and its altitude the perpendicular distance between them.

Let ABCD be the trapezoid, having its two sides AB, DC, parallel; and in AB produced take BE equal to DC, so that AE may be the sum of the two parallel sides; produce DC also, and let EF, GC, BH, be all three parallel to AD. Then is AF a parallelogram of the same altitude with the trapezoid ABCD, having its base AE equal to the sum of the parallel sides of the trapezoid; and it is to be proved that the trapezoid ABCD is equal to half the parallelogram AF.

Now, since triangles, or parallelograms, of equal bases and altitude, are equal (corol. 2, th. 25), the parallelogram DG is equal to the parallelogram HE, and the triangle CGB equal to the triangle CHB; consequently the line BC bisects, or equally divides, the parallelogram AF, and ABCD is the half of it. Q. E. D.

THEOREM XXX.

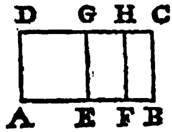
THE sum of all the rectangles contained under one whole line, and the several parts of another line, any way divided, is equal to the rectangle contained under the two whole lines.



Let AD be the one line, and AB the other, divided into the parts AE, EF, FB; then will the rectangle contained by AD and AB, be equal to the sum of the rectangles of AD and AE, and AD and EF, and AD and FB: thus expressed,

$$AD \cdot AB = AD \cdot AE + AD \cdot EF + AD \cdot FB.$$

For, make the rectangle AC of the two whole lines AD, AB; and draw EG, FH, perpendicular to AB, or parallel to AD, to which they are equal (th. 22). Then the whole rectangle AC is made up of all the other rectangles AG, EH,

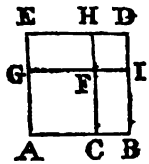


FC. But these rectangles are contained by AD and AE, EG and EF, FH and FB; which are equal to the rectangles of AD and AE, AD and EF, AD and FB, because AD is equal to each of the two EG, FH. Therefore the rectangle AD . AB is equal to the sum of all the other rectangles AD . AE, AD . EF, AD . FB. Q. E. D.

Corol. If a right line be divided into any two parts, the square on the whole line, is equal to both the rectangles of the whole line and each of the parts.

THEOREM XXXI.

THE square of the sum of two lines, is greater than the sum of their squares, by twice the rectangle of the said lines. Or, the square of a whole line, is equal to the squares of its two parts, together with twice the rectangle of those parts.



Let the line AB be the sum of any two lines AC, CB; then will the square of AB be equal to the squares of AC, CB, together with twice the rectangle of AC . CB. That is, $AB^2 = AC^2 + CB^2 + 2AC \cdot CB$.

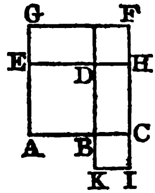
For, let ABDE be the square on the sum or whole line AB, and ACFG the square on the part AC. Produce CF and GF to the other sides at H and I.

From the lines CH, GI, which are equal, being each equal to the sides of the square AB or BD (th. 22), take the parts CF, GF, which are also equal, being the sides of the square AF, and there remains FH equal to FI, which are also equal to DH, DI, being the opposite sides of the parallelogram. Hence the figure HI is equilateral: and it has all its angles right ones (corol. 1, th. 22); it is therefore a square on the line FI, or the square of its equal CB. Also the figures EF, FB, are equal to two rectangles under AC and CB, because GF is equal to AC, and FH or FI equal to CB. But the whole square AD is made up of the four figures, viz. the two squares AF, FD, and the two equal rectangles EF, FB. That is, the square of AB is equal to the squares of AC, CB, together with twice the rectangle of AC, CB. Q. E. D.

Corol. Hence, if a line be divided into two equal parts; the square of the whole line will be equal to four times the square of half the line.

THEOREM XXXII.

The square of the difference of two lines, is less than the sum of their squares, by twice the rectangle of the said lines.



Let AC, BC, be any two lines, and AB their difference: then will the square of AB be less than the squares of AC, BC, by twice the rectangle of AC and BC. Or, $AB^2 = AC^2 + BC^2 - 2AC \cdot BC$.

For, let ABDE be the square on the difference AB, and ACFG the square on the line AC. Produce ED to H; also produce DB and HC, and draw KI, making BI the square of the other line BC.

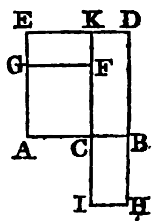
Now it is visible that the square AD is less than the two squares AF, BI, by the two rectangles EF, DI. But GF is equal to the one line AC, and GE or FH is equal to the other line BC; consequently the rectangle EF, contained under EG and GF, is equal to the rectangle of AC and BC.

Again, FH being equal to CI or BC or DH, by adding the common part HC, the whole HI will be equal to the whole FC, or equal to AC; and consequently the figure DI is equal to the rectangle contained by AC and BC.

Hence the two figures EF, DI, are two rectangles of the two lines AC, BC; and consequently the square of AB is less than the squares of AC, BC, by twice the rectangle AC . BC. Q. E. D.

THEOREM XXXIII.

The rectangle under the sum and difference of two lines, is equal to the difference of the squares of those lines*.



Let AB, AC, be any two unequal lines; then will the difference of the squares of AB, AC, be equal to a rectangle under their sum and difference. That is, $AB^2 - AC^2 = \overline{AB + AC} \cdot \overline{AB - AC}$.

For, let ABDE be the square of AB, and ACFG the square of AC. Produce DB till BH be equal to AC; draw HI parallel to AB or ED, and produce FC both ways to I and K.

---

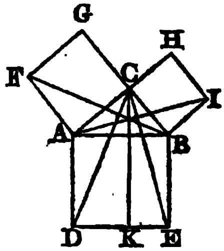
* This and the two preceding theorems, are evinced algebraically, by the three expressions

$$(a+b)^2 = a^2 + 2ab + b^2 = a^2 + b^2 + 2ab$$
$$(a-b)^2 = a^2 - 2ab + b^2 = a^2 + b^2 - 2ab$$
$$(a+b)(a-b) = a^2 - b^2.$$

Then the difference of the two squares AD, AF, is evidently the two rectangles EF, KB. But the rectangles EF, BI are equal, being contained under equal lines; for EK and BH are each equal to AC, and GE is equal to CB, being each equal to the difference between AB and AC, or their equals AE and AG. Therefore the two EF, KB, are equal to the two KB, BI, or to the whole KH; and consequently KH is equal to the difference of the squares AD, AF. But KH is a rectangle contained by DH, or the sum of AB and AC, and by KD, or the difference of AB and AC. Therefore the difference of the squares of AB, AC, is equal to the rectangle under their sum and difference. Q. E. D.

THEOREM XXXIV.

In any right-angled triangle, the square of the hypothenuse, is equal to the sum of the squares of the other two sides.



Let ABC be a right-angled triangle, having the right angle C; then will the square of the hypothenuse AB, be equal to the sum of the squares of the other two sides AC, CB. Or $AB^2 = AC^2 + BC^2$.

For, on AB describe the square AE, and on AC, CB, the squares AG, BH; then draw CK parallel to AD or BE; and join AI, BF, CD, CE.

Now, because the line AC meets the two CG, CB, so as to make two right angles, these two form one straight line GB (corol. 1, th. 6). And because the angle FAC is equal to the angle DAB, being each a right angle, or the angle of a square; to each of these equals add the common angle BAC, so will the whole angle or sum FAB, be equal to the whole angle or sum CAD. But the line FA is equal to the line AC, and the line AB to the line AD, being sides of the same square; so that the two sides FA, AB, and their included angle FAB, are equal to the two sides CA, AD, and the contained angle CAD, each to each: therefore the whole triangle AFB is equal to the whole triangle ACD (th. 1).

But the square AG is double the triangle AFB, on the same base FA, and between the same parallels FA, GB (th. 26); in like manner the parallelogram AK is double the triangle ACD, on the same base AD, and between the same parallels AD, CK. And since the doubles of equal things, are equal (by ax. 6); therefore the square AG is equal to the parallelogram AK.

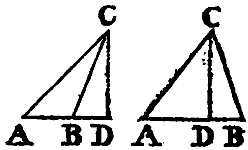
In like manner, the other square BH is proved equal to the other parallelogram BK. Consequently the two squares AG and BH together, are equal to the two parallelograms AK and BK together, or to the whole square AK. That is, the sum of the two squares on the two less sides, is equal to the square on the greatest side. Q. E. D.

Corol. 1. Hence, the square of either of the two less sides, is equal to the difference of the squares of the hypothenuse and the other side (ax. 3); or, equal to the rectangle contained by the sum and difference of the said hypothenuse and other side (th. 33).

Corol. 2. Hence also, if two right-angled triangles have two sides of the one equal to two corresponding sides of the other; their third sides will also be equal, and the triangles identical.

THEOREM XXXV.

In any triangle, the difference of the squares of the two sides, is equal to the difference of the squares of the segments of the base, or of the two lines, or distances, included between the extremes of the base and the perpendicular.



Let ABC be any triangle, having CD perpendicular to AB; then will the difference of the squares of AC, BC, be equal to the difference of the squares of AD, BD; that is, $AC^2 - BC^2 = AD^2 - BD^2$.

For, since AC^2 is equal to $AD^2 + CD^2$ and BC^2 is equal to $BD^2 + CD^2$ } (by th. 34);
Theref. the difference between AC^2 and BC^2,
is equal to the difference between $AD^2 + CD^2$ and $BD^2 + CD^2$,
or equal to the difference between AD^2 and BD^2,
by taking away the common square CD^2. Q. E. D.

Corol. The rectangle of the sum and difference of the two sides of any triangle, is equal to the rectangle of the sum and difference of the distances between the perpendicular and the two extremes of the base, or equal to the rectangle of the base and the difference or sum of the segments, according as the perpendicular falls within or without the triangle.

That is, $(AC+BC) \cdot (AC-BC) = (AD+BD) \cdot (AD-BD)$
Or, $(AC+BC) \cdot (AC-BC) = AB \cdot (AD-BD)$ in the 2d fig.
And $(AC+BC) \cdot (AC-BC) = AB \cdot (AD+BD)$ in the 1st fig.

THEOREM XXXVI.

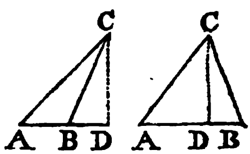
In any obtuse-angled triangle, the square of the side subtending the obtuse angle, is greater than the sum of the squares of the other two sides, by twice the rectangle of the base and the distance of the perpendicular from the obtuse angle.

Let ABC be a triangle, obtuse angled at B, and CD perpendicular to AB; then will the square of AC be greater than the squares of AB, BC, by twice the rectangle of AB, BD. That is, $AC^2 = AB^2 + BC^2 + 2AB \cdot BD$. See the 1st fig. above, or below.

For, $AD^2 = AB^2 + BD^2 + 2AB \cdot BD$ (th. 31).
And $AD^2 + CD^2 = AB^2 + BD^2 + CD^2 + 2AB \cdot BD$ (ax. 2).
But $AD^2 + CD^2 = AC^2$, and $BD^2 + CD^2 = BC^2$ (th. 34).
Therefore $AC^2 = AB^2 + BC^2 + 2AB \cdot BD$. Q. E. D.

THEOREM XXXVII.

In any triangle, the square of the side subtending an acute angle, is less than the squares of the base and the other side, by twice the rectangle of the base and the distance of the perpendicular from the acute angle.



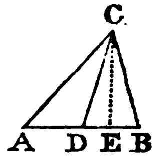
Let ABC be a triangle, having the angle A acute, and CD perpendicular to AB; then will the square of BC, be less than the squares of AB, AC, by twice the rectangle of AB, AD. That is, $BC^2 = AB^2 + AC^2 - 2AD \cdot AB$.

For $BD^2 = AD^2 + AB^2 - 2AD \cdot AB$ (th. 32).
And $BD^2 + DC^2 = AD^2 + DC^2 + AB^2 - 2AD \cdot AB$ (ax. 2).
Therefore $BC^2 = AC^2 + AB^2 - 2AD \cdot AB$ (th. 34). Q. E. D.

THEOREM XXXVIII.

In any triangle, the double of the square of a line drawn from the vertex to the middle of the base, together with double the square of the half base, is equal to the sum of the squares of the other two sides.

Let ABC be a triangle, and CD the line drawn from the vertex to the middle of the base AB, bisecting it into the two equal parts AD, DB; then will the sum of the squares of AC, CB, be equal to twice the sum of the squares of CD, AD; or $AC^2 + CB^2 = 2CD^2 + 2AD^2$.

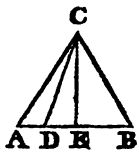


For $AC^2 = CD^2 + AD^2 + 2AD \cdot DE$ (th. 36).
And $BC^2 = CD^2 + BD^2 - 2AD \cdot DE$ (th. 37).
Therefore $AC^2 + BC^2 = 2CD^2 + AD^2 + BD^2$
$= 2CD^2 + 2AD^2$ (ax. 2). Q. E. D.

THEOREM XXXIX.

In an isosceles triangle, the square of a line drawn from the vertex to any point in the base, together with the rectangle of the segments of the base, is equal to the square of one of the equal sides of the triangle.

Let ABC be the isosceles triangle, and CD a line drawn from the vertex to any point D in the base: then will the square of AC, be equal to the square of CD, together with the rectangle of AD and DB. That is, $AC^2 = CD^2 + AD \cdot DB$.

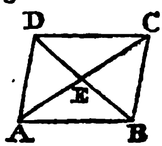


For $AC^2 - CD^2 = AE^2 - DE^2$ (th. 35).
$= AD \cdot DB$ (th. 33).
Therefore, $AC^2 = CD^2 + AD \cdot DB$ (ax. 2). Q. E. D.

THEOREM XL.

In any parallelogram, the two diagonals bisect each other; and the sum of their squares is equal to the sum of the squares of all the four sides of the parallelogram.

Let ABCD be a parallelogram, whose diagonals intersect each other in E: then will AE be equal to EC, and BE to ED; and the sum of the squares of AC, BD, will be equal to the sum of the squares of AB, BC, CD, DA. That is,



$AE = EC$, and $BE = ED$,
and $AC^2 + BD^2 = AB^2 + BC^2 + CD^2 + DA^2$.

For, the triangles AEB, DEC, are equiangular, because they have the opposite angles at E equal (th. 7), and the two lines AC, BD, meeting the parallels AB, DC, make the angle BAE equal to the angle DCE, and the angle ABE, equal to the angle CDE, and the side AB equal to the side DC (th. 22); therefore these two triangles are identical, and have their corresponding sides equal (th. 2), viz. AE = EC, and BE = ED.

Again, since AC is bisected in E, the sum of the squares $AD^2 + DC^2 = 2AE^2 + 2DE^2$ (th. 38).

In like manner, $AB^2 + BC^2 = 2AE^2 + 2BE^2$ or $2DE^2$.

Theref. $AB^2 + BC^2 + CD^2 + DA^2 = 4AE^2 + 4DE^2$ (ax. 2).

But, because the square of a whole line is equal to 4 times the square of half the line (cor. th. 31), that is, $AC^2 = 4AE^2$, and $BD^2 = 4DE^2$:

Theref. $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2$ (ax. 1).

Q. E. D.

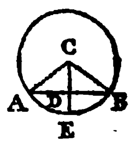
Cor. 1. If AD = DC, or the parallelogram be a rhombus; then $AD^2 = AE^2 + ED^2$, $CD^2 = DE^2 + CE^2$, &c.

Cor. 2. Hence, and by th. 34, the diagonals of a rhombus intersect at right angles.

THEOREM XLI.

IF a line, drawn through or from the centre of a circle, bisect a chord, it will be perpendicular to it; or, if it be perpendicular to the chord, it will bisect both the chord and the arc of the chord.

Let AB be any chord in a circle, and CD a line drawn from the centre C to the chord. Then, if the chord be bisected in the point D, CD will be perpendicular to AB.



Draw the two radii CA, CB. Then the two triangles ACD, BCD, having CA equal to CB (def. 44), and CD common, also AD equal to DB (by hyp.); they have all the three sides of the one, equal to all the three sides of the other, and so have their angles also equal (th. 5). Hence then, the angle ADC being equal to the angle BDC, these angles are right angles, and the line CD is perpendicular to AB (def. 11).

Again, if CD be perpendicular to AB, then will the chord

AB be bisected at the point D, or have AD equal to DB; and the arc AEB bisected in the point E, or have AE equal EB.

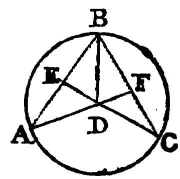
For, having drawn CA, CB, as before: Then, in the triangle ABC, because the side CA is equal to the side CB, their opposite angles A and B are also equal (th. 3). Hence then, in the two triangles ACD, BCD, the angle A is equal to the angle B, and the angles at D are equal (def. 11); therefore the third angles are also equal (corol. 1. th. 17). And having the side CD common, they have also the side AD equal to the side DB (th. 2).

Also, since the angle ACE is equal to the angle BCE, the arc AE, which measures the former (def. 57), is equal to the arc BE, which measures the latter, since equal angles must have equal measures.

Corol. Hence a line bisecting any chord at right angles, passes through the centre of the circle.

THEOREM XLII.

If more than two equal lines can be drawn from any point within a circle to the circumference, that point will be the centre.



Let ABC be a circle, and D a point within it: then if any three lines, DA, DB, DC, drawn from the point D to the circumference, be equal to each other, the point D will be the centre.

Draw the chords AB, BC, which let be bisected in the points E, F, and join DE, DF.

Then, the two triangles, DAE, DBE, have the side DA equal to the side DB by supposition, and the side AE equal to the side EB by hypothesis, also the side DE common: therefore these two triangles are identical, and have the angles at E equal to each other (th. 5); consequently DE is perpendicular to the middle of the chord AB (def. 11), and therefore passes through the centre of the circle (corol. th. 41).

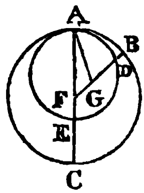
In like manner, it may be shown that DF passes through the centre. Consequently the point D is the centre of the circle, and the three equal lines DA, DB, DC, are radii.

Q. E. D.

THEOREM XLIII.

If two circles placed one within another, touch, the centres of the circles and the point of contact will be all in the same right line.

Let the two circles ABC, ADE, touch one another internally in the point A; then will the point A and the centres of those circles be all in the same right line.



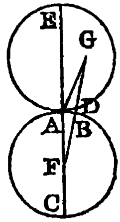
Let F be the centre of the circle ABC, through which draw the diameter AFC. Then, if the centre of the other circle can be out of this line AC, let it be supposed in some other point as G; through which draw the line FG, cutting the two circles in B and D.

Now in the triangle AFG, the sum of the two sides FG, GA, is greater than the third side AF (th. 10), or greater than its equal radius FB. From each of these take away the common part FG, and the remainder GA will be greater than the remainder GB. But the point G being supposed the centre of the inner circle, its two radii, GA, GD, are equal to each other; consequently GD will also be greater than GB. But ADE being the inner circle, GD is necessarily less than GB. So that GD is both greater and less than GB; which is absurd. Consequently the centre G cannot be out of the line AFC. Q. E. D.

THEOREM XLIV.

If two circles touch one another externally, the centres of the circles and the point of contact will be all in the same right line.

Let the two circles ABC, ADE, touch one another externally at the point A; then will the point of contact A and the centres of the two circles be all in the same right line.

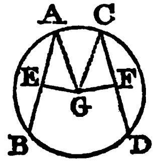


Let F be the centre of the circle ABC, through which draw the diameter AFC, and produce it to the other circle at E. Then, if the centre of the other circle ADE can be out of the line FE, let it, if possible, be supposed in some other point as G; and draw the lines AG, FBDG, cutting the two circles in B and D.

Then, in the triangle AFG, the sum of the two sides AF, AG, is greater than the third side FG (th. 10). But, F and G being the centres of the two circles, the two radii GA, GD, are equal, as are also the two radii AF, FB. Hence the sum of GA, AF, is equal to the sum of GD, BF; and therefore this latter sum also, GD, BF, is greater than GF, which is absurd. Consequently the centre G cannot be out of the line EF. Q. E. D.

THEOREM XLV.

Any chords in a circle, which are equally distant from the centre, are equal to each other; or if they be equal to each other, they will be equally distant from the centre.



Let AB, CD, be any two chords at equal distances from the centre G; then will these two chords AB, CD, be equal to each other.

Draw the two radii GA, GC, and the two perpendiculars GE, GF, which are the equal distances from the centre G. Then, the two right-angled triangles, GAE, GCF, having the side GA equal the side GC, and the side GE equal the side GF, and the angle at E equal to the angle at F, therefore those two triangles are identical (cor. 2, th. 34), and have the line AE equal to the line CF. But AB is the double of AE, and CD is the double of CF (th. 41); therefore AB is equal to CD (by ax. 6). Q. E. D.

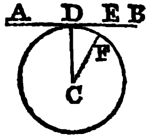
Again, if the chord AB be equal to the chord CD; then will their distances from the centre, GE, GF, also be equal to each other.

For, since AB is equal CD by supposition, the half AE is equal the half CF. Also the radii GA, GC, being equal, as well as the right angles E and F, therefore the third sides are equal (cor. 2, th. 34), or the distance GE equal the distance GF. Q. E. D.

THEOREM XLVI.

A line perpendicular to the extremity of a radius, is a tangent to the circle.

Let the line ADB be perpendicular to the radius CD of a circle; then shall AB touch the circle in the point D only.



From any other point E in the line AB draw CFE to the centre, cutting the circle in F.

Then, because the angle D, of the triangle CDE, is a right angle, the angle at E is acute (cor. 3, th. 17), and consequently less than the angle D. But the greater side is always opposite to the greater angle (th. 9); therefore the side CE is greater than the side CD, or greater than its equal CF. Hence the point E is without the circle; and the same for every other point in the line AB. Consequently the whole line is without the circle, and meets it in the point D only.

THEOREM XLVII.

When a line is a tangent to a circle, a radius drawn to the point of a contact is perpendicular to the tangent.

Let the line AB touch the circumference of a circle at the point D; then will the radius CD be the perpendicular to the tangent AB. [See the last figure.]

For the line AB being wholly without the circumference except at the point D, every other line, as CE, drawn from the centre C to the line AB, must pass out of the circle to arrive at this line. The line CD is therefore the shortest that can be drawn from the point C to the line AB, and consequently (th. 21) it is perpendicular to that line.

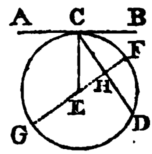
Corol. Hence, conversely, a line drawn perpendicular to a tangent, at the point of contact, passes through the centre of the circle.

THEOREM XLVIII.

The angle formed by a tangent and chord is measured by half the arc of that chord.

Let AB be a tangent to a circle, and CD a chord drawn from the point of contact C; then is the angle BCD measured by half the arc CFD, and the angle ACD measured by half the arc CGD.

Draw the radius EC to the point of contact, and the radius EF perpendicular to the chord at H.



Then the radius EF, being perpendicular to the chord CD, bisects the arc CFD (th. 41). Therefore CF is half the arc CFD.

In the triangle CEH, the angle H being a right one, the sum of the two remaining angles E and C is equal to a right angle (cor. 3, th. 17), which is equal to the angle BCE, because the radius CE is perpendicular to the tangent. From each of these equals take the common part or angle C, and there remains the angle E equal to the angle BCD. But the angle E is measured by the arc CF (def. 57), which is the half of CFD; therefore the equal angle BCD must also have the same measure, namely, half the arc CFD of the chord CD.

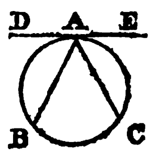
Again, the line GEF, being perpendicular to the chord CD, bisects the arc CGD (th. 41). Therefore CG is half the arc CGD. Now, since the line CE, meeting FG, makes the sum of the two angles at E equal to two right angles (th. 6), and the line CD makes with AB the sum of the two angles at C equal to two right angles; if from these two equal sums there be taken away the parts or angles CEH and BCH, which have been proved equal, there remains the angle CEG equal to the angle ACH. But the former of these, CFG, being an angle at the centre, is measured by the arc CG (def. 57); consequently the equal angle ACD must also have the same measure CG, which is half the arc CGD of the chord CD. Q. E. D.

Corol. 1. The sum of two right angles is measured by half the circumference. For the two angles BCD, ACD, which make up two right angles, are measured by the arcs CF, CG, which make up half the circumference, FG being a diameter.

Corol. 2. Hence also one right angle must have for its measure a quarter of the circumference, or 90 degrees.

THEOREM XLIX.

An angle at the circumference of a circle is measured by half the arc that subtends it.

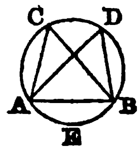


Let BAC be an angle at the circumference; it has for its measure, half the arc BC which subtends it.

For, suppose the tangent DE passing through the point of contact A; then, the angle DAC being measured by half the arc ABC, and the angle DAB by half the arc AB (th. 48); it follows, by equal subtraction, that the difference, or angle BAC, must be measured by half the arc BC, which it stands upon. Q. E. D.

THEOREM L.

All angles in the same segment of a circle, or standing on the same arc, are equal to each other.

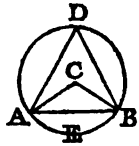


Let C and D be two angles in the same segment ACDB, or, which is the same thing, standing on the supplemental arc AEB; then will the angle C be equal to the angle D.

For each of these angles is measured by half the arc AEB; and thus, having equal measures, they are equal to each other (ax. 11).

THEOREM LI.

An angle at the centre of a circle is double the angle at the circumference, when both stand on the same arc.

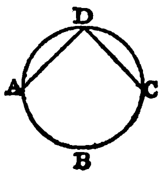


Let C be an angle at the centre C, and D an angle at the circumference, both standing on the same arc or same chord AB: then will the angle C be double of the angle D, or the angle D equal to half the angle C.

For, the angle at the centre C is measured by the whole arc AEB (def. 57), and the angle at the circumference D is measured by half the same arc AEB (th. 49); therefore the angle D is only half the angle C, or the angle C doubles the angle D.

THEOREM LII.

An angle in a semicircle, is a right angle.

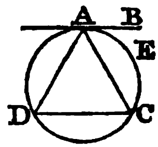


If ABC or ADC be a semicircle; then any angle D in that semicircle, is a right angle.

For, the angle D, at the circumference, is measured by half the arc ABC (th. 49), that is, by a quadrant of the circumference. But a quadrant is the measure of a right angle (cor. 4, th. 6; or cor. 2, th. 48). Therefore the angle D is a right angle.

THEOREM LIII.

The angle formed by a tangent to a circle, and a chord drawn from the point of contact, is equal to the angle in the alternate segment.

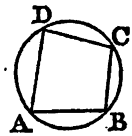


If AB be a tangent, and AC a chord, and D any angle in the alternate segment ADC; then will the angle D be equal to the angle BAC made by the tangent and chord of the arc AEC.

For the angle D, at the circumference, is measured by half the arc AEC (th. 49); and the angle BAC, made by the tangent and chord, is also measured by the same half arc AEC (th. 48); therefore these two angles are equal (ax. 11).

THEOREM LIV.

The sum of any two opposite angles of a Quadrangle inscribed in a circle, is equal to two right angles.

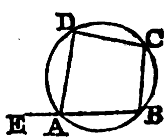


Let ABCD be any quadrilateral inscribed in a circle; then shall the sum of the two opposite angles A and C, or B and D, be equal to two right angles.

For the angle A is measured by half the arc DCB, which it stands on, and the angle C by half the arc DAB (th. 49); therefore the sum of the two angles A and C is measured by half the sum of these two arcs, that is, by half the circumference. But half the circumference is the measure of two right angles (cor. 4, th. 6); therefore the sum of the two opposite angles A and C is equal to two right angles. In like manner it is shown, that the sum of the other two opposite angles, D and B, is equal to two right angles. Q. E. D.

THEOREM LV.

If any side of a quadrangle, inscribed in a circle, be produced out, the outward angle will be equal to the inward opposite angle.

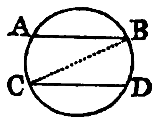


If the side AB, of the quadrilateral ABCD, inscribed in a circle, be produced to E; the outward angle DAE will be equal to the inward opposite angle C.

For, the sum of the two adjacent angles DAE and DAB is equal to two right angles (th. 6); and the sum of the two opposite angles C and DAB is also equal to two right angles (th. 54); therefore the former sum, of the two angles DAE and DAB, is equal to the latter sum, of the two C and DAB (ax. 1). From each of these equals taking away the common angle DAB, there remains the angle DAE equal the angle C. Q. E. D.

THEOREM LVI.

Any two parallel chords intercept equal arcs.

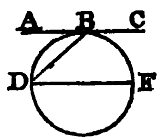


Let the two chords AB, CD, be parallel: then will the arcs AC, BD, be equal; or AC = BD.

Draw the line BC. Then, because the lines AB, CD, are parallel, the alternate angles B and C are equal (th. 12). But the angle at the circumference B, is measured by half the arc AC (th. 49); and the other equal angle at the circumference C is measured by half the arc BD: therefore the halves of the arcs AC, BD, and consequently the arcs themselves, are also equal. Q. E. D.

THEOREM LVII.

When a tangent and chord are parallel to each other, they intercept equal arcs.

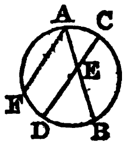


Let the tangent ABC be parallel to the chord DF; then are the arcs BD, BF, equal; that is, BD = BF.

Draw the chord BD. Then, because the lines AB, DF, are parallel, the alternate angles D and B are equal (th. 12). But the angle B, formed by a tangent and chord, is measured by half the arc BD (th. 48); and the other angle at the circumference D is measured by half the arc BF (th. 49); therefore the arcs BD, BF, are equal. Q. E. D.

THEOREM LVIII.

THE angle formed, within a circle, by the intersection of two chords, is measured by half the sum of the two intercepted arcs.



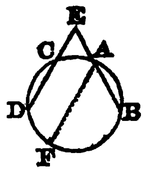
Let the two chords AB, CD, intersect at the point E: then the angle AEC, or DEB, is measured by half the sum of the two arcs AC, DB.

Draw the chord AF parallel to CD. Then because the lines AF, CD, are parallel, and AB cuts them, the angles on the same side A and DEB are equal (th. 14). But the angle at the circumference A is measured by half the arc BF, or of the sum of FD and DB (th. 49); therefore the angle E is also measured by half the sum of FD and DB.

Again, because the chords AF, CD, are parallel, the arcs AC, FD, are equal (th. 56); therefore the sum of the two arcs AC, DB, is equal to the sum of the two FD, DB; and consequently the angle E, which is measured by half the latter sum, is also measured by half the former. Q. E. D.

THEOREM LIX.

THE angle formed, out of a circle, by two secants, is measured by half the difference of the intercepted arcs.



Let the angle E be formed by two secants EAB and ECD; this angle is measured by half the difference of the two arcs AC, DB, intercepted by the two secants.

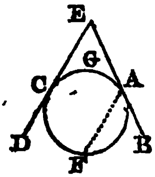
Draw the chord AF parallel to CD. Then, because the lines AF, CD, are parallel, and AB cuts them, the angles on the same side A and BED are equal (th. 14). But the angle A, at the circumference, is measured by half the arc BF (th. 49), or of the difference of DF and DB: therefore the equal angle E is also measured by half the difference of DF, DB.

Again, because the chords, AF, CD, are parallel, the arcs AC, FD, are equal (th. 56); therefore the difference of the two arcs AC, DB, is equal to the difference of the two DF, DB. Consequently the angle E, which is measured by half the latter difference, is also measured by half the former.

Q. E. D.

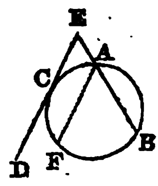
THEOREM LX.

The angle formed by two tangents, is measured by half the difference of the two intercepted arcs.



Let EB, ED, be two tangents to a circle at the points A, C; then the angle E is measured by half the difference of the two arcs CFA, CGA.

Draw the chord AF parallel to ED. Then, because the lines, AF, ED, are parallel, and EB meets them, the angles on the same side A and E are equal (th. 14). But the angle A, formed by the chord AF and tangent AB, is measured by half the arc AF (th. 48); therefore the equal angle E is also measured by half the same arc AF, or half the difference of the arcs CFA and CF, or CGA (th. 57).

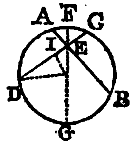


Corol. In like manner it is proved, that the angle E, formed by a tangent ECD, and a secant EAB, is measured by half the difference of the two intercepted arcs CA and CFB.

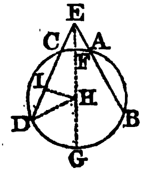
THEOREM LXI.

When two lines, meeting a circle each in two points, cut one another, either within it or without it; the rectangle of the parts of the one, is equal to the rectangle of the parts of the other; the parts of each being measured from the point of meeting to the two intersections with the circumference.

Let the two lines AB, CD, meet each other in E; then the rectangle of AE, EB, will be equal to the rectangle of CE, ED. Or, AE . EB = CE . ED.



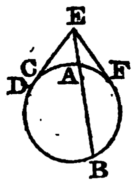
For, through the point E draw the diameter FG; also, from the centre H draw the radius DH, and draw HI perpendicular to CD.



Then, since DEH is a triangle, and the perp. HI bisects the chord CD (th. 41), the line CE is equal to the difference of the segments DI, EI, the sum of them being DE. Also, because H is the centre of the circle, and the radii DH, FH, GH, are all equal, the line EG is equal to the sum of the sides DH, HE; and EF is equal to their difference.

But the rectangle of the sum and difference of the two sides of a triangle is equal to the rectangle of the sum and difference of the segments of the base (th. 35); therefore the rectangle of FE, EG, is equal to the rectangle of CE, ED. In like manner it is proved, that the same rectangle of FE, EG, is equal to the rectangle of AE, EB. Consequently the rectangle of AE, EB, is also equal to the rectangle of CE, ED (ax. 1). Q. E. D.

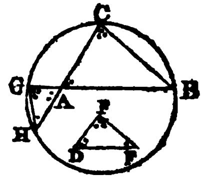
Corol. 1. When one of the lines in the second case, as DE, by revolving about the point E, comes into the position of the tangent EC or ED, the two points C and D running into one; then the rectangle of CE, ED, becomes the square of CE, because CE and DE are then equal. Consequently the rectangle of the parts of the secant, AE . EB, is equal to the square of the tangent, CE^2.



Corol. 2. Hence both the tangents EC, EF, drawn from the same point E, are equal; since the square of each is equal to the same rectangle or quantity AE . EB.

THEOREM LXII.

In equiangular triangles, the rectangles of the corresponding or like sides, taken alternately, are equal.



Let ABC, DEF, be two equiangular triangles, having the angle A = the angle D, the angle B = the angle E, and the angle C = the angle F; also the like sides AB, DE, and AC, DF, being those opposite the equal angles: then will the rectangle of AB, DF, be equal to the rectangle of AC, DE.

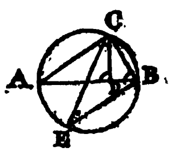
In BA produced take AG equal to DF; and through the three points B, C, G, conceive a circle BCGH to be described, meeting CA produced at H, and join GH.

Then the angle G is equal to the angle C on the same arc BH, and the angle H equal to the angle B on the same arc CG (th. 50); also the opposite angles at A are equal (th. 7): therefore the triangle AGH is equiangular to the triangle ACB, and consequently to the triangle DFE also. But the two like sides AG, DF, are also equal by supposition; consequently the two triangles AGH, DFE, are identical (th. 2), having the two sides AG, AH, equal to the two DF, DE, each to each.

But the rectangle GA . AB is equal to the rectangle HA . AC (th. 61): consequently the rectangle DF . AB is equal to the rectangle DE . AC. Q. E. D.

THEOREM LXIII.

The rectangle of the two sides of any triangle, is equal to the rectangle of the perpendicular on the third side and the diameter of the circumscribing circle.

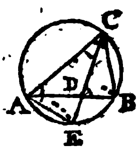


Let CD be the perpendicular, and CE the diameter of the circle about the triangle ABC; then the rectangle CA . CB is = the rectangle CD . CE.

For, join BE: then in the two triangles ACD, ECB, the angles A and E are equal, standing on the same arc BC (th. 50); also the right angle D is equal the angle B, which is also a right angle, being in a semicircle (th. 52): therefore these two triangles have also their third angles equal, and are equiangular. Hence, AC, CE, and CD, CB, being like sides, subtending the equal angles, the rectangle AC . CB. of the first and last of them, is equal to the rectangle CE . CD, of the other two (th. 62).

THEOREM LXIV.

The square of a line bisecting any angle of a triangle, together with the rectangle of the two segments of the opposite side, is equal to the rectangle of the two other sides including the bisected angle.



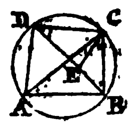
Let CD bisect the angle C of the triangle ABC; then the square CD^2 + the rectangle AD . DB is = the rectangle AC . CB.

For, let CD be produced to meet the circumscribing circle at E, and join AE.

Then the two triangles ACE, BCD, are equiangular: for the angles at C are equal by supposition, and the angles B and E are equal, standing on the same arc AC (th. 50); consequently the third angles at A and D are equal (cor. 1, th. 17): also AC, CD, and CE, CB, are like or corresponding sides, being opposite to equal angles: therefore the rectangle AC . CB is = the rectangle CD . CE (th. 62). But the latter rectangle CD . CE is = CD^2 + the rectangle CD . DE (th. 30); therefore the former rectangle AC . CB is also = CD^2 + CD . DE, or equal to CD^2 + AD . DB, since CD . DE is = AD . DB (th. 61). Q. E. D.

THEOREM LXV.

The rectangle of the two diagonals of any quadrangle inscribed in a circle, is equal to the sum of the two rectangles of the opposite sides.



Let ABCD be any quadrilateral inscribed in a circle, and AC, BD, its two diagonals: then the rectangle AC . BD is = the rectangle AB . DC + the rectangle AD . BC.

For, let CE be drawn, making the angle BCE equal to the angle DCA. Then the two triangles ACD, BCE, are equiangular; for the angles A and B are equal, standing on the same arc DC; and the angles DCA, BCE, are equal by supposition; consequently the third angles ADC, BEC, are also equal: also, AC, BC, and AD, BE, are like or corresponding sides, being opposite to the equal angles: therefore the rectangle AC . BE is = the rectangle AD . BC (th. 62).

Again, the two triangles ABC, DEC, are equiangular: for the angles BAC, BDC, are equal, standing on the same arc BC; and the angle DCE is equal to the angle BCA, by adding the common angle ACE to the two equal angles DCA, BCE; therefore the third angles E and ABC are also equal: but AC, DC, and AB, DE, are the like sides: therefore the rectangle AC . DE is = the rectangle AB . DC (th. 62).

Hence, by equal additions, the sum of the rectangles AC . BE + AC . DE is = AD . BC + AB . DC. But the former sum of the rectangles AC . BE + AC . DE is = the rectangle AC . BD (th. 30): therefore the same rectangle AC . BD is equal to the latter sum, the rect. AD . BC + the rect. AB . DC (ax. 1). Q. E. D.

Corol. Hence, if ABD be an equilateral triangle, and C any point in the arc BCD of the circumscribing circle, we have AC = BC + DC. For AC . BD being = AD . BC + AB . DC; dividing by BD = AB = AD, there results AC = BC + DC.

OF RATIOS AND PROPORTIONS.

DEFINITIONS.

DEF. 76. RATIO is the proportion or relation which one magnitude bears to another magnitude of the same kind, with respect to quantity.

Note. The measure, or quantity, of a ratio, is conceived, by considering what part or parts the leading quantity, called the Antecedent, is of the other, called the Consequent; or what part or parts the number expressing the quantity of the former, is of the number denoting in like manner the latter. So, the ratio of a quantity expressed by the number 2, to a like quantity expressed by the number 6, is denoted by 2 divided by 6, or $\frac{2}{6}$ or $\frac{1}{3}$: the number 2 being 3 times contained in 6, or the third part of it. In like manner, the ratio of the quantity 3 to 6, is measured by $\frac{3}{6}$ or $\frac{1}{2}$; the ratio of 4 to 6 is $\frac{4}{6}$ or $\frac{2}{3}$; that of 6 to 4 is $\frac{6}{4}$ or $\frac{3}{2}$; &c.

77. Proportion is an equality of ratios. Thus,

78. Three quantities are said to be proportional, when the ratio of the first to the second is equal to the ratio of the

second to the third. As of the three quantities A (2), B (4), C (8), where $\frac{2}{4} = \frac{4}{8} = \frac{1}{2}$, both the same ratio.

79. Four quantities are said to be proportional, when the ratio of the first to the second, is the same as the ratio of the third to the fourth. As of the four, A (4), B (2), C (10), D (5), where $\frac{4}{2} = \frac{10}{5} = 2$, both the same ratio.

Note. To denote that four quantities, A, B, C, D, are proportional, they are usually stated or placed thus, A : B : : C : D; and read thus, A is to B as C is to D. But when three quantities are proportional, the middle one is repeated, and they are written thus, A : B : : B : C.

The proportionality of quantities may also be expressed very generally by the equality of fractions, as at pa. 118.

Thus, if $\frac{A}{B} = \frac{C}{D}$, then A : B : : C : D, also B : A : : C : D, and A : C : : B : D, and C : A : : B : D.

80. Of three proportional quantities, the middle one is said to be a Mean Proportional between the other two; and the last, a Third Proportional to the first and second.

81. Of four proportional quantities, the last is said to be a Fourth Proportional to the other three, taken in order.

82. Quantities are said to be Continually Proportional, or in Continued Proportion, when the ratio is the same between every two adjacent terms, viz. when the first is to the second, as the second to the third, as the third to the fourth, as the fourth to the fifth, and so on, all in the same common ratio.

As in the quantities 1, 2, 4, 8, 16, &c.; where the common ratio is equal to 2.

83. Of any number of quantities, A, B, C, D, the ratio of the first A, to the last D, is said to be Compounded of the ratios of the first to the second, of the second to the third, and so on to the last.

84. Inverse ratio is, when the antecedent is made the consequent, and the consequent the antecedent.—Thus, if 1 : 2 : : 3 : 6; then inversely, 2 : 1 : : 6 : 3.

85. Alternate proportion is, when antecedent is compared with antecedent, and consequent with consequent.—As, if 1 : 2 : : 3 : 6; then, by alternation, or permutation, it will be 1 : 3 : : 2 : 6.

86. Compound ratio is, when the sum of the antecedent and consequent is compared, either with the consequent, or

with the antecedent.—Thus, if 1 : 2 : : 3 : 6, then by composition, 1 + 2 : 1 : : 3 + 6 : 3, and 1 + 2 : 2 : : 3 + 6 : 6.

87. Divided ratio, is when the difference of the antecedent and consequent is compared, either with the antecedent or with the consequent.—Thus, if 1 : 2 : : 3 : 6, then, by division, 2 — 1 : 1 : : 6 — 3 : 3, and 2 — 1 : 2 : : 6 — 3 : 6.

Note. The term Divided, or Division, here means subtracting, or parting; being used in the sense opposed to compounding, or adding, in def. 86.

THEOREM LXVI.

EQUIMULTIPLES of any two quantities have the same ratio as the quantities themselves.

Let A and B be any two quantities, and mA, mB, any equimultiples of them, m being any number whatever: then will mA and mB have the same ratio as A and B, or A : B : : mA : mB.

For $\frac{mB}{mA} = \frac{B}{A}$, the same ratio.

Corol. Hence, like parts of quantities have the same ratio as the wholes; because the wholes are equimultiples of the like parts, or A and B are like parts of mA and mB.

THEOREM LXVII.

IF four quantities, of the same kind, be proportionals; they will be in proportion by alternation or permutation, or the antecedents will have the same ratio as the consequents*.

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* The author's object in these propositions was to simplify the doctrine of ratios and proportions, by imagining that the antecedents and consequences may always be divided into parts that are commensurable. But it is known to mathematicians that there are certain quantities or magnitudes, such as the side and the diagonal of a square, which cannot possibly be divided in that manner by means of a *common measure*. The theorems themselves are true, nevertheless, when applied to these *incommensurables*; since no two quantities of the same kind can possibly be assigned, whose ratio cannot be expressed by that of two numbers, so near, that the difference shall be less than the least number that can be named. From the greater of two unequal magnitudes we may take, or suppose taken, its *half*, from the remaining half, *its* half,

Let A : B : : mA : mB ; then will A : mA : : B : mB.

For $\frac{mA}{A} = \frac{m}{1}$, and $\frac{mB}{B} = \frac{m}{1}$, both the same ratio.

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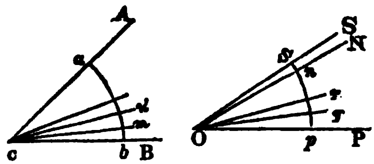
and so on, by continual bisections, until there shall at length be left a magnitude less than the least of two magnitudes; or, indeed, less than the least magnitude that can be assigned; and this principle furnishes a ground of reasoning.

Or, somewhat differently, let A and B be two constant quantities, a and b two variable quantities, which we can render as small as we please, if we have an equality between $A + a$, and $B + b$, or, in other words, if the equation $A + a = B + b$ holds good whatever are the values of a and b, it may be divided into two others, $A = B$, between the constant quantities, and, $a = b$, between the variable quantities, and which latter must obtain for all their states of magnitude. For if, on the contrary, we suppose $A = B \pm Q$, we shall have $A - B = b - a = \pm Q$, an absurd result; since the quantities a and b being susceptible of diminishing indefinitely, their difference cannot always be $= Q$. This is the principle which constitutes the *method of limits.* In general, one magnitude is called a *limit* of another, *when we can make this latter approach so near to the former, that their difference shall be less than any given magnitude, and yet so that the two magnitudes shall never become strictly equal.*

Let us here apply the principle to the demonstration of this proposition, that the ratio of two angles ACB, NOP, is equal to that of the arcs, ab, np, comprised between their sides, and drawn from their respective summits as centres with equal radii.



If the arcs pn, ba, are commensurable, their common measure bm will be contained n times in pn, r times in ba; so that we shall have the equal ratios $\frac{pn}{ba} = \frac{n}{r}$. Through each point of division, m, n', &c. draw the lines mc, $n'c$, &c. to the summits c, and o, the angles proposed will be divided into n, and r, equal angles, bcm, mcn', poq, qor, &c. We shall, therefore, have $\frac{PON}{BCA} = \frac{n}{r}$. Hence $\frac{PON}{BCA}$ is $= \frac{pn}{ba}$, since each of them is equal to the ratio $\frac{n}{r}$.

If the arcs are incommensurable, divide one of them, ba, into a number r of equal parts, bm, mn', &c. and set off equal parts pq, qr, &c. upon the other arc pn; and let s be the point of division that falls nearest to n. Draw OSS. Then, by the preceding, ba, ps, being commensurable, we shall have $\frac{POS}{BCA} = \frac{ps}{ba}$ the angle $POS = PON + NOS$, arc $ps = pn + ns$. Therefore,

$$\frac{PON}{BCA} + \frac{NOS}{BCA} = \frac{pn}{ba} + \frac{ns}{ba}.$$

Here NOS and ns are susceptible of indefinite variation, according as we change the common measure, bm, of ba; they may, therefore, be

Otherwise. Let $A : B :: C : D$; then shall $B : A :: C : D$.

For, let $\frac{A}{B} = \frac{C}{D} = r$; then $A = Br$, and $C = Dr$: therefore $B = \frac{A}{r}$, and $D = \frac{C}{r}$. Hence $\frac{B}{A} = \frac{1}{r}$, and $\frac{D}{C} = \frac{1}{r}$.

Whence it is evident that $\frac{B}{A} = \frac{D}{C}$ (ax. 1), or $B : A :: D : C$.

In a similar manner may most of the other theorems be demonstrated.

THEOREM LXVIII.

If four quantities be proportional; they will be in proportion by inversion, or inversely.

Let $A : B :: mA : mB$; then will $B : A :: mB : mA$.

For $\frac{mA}{mB} = \frac{A}{B}$, both the same ratio.

THEOREM LXIX.

If four quantities be proportional; they will be in proportion by composition and division.

Let $A : B :: mA : mB$;
Then will $B \pm A : A :: mB \pm mA : mA$,
and $B \pm A : B :: mB \pm mA : mB$.

For $\frac{mA}{mB \pm mA} = \frac{A}{B \pm A}$; and $\frac{mB}{mB \pm mA} = \frac{B}{B \pm A}$.

Corol. It appears from hence, that the sum of the greatest and least of four proportional quantities, of the same kind, exceeds the sum of the other two. For, since - - - - $A : A + B :: mA : mA + mB$, where A is the least, and $mA + mB$ the greatest; then $\overline{m + 1} . A + mB$, the sum of the greatest and least, exceeds $\overline{m + 1} . A + B$, the sum of the two other quantities.

THEOREM LXX.

If, of four proportional quantities, there be taken any equimultiples whatever of the two antecedents, and any equi-

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rendered as small as we please, while the other quantities remain the same. Consequently, by the nature of limits, as above explained, we have the equal ratios $\frac{PON}{BCA} = \frac{pn}{ba}$, or $PON : BAC :: pn : ba$.

multiples whatever of the two consequents; the quantities resulting will still be proportional.

Let $A : B :: mA : mB$; also, let pA and pmA be any equimultiples of the two antecedents, and qB and qmB any equimultiples of the two consequents; then will . . . . $pA : qB :: pmA : qmB$.

For $\frac{qmB}{pmA} = \frac{qB}{pA}$, both the same ratio.

THEOREM LXXI.

If there be four proportional quantities, and the two consequents be either augmented or diminished by quantities that have the same ratio as the respective antecedents; the results and the antecedents will still be proportionals.

Let $A : B :: mA : mB$, and nA and nmA any two quantities having the same ratio as the two antecedents; then will $A : B \pm nA :: mA : mB \pm nmA$.

For $\frac{mB \pm nmA}{mA} = \frac{B \pm nA}{A}$, both the same ratio.

THEOREM LXXII.

If any number of quantities be proportional, then any one of the antecedents will be to its consequent, as the sum of all the antecedents, is to the sum of all the consequents.

Let $A : B :: mA : mB :: nA : nB$, &c.; then will . . . $A : B :: A + mA + nA : B + mB + nB$, &c.

For $\frac{B+mB+nB}{A+mA+nA} = \frac{(1+m+n)B}{(1+m+n)A} = \frac{B}{A}$, the same ratio.

THEOREM LXXIII.

If a whole magnitude be to a whole, as a part taken from the first, is to a part taken from the other; then the remainder will be to the remainder, as the whole to the whole.

Let $A : B :: \frac{m}{n}A : \frac{m}{n}B$;

then will $A : B :: A - \frac{m}{n}A : B - \frac{m}{n}B$.

For $\frac{B - \frac{m}{n}B}{A - \frac{m}{n}A} = \frac{B}{A}$, both the same ratio.

THEOREM LXXIV.

If any quantities be proportional; their squares, or cubes, or any like powers, or roots, of them, will also be proportional.

Let $A : B :: mA : mB$; then will $A^n : B^n :: m^nA^n : m^nB^n$.

For $\frac{m^nB^n}{m^nA^n} = \frac{B^n}{A^n}$, both the same ratio.

See also, th. VIII. pa. 118.

THEOREM LXXV.

If there be two sets of proportionals; then the products or rectangles of the corresponding terms will also be proportional.

Let $A : B :: mA : mB$,
and $C : D :: nC : nD$;
then will $AC : BD :: mnAC : mnBD$.

For $\frac{mnBD}{mnAC} = \frac{BD}{AC}$, both the same ratio.

THEOREM LXXVI.

If four quantities be proportional; the rectangle or product of the two extremes, will be equal to the rectangle or product of the two means. And the converse.

Let $A : B :: mA : mB$;
then is $A \times mB = B \times mA = mAB$, as is evident.

THEOREM LXXVII.

If three quantities be continued proportionals; the rectangle or product of the two extremes, will be equal to the square of the mean. And the converse.

Let A, mA, m^2A be three proportionals,
or $A : mA :: mA : m^2A$;
then is $A \times m^2A = m^2A^2$, as is evident.

THEOREM LXXVIII.

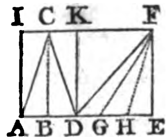
If any number of quantities be continued proportionals; the ratio of the first to the third, will be duplicate or the square of the ratio of the first and second; and the ratio of the first and fourth will be triplicate or the cube of that of the first and second; and so on.

Let A, mA, m^2A, m^3A, &c. be proportionals;

then is $\frac{A}{mA} = \frac{1}{m}$; but $\frac{A}{m^2A} = \frac{1}{m^2}$; and $\frac{A}{m^3A} = \frac{1}{m^3}$; &c.

THEOREM LXXIX.

Triangles, and also parallelograms, having equal altitudes, are to each other as their bases.



Let the two triangles ADC, DEF, have the same altitude, or be between the same parallels AE, CE; then is the surface of the triangle ADC, to the surface of the triangle DEF, as the base AD is to the base DE. Or, AD : DE : : the triangle ADC : the triangle DEF.

For, let the base AD be to the base DE, as any one number m (2), to any other number n (3); and divide the respective bases into those parts, AB, BD, DG, GH, HE, all equal to one another; and from the points of division draw the lines BC, FG, FH, to the vertices C and F. Then will these lines divide the triangles ADC, DEF, into the same number of parts as their bases, each equal to the triangle ABC, because those triangular parts have equal bases and altitude (cor. 2, th. 25); namely, the triangle ABC equal to each of the triangles BDC, DFG, GFH, HFE. So that the triangle ADC, is to the triangle DFE, as the number of parts m (2) of the former, to the number n (3) of the latter, that is, as the base AD to the base DE (def. 79)*.

In like manner, the parallelogram ADKI is to the parallelogram DEFK, as the base AD is to the base DE; each of these having the same ratio as the number of their parts, m to n. Q. E. D.

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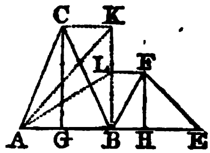
* If the bases AD, DE, of two triangles that have a common vertex C, are incommensurable to each other, the ratio of the triangles is, notwithstanding, equal to that of their bases.

THEOREM LXXX.

Triangles, and also parallelograms having equal bases, are to each other as their altitudes.



Let ABC, BEF, be two triangles having the equal bases AB, BE, and whose altitudes are the perpendiculars CG, FH; then will the triangle ABC : the triangle BEF : : CG : FH.

For, let BK be perpendicular to AB, and equal to CG; in which let there be taken BL = FH; drawing AK and AL.

Then triangles of equal bases and heights being equal (cor. 2, th. 25), the triangle ABK is = ABC, and the triangle ABL = BEF. But, considering now ABK, ABL, as two triangles on the bases BK, BL, and having the same altitude AB, these will be as their bases (th. 79), namely, the triangle ABK : the triangle ABL : : BK : : BL.

But the triangle ABK = ABC, and the triangle ABL = BEF, also BK = CG, and BL = FH.
Theref. the triangle ABC : triangle BEF : : CG : FH.

And since parallelograms are the doubles of these triangles, having the same bases and altitudes, they will likewise have to each other the same ratio as their altitudes. Q. E. D.

Corol. Since, by this theorem, triangles and parallelograms, when their bases are equal, are to each other as their altitudes; and by the foregoing one, when their altitudes are equal, they are to each other as their bases; therefore uni-

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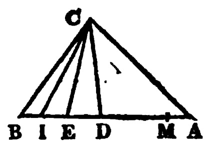


For, first, if possible, let the triangle ECD be to the triangle ACD, not as ED to AD, but as some other line BD *greater* than ED, is to AD.

Let AM be a part, or measure of AD, less than BE, and let DI be that multiple of AM, which least exceeds DE, and which by the note to th. 67, may be made as small as we please. Let CB, CI, be drawn. I evidently falls between E and B, because (by hyp.) EI is less than AM. But ICD : ACD : : ID : AD, by th. 79. Also, by hyp. ECD : ACD : : BD : AD, greater than the ratio of ID : AD, or of ICD : ACD; and consequently, ECD is greater than ICD: which is *impossible*. By a like reasoning it may be shown, that ECD cannot be to ACD, as a line *less* than ED, is to AD. Consequently, it must be ECD : ACD : : ED : AD.

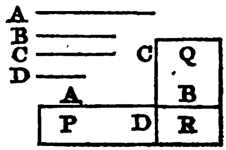
Similar reasoning, founded upon the preceding note, applies also to the case of parallelograms.

versally, when neither are equal, they are to each other in the compound ratio, or as the rectangle or product of their bases and altitudes.

THEOREM LXXXI.

If four lines be proportional; the rectangle of the extremes will be equal to the rectangle of the means. And, conversely, if the rectangle of the extremes, of four lines, be equal to the rectangle of the means, the four lines, taken alternately, will be proportional.

Let the four lines A, B, C, D, be proportionals, or A : B : : C : D; then will the rectangle of A and D be equal to the rectangle of B and C; or the rectangle A . D = B . C.



For, let the four lines be placed with their four extremities meeting in a common point, forming at that point four right angles; and draw lines parallel to them to complete the rectangles P, Q, R, where P is the rectangle of A and D, Q the rectangle of B and C, and R the rectangle of B and D.

Then the rectangles P and R, being between the same parallels are to each other as their bases A and B (th. 79); and the rectangles Q and R, being between the same parallels, are to each other as their bases C and D. But the ratio of A to B, is the same as the ratio of C to D, by hypothesis: therefore the ratio of P to R, is the same as the ratio of Q to R; and consequently the rectangles P and Q are equal.

Q. E. D.

Again, if the rectangle of A and D, be equal to the rectangle of B and C; these lines will be proportional, or A : B : : C : D.]

For, the rectangles being placed the same as before: then, because parallelograms between the same parallels, are to one another as their bases, the rectangle P : R : : A : B, and Q : R : : C : D. But as P and Q are equal, by supposition, they have the same ratio to R, that is, the ratio of A to B is equal to the ratio of C to D, or A : B : : C : D. Q. E. D.

Corol. 1. When the two means, namely, the second and third terms, are equal, their rectangle becomes a square of the second term, which supplies the place of both the second and third, And hence it follows, that when three lines are proportionals, the rectangle of the two extremes is equal to the square of the mean ; and, conversely, if the rectangle of the extremes be equal to the square of the mean, the three lines are proportionals.

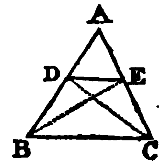
Corol. 2. Since it appears, by the rules of proportion in Arithmetic and Algebra, that when four quantities are proportional, the product of the extremes is equal to the product of the two means ; and, by this theorem, the rectangle of the extremes is equal to the rectangle of the two means ; it follows, that the area or space of a rectangle is represented or expressed by the product of its length and breadth multiplied together. And, in general, a rectangle in geometry is similar to the product of the measures of its two dimensions of length and breadth, or base and height. Also, a square is similar to, or represented by, the measure of its side multiplied by itself. So that, what is shown of such products, is to be understood of the squares and rectangles.

Corol. 3. Since the same reasoning, as in this theorem, holds for any parallelograms whatever, as well as for the rectangles, the same property belongs to all kinds of parallelograms, having equal angles, and also to triangles, which are the halves of parallelograms ; namely, that if the sides about the equal angles of parallelograms, or triangles, be reciprocally proportional, the parallelograms or triangles will be equal ; and, conversely, if the parallelograms or triangles be equal, their sides about the equal angles will be reciprocally proportional.

Corol. 4. Parallelograms, or triangles, having an angle in each equal, are in proportion to each other as the rectangles of the sides which are about these equal angles.

THEOREM LXXXII.

If a line be drawn in a triangle parallel to one of its sides, it will cut the other two sides proportionally.



Let DE be parallel to the side BC of the triangle ABC ; then will AD : DB : : AE : EC.

For, draw BE and CD. Then the triangles DBE, DCE, are equal to each other, because they have the same base DE, and are between the same parallels DE, BC (th. 25). But the two triangles, ADE, BDE, on the bases AD, DB, have the same altitude ; and the two triangles ADE, CDE, on the bases AE, EC, have also the same altitude ; and because triangles of the same altitude are to each other as their bases, therefore

the triangle ADE : BDE : : AD : DB,
and triangle ADE : CDE : : AE : EC.

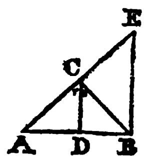
But BDE is = CDE ; and equals must have to equals the same ratio ; therefore AD : DB : : AE : EC. Q. E. D.

Corol. Hence, also, the whole lines AB, AC, are proportional to their corresponding proportional segments (corol. th. 66),

viz. AB : AC : : AD : AE,
and AB : AC : : BD : CE.

THEOREM LXXXIII.

A Line which bisects any angle of a triangle, divides the opposite side into two segments, which are proportional to the two other adjacent sides.



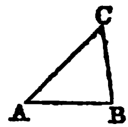
Let the angle ACB, of the triangle ABC, be bisected by the line CD, making the angle *r* equal to the angle *s* : then will the segment AD be to the segment DB, as the side AC is to the side CB. Or, - - - -
AD : DB : : AC : CB.

For, let BE be parallel to CD, meeting AC produced at E. Then, because the line BC cuts the two parallels CD, BE, it makes the angle CBE equal to the alternate angle *s* (th. 12), and therefore also equal to the angle *r*, which is equal to *s* by the supposition. Again, because the line AE cuts the two parallels DC, BE, it makes the angle E equal to the angle *r* on the same side of it (th. 14). Hence, in the triangle BCE, the angles B and E, being each equal to the angle *r*, are equal to each other, and consequently their opposite sides CB, CE, are also equal (th. 3).

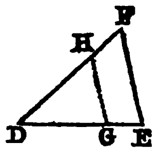
But now, in the triangle ABE, the line CD, being drawn parallel to the side BE, cuts the two other sides AB, AE, proportionally (th. 82), making AD to DB, as is AC to CE or to its equal CB. Q. E. D.

THEOREM LXXXIV.

Equiangular triangles are similar, or have their like sides proportional.



Let ABC, DEF, be two equiangular triangles, having the angle A equal to the angle D, the angle B to the angle E, and consequently the angle C to the angle F; then will AB : AC : : DE : DF.

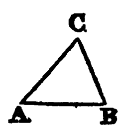


For, make DG = AB, and DH = AC, and join GH. Then the two triangles ABC, DGH, having the two sides AB, AC, equal to the two DG, DH, and the contained angles A and D also equal, are identical, or equal in all respects (th. 1), namely, the angles B and C are equal to the angles G and H. But the angles B and C are equal to the angles E and F by the hypothesis; therefore also the angles G and H are equal to the angles E and F (ax. 1), and consequently the line GH is parallel to the side EF (cor. 1, th. 14).

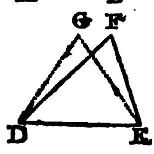
Hence then, in the triangle DEF, the line GH, being parallel to the side EF, divides the two other sides proportionally, making DG : DH : : DE : DF (cor. th. 82). But DG and DH are equal to AB and AC; therefore also - - - - - AB : AC : : DE : DF. Q. E. D.

THEOREM LXXXV.

Triangles which have their sides proportional, are equiangular.



In the two triangles ABC, DEF, if AB : DE : : AC : DF : : BC : EF; the two triangles will have their corresponding angles equal.



For, if the triangle ABC be not equiangular with the triangle DEF, suppose some other triangle, as DEG, to be equiangular with ABC. But this is impossible: for if the two triangles ABC, DEG, were equiangular, their sides would be proportional (th. 84). So that, AB being to DE as AC to DG, and AB to DE as BC to EG, it follows that DG and EG, being fourth proportionals to the same three quantities, as well as the two DF, EF, the former, DG, EG, would be equal to the latter, DF, EF. Thus, then, the two triangles DEF, DEG, having their three sides equal, would be identical (th. 5); which is absurd, since their angles are unequal.

THEOREM LXXXVI.

TRIANGLES, which have an angle in the one equal to an angle in the other, and the sides about these angles proportional, are equiangular.

Let ABC, DEF, be two triangles, having the angle A = the angle D, and the sides AB, AC, proportional to the sides DE, DF: then will the triangle ABC be equiangular with the triangle DEF.

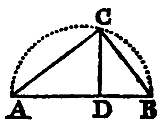
For, make DG = AB, and DH = AC, and join GH.

Then, the two triangles ABC, DGH, having two sides equal, and the contained angles A and D equal, are identical and equiangular (th. 1), having the angles G and H equal to the angles B and C. But, since the sides DG, DH, are proportional to the sides DE, DF, the line GH is parallel to EF (th. 82); hence the angles E and F are equal to the angles G and H (th. 14), and consequently to their equals B and C. Q. E. D.

[See fig. th. LXXXIV.]

THEOREM LXXXVII.

IN a right-angled triangle, a perpendicular from the right angle, is a mean proportional between the segments of the hypothenuse; and each of the sides, about the right angle, is a mean proportional between the hypothenuse and the adjacent segment.



Let ABC be a right-angled triangle, and CD a perpendicular from the right angle C to the hypothenuse AB; then will

CD be a mean proportional between AD and DB;
AC a mean proportional between AB and AD;
BC a mean proportional between AB and BD.

Or, AD : CD : : CD : DB; and AB : BC : : BC : BD; and AB : AC : : AC : AD.

For, the two triangles, ABC, ADC, having the right angles at C and D equal, and the angle A common, have their third angles equal, and are equiangular, (cor. 1, th. 17). In like manner, the two triangles ABC, BDC, having the right angles at C and D equal, and the angle B common, have their third angles equal, and are equiangular.

Hence then, all the three triangles, ABC, ADC, BDC, being equiangular, will have their like sides proportional (th. 84);

viz. AD : CD : : CD : DB;
and AB : AC : : AC : AD;
and AB : BC : : BC : BD. Q. E. D.

Corol. 1. Because the angle in a semicircle is a right angle (th. 52); it follows, that if, from any point C in the periphery of the semicircle, a perpendicular be drawn to the diameter AB; and the two chords CA, CB, be drawn to the extremities of the diameter: then are AC, BC, CD, the mean proportionals as in this theorem, or (by th. 77), ... $CD^2 = AD . DB$; $AC^2 = AB . AD$; and $BC^2 = AB . BD$.

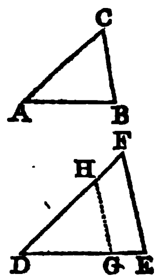
Corol. 2. Hence $AC^2 : BC^2 :: AD : BD$.

Corol. 3. Hence we have another demonstration of th. 34.

For since $AC^2 = AB . AD$, and $BC^2 = AB . BD$;
By addition $AC^2 + BC^2 = AB (AD + BD) = AB^2$.

THEOREM LXXXVIII.

EQUIANGULAR or similar triangles, are to each other as the squares of their like sides.



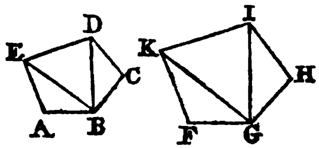
Let ABC, DEF, be two equiangular triangles, AB and DE being two like sides: then will the triangle ABC be to the triangle DEF, as the square of AB is to the square of DE, or as AB^2 to DE^2.

For, the triangles being similar, they have their like sides proportional (th. 84), and are to each other as the rectangles of the like pairs of their sides (cor. 4, th. 81);

theref. AB : DE : : AC : DF (th. 84),
and AB : DE : : AB : DE of equality:
theref $AB^2 : DE^2 :: AB . AC : DE . DF$ (th. 75).
But $\triangle$ ABC : $\triangle$ DEF : : AB . AC : DE . DF (cor. 4, th. 81),
theref. $\triangle$ ABC : $\triangle$ DEF : : $AB^2 : DE^2$. Q. E. D.

THEOREM LXXXIX.

All similar figures are to each other, as the squares of their like sides.



Let ABCDE, FGHIK, be any two similar figures, the like sides being AB, FG, and BC, GH, and so on in the same order: then will the figure ABCDE be to the figure FGHIK, as the square of AB to the square of FG, or as AB^2 to FG^2.

For, draw BE, BD, GK, GI, dividing the figures into an equal number of triangles, by lines from two equal angles B and G.

The two figures being similar (by suppos.), they are equiangular, and have their like sides proportional (def. 67).

Then, since the angle A is = the angle F, and the sides AB, AE, proportional to the sides FG, FK, the triangles ABE, FGK, are equiangular (th. 86). In like manner, the two triangles BCD GHI, having the angle C = the angle H, and the sides BC, CD, proportional to the sides GH, HI, are also equiangular. Also, if from the equal angles AED, FKI, there be taken the equal angles AEB, FKG, there will remain the equals BED, GKI; and if from the equal angles CDE, HIK, be taken away the equals CDB, HIG, there will remain the equals BDE, GIK; so that the two triangles BDE, GIK, having two angles equal, are also equiangular. Hence each triangle of the one figure, is equiangular with each corresponding triangle of the other.

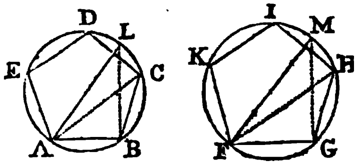
But equiangular triangles are similar, and are proportional to the squares of their like sides (th. 88).

Therefore the $\triangle$ ABE : $\triangle$ FGK : : $AB^2 : FG^2$,
and $\triangle$ BCD : $\triangle$ GHI : : $BC^2 : GH^2$,
and $\triangle$ BDE : $\triangle$ GIK : : $DE^2 : IK^2$.

But as the two polygons are similar, their like sides are proportional, and consequently their squares also proportional; so that all the ratios AB^2 to FG^2, and BC^2 to GH^2, and DE^2 to IK^2, are equal among themselves, and consequently the corresponding triangles also, ABE to FGH, and BCD to GHI, and BDE to GIK, have all the same ratio, viz. that of AB^2 to FG^2: and hence all the antecedents, or the figure ABCDE, have to all the consequents, or the figure FGHIK, still the same ratio, viz. that of AB^2 to FG^2 (th. 72). Q. E. D.

THEOREM XC.

SIMILAR figures inscribed in circles, have their like sides, and also their whole perimeters, in the same ratio as the diameters of the circles in which they are inscribed.



Let ABCDE, FGHIK, be two similar figures, inscribed in the circles whose diameters are AL and FM; then will each side AB, BC, &c. of the one figure be to the like side GF, GH, &c. of the other figure, or the whole perimeter AB + BC + &c. of the one figure, to the whole perimeter FG + GH + &c. of the other figure, as the diameter AL to the diameter FM.

For, draw the two corresponding diagonals AC, FH, as also the lines BL, GM. Then, since the polygons are similar, they are equiangular, and their like sides have the same ratio (def. 67); therefore the two triangles ABC, FGH, have the angle B = the angle G, and the sides AB, BC, proportional to the two sides FG, GH, consequently these two triangles are equiangular (th. 86), and have the angle ACB = FHG. But the angle ACB = ALB, standing on the same arc AB; and the angle FHG = FMG, standing on the same arc FG; therefore the angle ALB = FMG (ax. 1). And since the angle ABL = FGM, being both right angles, because in a semicircle; therefore the two triangles ABL, FGM, having two angles equal, are equiangular; and consequently their like sides are proportional (th. 84); hence AB : FG :: the diameter AL : the diameter FM.

In like manner, each side BC, CD, &c. has to each side GH, HI, &c. the same ratio of AL to FM; and consequently the sums of them are still in the same ratio, viz. AB + BC + CD, &c. : FG + GH + HI, &c. :: the diam. AL : the diam. FM (th. 72). Q. E. D.

THEOREM XCI.

SIMILAR figures inscribed in circles, are to each other as the squares of the diameters of those circles.

Let ABCDE, FGHIK, be two similar figures, inscribed in the circles whose diameters are AL and FM; then the surface of the polygon ABCDE will be to the surface of the polygon FGHIK, as AL^2 to FM^2.

For, the figures being similar, are to each other as the squares of their like sides, AB^2 to FG^2 (th. 88). But, by the last theorem, the sides AB, FG, are as the diameters AL, FM; and therefore the squares of the sides AB^2 to FG^2, as the squares of the diameters AL^2 to FM^2 (th. 74). Consequently the polygons ABCDE, FGHIK, are also to each other as the squares of the diameters AL^2 to FM^2 (ax. 1). Q. E. D.

[See fig. th. XC.]

THEOREM XCII.

THE circumferences of all circles are to each other as their diameters*.

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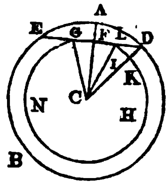
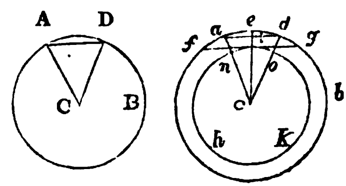
* The truth of theorems 92, 93, and 94, may be established more satisfactorily than in the text, upon principles analogous to those of the two last notes.

THEOREM. The area of any circle ABD is equal to the rectangle contained by the radius, and a straight line equal to half the circumference.



If not, let the rectangle be *less* than the circle ABD, or equal to the circle FNH: and imagine ED drawn to touch the interior circle in F, and meet the circumference ABD in E and D. Join CD, cutting the arc of the interior circle in K. Let FH be a quadrantal arc of the inner circle, and from it take its half, from the remainder its half, and so on, until an arc FI is obtained, less than FK. Join CI, produce it to cut ED in L, and make FG = FL: so shall LG be the side of a regular polygon circumscribing the circle FNH. It is manifest that this polygon is less than the circle ABD, because it is contained within it. Because the triangle GCL is half the rectangle of base GL and altitude CF, the whole polygon of which GCL is a constituent triangle, is equal to half the rectangle whose base is the perimeter of that polygon and altitude CF. But that perimeter is less than the circumference ABD, because each portion of it, such as GL, is less than the corresponding arch of circle having radius CL, and therefore, *a fortiori*, less than the corresponding arch of circle with radius CA. Also CE is less than CA. Therefore the polygon of which one side is GL, is less than the rectangle whose base is half the circumference ABD and altitude CA; that is, (by hyp.) less than the circle FNH, which it contains: which is absurd. Therefore, the rectangle under the radius and half the circumference is not *less* than the circle ABD. And by a similar process it may be shown that it is not *greater*. Consequently, it is equal to that rectangle. Q. E. D.

THEOREM. The circumferences of two circles ABD, *abd*, are as their radii.



If possible, let the radius AC, be to the radius *ac*, as the circumference ABD to a circumference *ihk* less than *abd*. Draw the radius *cie*, and the straight line *fig* a chord to the circle *abd*, and a tangent to the circle *ihk* in *i*. From *eb*, a quarter of the circumference of *abd*, take

Let D, d, denote the diameters of two circles, and C, c, their circumferences;

then will D : d : : C : c, or D : C : : d : c.

For (by theor. 90), similar polygons inscribed in circles have their perimeters in the same ratio as the diameters of those circles.

Now as this property belongs to all polygons, whatever the number of the sides may be; conceive the number of the sides to be indefinitely great, and the length of each indefinitely small, till they coincide with the circumference of the circle, and be equal to it, indefinitely near. Then the perimeter of the polygon of an infinite number of sides, is the same thing as the circumference of the circle. Hence it appears that the circumferences of the circles, being the same as the perimeters of such polygons, are to each other in the same ratio as the diameters of the circles. Q. E. D.

THEOREM XCIII.

THE areas or spaces of circles, are to each other as the squares of their diameters, or of their radii.

Let A, a, denote the areas or spaces of two circles, and D, d, their diameters; then A : a : : D^2 : d^2.

For (by theorem 91) similar polygons inscribed in circles are to each other as the squares of the diameters of the circles.

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away its half, and then the half of the remainder, and so on, until there be obtained an arc ed less than eg; and from d draw ad parallel to fg, it will be the side of a regular polygon inscribed in the circle abd, yet evidently greater than the circle ihk, because each of its constituent triangles, as acd contains the corresponding circular sector cno. Let AD be the side of a similar polygon inscribed in the circle ADB, and join AC, CD, similarly to ac, cd. The similar triangles ACD, acd, give AC : ac : : AD : ad, and : : perim. of polygon in ABD : perim. of polygon in abd. But, by the preceding theorem, AC : ac : : circumf. ABD : circumf. abd. The perimeters of the polygons are, therefore, as the circumferences of the circles. But, this is impossible; because, (by hyp.) the perim. of polygon in ABD is *less* than the circumf.; while, on the contrary, the perim. of polygon in adb is *greater* than the circumf. ihk. Consequently, AC is not to ac, as circumf. ADB, to a circumference *less* than adb. And by a similar process it may be shown, that ac is not to AC, as the circumf. abd, to a circumference *less* than ABD. Therefore AC : ac : : circumf. ABD : circumf. abd. Q. E. D.

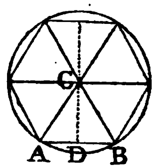
Corol. Since by this theorem, we have C : c : : R : r, or, if $C = \pi R$, $c = \pi r$; and, by the former, area (A) : area (a) : : $\frac{1}{2}RC : \frac{1}{2}rc$: we have A : a : : $\frac{1}{2}\pi R^2 : \frac{1}{2}\pi r^2$: : $R^2 : r^2$: : $D^2 : d^2$: : $C^2 : c^2$.

Hence, conceiving the number of the sides of the polygons to be increased more and more, or the length of the sides to become less and less, the polygon approaches nearer and nearer to the circle, till at length, by an infinite approach, they coincide, and become in effect equal ; and then it follows, that the spaces of the circles, which are the same as of the polygons, will be to each other as the squares of the diameters of the circles. Q. E. D.

Corol. The spaces of circles are also to each other as the squares of the circumferences ; since the circumferences are in the same ratio as the diameters (by theorem 92).

THEOREM XCIV.

The area of any circle, is equal to the rectangle of half its circumference and half its diameter.



Conceive a regular polygon to be inscribed in the circle ; and radii drawn to all the angular points, dividing it into as many equal triangles as the polygon has sides, one of which is ABC, of which the altitude is the perpendicular CD from the centre to the base AB.

Then the triangle ABC, being equal to a rectangle of half the base and equal altitude (th. 26, cor. 2), is equal to the rectangle of the half base AD and the altitude CD ; consequently the whole polygon, or all the triangles added together which compose it, is equal to the rectangle of the common altitude CD, and the halves of all the sides, or the half perimeter of the polygon.

Now conceive the number of sides of the polygon to be indefinitely increased ; then will its perimeter coincide with the circumference of the circle, and consequently the altitude CD will become equal to the radius, and the whole polygon equal to the circle. Consequently the space of the circle, or of the polygon in that state, is equal to the rectangle of the radius and half the circumference. Q. E. D.

OF PLANES AND SOLIDS.

DEFINITIONS.

Def. 88. The Common Section of two Planes, is the line in which they meet, or cut each other.

89. A Line is Perpendicular to a Plane, when it is perpendicular to every line in that plane which meets it.

90. One Pláne is Perpendicular to Another, when every line of the one, which is perpendicular to the line of their common section, is perpendicular to the other.

91. The Inclination of one Plane to another, or the angle they form between them, is the angle contained by two lines, drawn from any point in the common section, and at right angles to the same, one of these lines in each plane.

92. Parallel Planes, are such as being produced ever so far both ways, will never meet, or which are every where at an equal perpendicular distance.

93. A Solid Angle, is that which is made by three or more plane angles, meeting each other in the same point.

94. Similar Solids, contained by plane figures, are such as have all their solid angles equal, each to each, and are bounded by the same number of similar planes, alike placed.

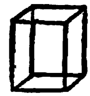
95. A Prism, is a solid whose ends are parallel, equal, and like plane figures, and its sides, connecting those ends, are parallelograms.

96. A Prism takes particular names according to the figure of its base or ends, whether triangular, square, rectangular, pentagonal, hexagonal, &c.

97. A Right or Upright Prism, is that which has the planes of the sides perpendicular to the planes of the ends or base.

98. A Parallelopiped, or Parallelopipedon, is a prism bounded by six parallelograms, every opposite two of which are equal, alike, and parallel.

99. A Rectangular Parallelopipedon, is that whose bounding planes are all rectangles, which are perpendicular to each other.



100. A Cube, is a square prism, being bounded by six equal square sides or faces, and are perpendicular to each other.

101. A Cylinder is a round prism, having circles for its ends; and is conceived to be formed by the rotation of a right line about the circumferences of two equal and parallel circles, always parallel to the axis.

102. The Axis of a Cylinder, is the right line joining the centres of the two parallel circles, about which the figure is described.

103. A Pyramid, is a solid, whose base is any right-lined plane figure, and its sides triangles, having all their vertices meeting together in a point above the base, called the vertex of the pyramid.

104. A pyramid, like the prism, takes particular names from the figure of the base.

105. A Cone, is a round pyramid, having a circular base, and is conceived to be generated by the rotation of a right line about the circumference of a circle, one end of which is fixed at a point above the plane of that circle.

106. The Axis of a cone, is the right line, joining the vertex, or fixed point, and the centre of the circle about which the figure is described.

107. Similar Cones and Cylinders, are such as have their altitudes and the diameters of their bases proportional.

108. A Sphere, is a solid bounded by one curve surface, which is every where equally distant from a certain point within, called the Centre. It is conceived to be generated by the rotation of a semicircle about its diameter, which remains fixed.

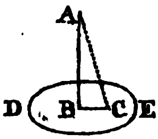
109. The Axis of a Sphere, is the right line about which the semicircle revolves; and the centre is the same as that of the revolving semicircle.

110. The Diameter of a Sphere, is any right line passing through the centre, and terminated both ways by the surface.

111. The Altitude of a solid, is the perpendicular drawn from the vertex to the opposite side or base.

THEOREM XCV.

A PERPENDICULAR is the shortest line which can be drawn from any point to a plane.



Let AB be perpendicular to the plane DE; then any other line, as AC, drawn from the same point A to the plane, will be longer than the line AB.

In the plane draw the line BC, joining the points BC.

Then, because the line AB is perpendicular to the plane DE, the angle B is a right angle (def. 90), and consequently greater than the angle C; therefore the line AB, opposite to the less angle, is less than any other line AC, opposite the greater angle (th. 21). Q. E. D.

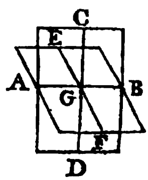
THEOREM XCVI.

A PERPENDICULAR measures the distance of any point from a plane.

The distance of one point from another is measured by a right line joining them, because this is the shortest line which can be drawn from one point to another. So, also, the distance from a point to a line, is measured by a perpendicular, because this line is the shortest which can be drawn from the point to the line. In like manner, the distance from a point to a plane, must be measured by a perpendicular drawn from that point to the plane, because this is the shortest line which can be drawn from the point to the plane.

THEOREM XCVII.

THE common section of two planes, is a right line.



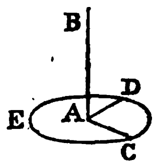
Let ACBDA, AEBFA, be two planes cutting each other, and A, B, two points in which the two planes meet; drawing the line AB, this line will be the common intersection of the two planes.

For, because the right line AB touches the two planes in the points A and B, it touches them in all other points (def. 20) ; this line is therefore common to the two planes. That is, the common intersection of the two planes is a right line. Q. E. D.

Corol. From the same point in a plane, there cannot be drawn two perpendiculars to the plane on the same side of it. For, if it were possible, each of these lines would be perpendicular to the straight line which is the common intersection of the plane and another plane passing through the two perpendiculars, which is impossible.

THEOREM XCVIII.

If a line be perpendicular to two other lines, at their common point of meeting ; it will be perpendicular to the plane of those lines.

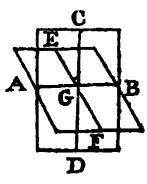


Let the line AB make right angles with the lines AC, AD ; then will it be perpendicular to the plane CDE which passes through these lines.

If the line AB were not perpendicular to the plane CDE, another plane might pass through the point A, to which the line AB would be perpendicular. But this is impossible ; for, since the angles BAC, BAD, are right angles, this other plane must pass through the points C, D. Hence, this plane passing through the two points A, C, of the line AC, and through the two points A, D, of the line AD, it will pass through both these two lines, and therefore be the same plane with the former. Q. E. D.

THEOREM XCIX.

If two planes cut each other at right angles, and a line be drawn in one of the planes perpendicular to their common intersection, it will be perpendicular to the other plane.



Let the two planes ACBD, AEBF, cut each other at right angles ; and the line CG be perpendicular to their common section AB ; then will CG be also perpendicular to the other plane AEBF.

For, draw EG perpendicular to AB. Then, because the two lines, GC, GE, are perpendicular to the common intersection

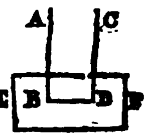
AB, the angle CGE is the angle of inclination of the two planes (def. 92). But since the two planes cut each other perpendicularly, the angle of inclination CGE is a right angle. And since the line CG is perpendicular to the two lines GA, GE, in the plane AEBF, it is therefore perpendicular to that plane (th. 98). Q. E. D.

Corol. 1. Every plane, ACB, passing through a perpendicular CG to another plane AEBF, will be perpendicular to that other plane. For, if ACB be not perpendicular to the plane AEBF, some other plane on the same side of AEBF, and passing through AB, will be perpendicular to it. Then, if from the point G a straight line be drawn in this other plane perpendicular to the common intersection, it will be perpendicular to the plane AEBF. But (hyp.) CG is perpendicular to that plane. Therefore, there will be, from the same point G, two perpendiculars to the same plane on the same side of it, which is impossible (cor. 97).

Corol. 2. If from any point G in the common intersection of the two planes ACB and AEBF perpendicular to each other, a line be drawn perpendicular to either plane, that line will be in the other plane.

THEOREM C.

If two lines be perpendicular to the same plane, they will be parallel to each other.



Let the two lines AB, CD, be both perpendicular to the same plane EBDF; then will AB be parallel to CD.

For, join B, D, by the line BD in the plane. The plane ABD is perpendicular to the plane EF (cor. 1, th. 99); and therefore the line CD, drawn from a point in the common intersection of the two planes, perpendicular to EF, will be in the plane ABD (cor. 2, th. 99). But, because the lines AB, CD, are perpendicular to the plane EF, they are both perpendicular to the line BD in that plane, and they have been proved to be in the same plane ABD; consequently, they are parallel to each other (cor. th. 13). Q. E. D.

Corol. If two lines be parallel, and if one of them be perpendicular to any plane, the other will also be perpendicular to the same plane.

THEOREM CI.

If one plane meet another plane, it will make angles with that other plane, which are together equal to two right angles.

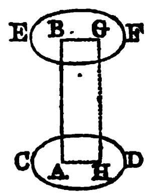
Let the plane ACBD meet the plane AEBF; these planes make with each other two angles whose sum is equal to two right angles.

For, through any point G, in the common section AB, draw CD, EF, perpendicular to AB. Then, the line CG makes with EF two angles together equal to two right angles. But these two angles are (by def. 92) the angles of inclination of the two planes. Therefore the two planes make angles with each other, which are together equal to two right angles.

Corol. In like manner, it may be demonstrated, that planes which intersect have their vertical or opposite angles equal; also, that parallel planes have their alternate angles equal; and so on, as in parallel lines.

THEOREM CII.

If two planes be parallel to each other; a line which is perpendicular to one of the planes, will also be perpendicular to the other.



Let the two planes CD, EF, be parallel, and let the line AB be perpendicular to the plane CD; then shall it also be perpendicular to the other plane EF.

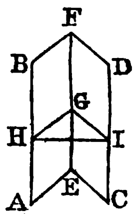
For, from any point G, in the plane EF, draw GH perpendicular to the plane CD, and draw AH, BG.

Then, because BA, GH, are both perpendicular to the plane CD, the angles A and H are both right angles. And because the planes CD, EF, are parallel, the perpendiculars BA, GH, are equal (def. 93). Hence it follows that the lines BG, AH, are parallel (def. 9). And the line AB being perpendicular to the line AH, is also perpendicular to the parallel line BG (cor. th. 12).

In like manner it is proved, that the line AB is perpendicular to all other lines which can be drawn from the point B in the plane EF. Therefore the line AB is perpendicular to the whole plane EF (def. 90). Q. E. D.

THEOREM CIII.

If two lines be parallel to a third line, though not in the same plane with it; they will be parallel to each other.



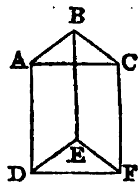
Let the lines AB, CD, be each of them parallel to the third line EF, though not in the same plane with it; then will AB be parallel to CD.

For, from any point G in the line EF, let GH, GI, be each perpendicular to EF, in the planes EB, ED, of the proposed parallels.

Then, since the line EF is perpendicular to the two lines GH, GI, it is perpendicular to the plane GHI of those lines (th. 98). And because EF is perpendicular to the plane GHI, its parallel AB is also perpendicular to that plane (cor. th. 99). For the same reason, the line CD is perpendicular to the same plane GHI. Hence, because the two lines AB, CD, are perpendicular to the same plane, these two lines are parallel (th. 99). Q. E. D.

THEOREM CIV.

If two lines, that meet each other, be parallel to two other lines that meet each other, though not in the same plane with them; the angles contained by those lines will be equal.



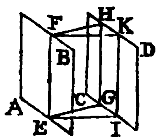
Let the two lines AB, BC, be parallel to the two lines, DE, EF; then will the angle ABC be equal to the angle DEF.

For, make the lins AB, BC, DE, EF, all equal to each other, and join AC, DF, AD, BE, CF.

Then, the lines AD, BE, joining the equal and parallel lines AB, DE, are equal and parallel (th. 24). For the same reason, CF, BE, are equal and parallel. Therefore AD, CF, are equal and parallel (th. 15); and consequently also AC, DF (th. 24). Hence, the two triangles ABC, DEF, having all their sides equal, each to each, have their angles also equal, and consequently the angle ABC = the angle DEF. Q. E. D.

THEOREM CV.

The sections made by a plane cutting two other parallel planes, are also parallel to each other.



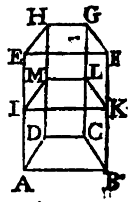
Let the two parallel planes AB, CD, be cut by the third plane EFHG, in the lines EF, GH: these two sections EF, GH, will be parallel.

Suppose EG, FH, be drawn parallel to each other in the plane EFHG; also let EI, FK, be perpendicular to the plane CD; and let IG, KH, be joined.

Then EG, FH, being parallels, and EI, FK, being both perpendicular to the plane CD, are also parallel to each other (th. 99); consequently the angle HFK is equal to the angle GEI (th. 104). But the angle FKH is also equal to the angle EIG, being both right angles; therefore the two triangles are equiangular (cor. 1, th. 17); and the sides FK, EI, being the equal distances between the parallel planes (def. 93), it follows that the sides FH, EG, are also equal (th. 2). But these two lines are parallel (by suppos.), as well as equal; consequently the two lines EF, GH, joining those two equal parallels, are also parallel (th. 24). Q. E. D.

THEOREM CVI.

If any prism be cut by a plane parallel to its base, the section will be equal and like to the base.

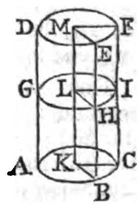


Let AG be any prism, and IL a plane parallel to the base, AC; then will the plane IL be equal and like to the base AC, or the two planes will have all their sides and all their angles equal.

For, the two planes AC, IL, being parallel by hypothesis; and two parallel planes, cut by a third plane, having parallel sections (th. 105); therefore IK is parallel to AB, and KL to BC, and LM to CD, and IM to AD. But AI and BK are parallels (by def. 95); consequently AK is a parallelogram; and the opposite sides AB, IK, are equal (th. 22). In like manner, it is shown that KL is = BC, and LM = CD, and IM = AD, or the two planes AC, IL, are mutually equilateral. But these two planes having their corresponding sides parallel, have the angles contained by them also equal (th. 104), namely, the angle A = the angle I, the angle B = the angle K, the angle C = the angle L, and the angle D = the angle M. So that the two planes AC, IL, have all their corresponding sides and angles equal, or they are equal and like. Q. E. D.

THEOREM CVII.

If a cylinder be cut by a plane parallel to its base, the section will be a circle, equal to the base.



Let AF be a cylinder, and GHI any section parallel to the base ABC; then will GHI be a circle, equal to ABC.

For, let the planes KE, KF, pass through the axis of the cylinder MK, and meet the section GHI in the three points H, I, L; and join the points as in the figure.

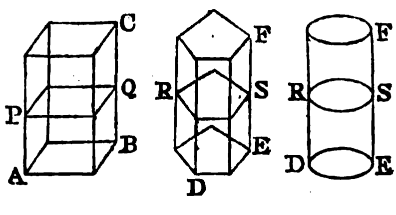
Then, since KL, CI, are parallel (by def. 102); and the plane KI, meeting the two parallel planes ABC, GHI, makes the two sections KC, LI, parallel (th. 105); the figure KLIC is therefore a parallelogram, and consequently has the opposite sides LI, KC, equal, where KC is a radius of the circular base.

In like manner it is shown that LH is equal to the radius KB; and that any other lines, drawn from the point L to the circumference of the section GHI, are all equal to radii of the base; consequently GHI is a circle, and equal to ABC.

Q. E. D.

THEOREM CVIII.

All prisms and cylinders, of equal bases and altitudes, are equal to each other.



Let AC, DF, be two prisms, and a cylinder, on equal bases, AB, DE, and having equal altitudes BC, EF; then will the solids AC, DF, be equal*.

For, let PQ, RS, be

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* This, and some other demonstrations relative to solids, are upon the defective principle of *Indivisibles*, introduced by *Cavalerius* in the year 1635. Unfortunately, demonstrations upon sounder principles would not accord with the brevity of this Course.

any two sections parallel to the bases, and equidistant from them. Then, by the last two theorems, the section PQ is equal to the base, AB, and the section RS equal to the base DE. But the bases, AB, DE, are equal, by the hypothesis; therefore the sections PQ, RS, are equal also. In like manner, it may be shown, that any other corresponding sections are equal to one another.

Since then every section in the prism AC is equal to its corresponding section in the prism or cylinder DF, the prisms and cylinder themselves, which are composed of an equal number of all those equal sections, must also be equal.

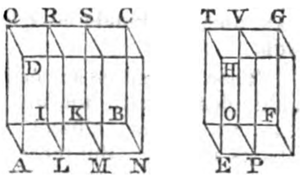
Q. E. D.

Corol. Every prism, or cylinder, is equal to a rectangular parallelopipedon, of an equal base and altitude.

THEOREM CIX.

RECTANGULAR parallelopipedons, of equal altitudes, are to each other as their bases*.

Let AC, EG, be two rectangular parallelopipedons, having the equal altitudes AD, EH; then will the solid AC be to the solid EG, as the base AB is to the base EF.



For, let the proportion of the base AB to the base EF, be that of any one number *m* (3) to any other number *n* (2). And conceive AB to be divided into *m* equal parts, or rectangles, AI, LK, MB (by dividing AN into that number of equal parts, and drawing IL, KM, parallel to BN). And let EF be divided, in like manner, into *n* equal parts, or rectangles, EO, PF: all of these parts, of both bases, being mutually equal among themselves. And through the lines of division let the plane sections LR, MS, PV, pass parallel to AQ, ET.

Then the parallelopipedons AR, LS, MC, EV, PG, are all equal, having equal bases and altitudes. Therefore the solid AC is to the solid EG, as the number of parts in the former, to the number of equal parts in the latter; or as the number

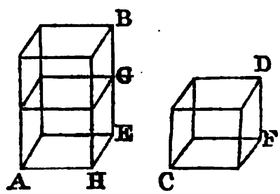
* Here, also, the principle of former notes may readily be applied in the case of incommensurables.

of parts in AB to the number of equal parts in EF, that is, as the base AB to the base EF. Q. E. D.

Corol. From this theorem, and the corollary to the last, it appears that all prisms and cylinders of equal altitudes, are to each other as their bases; every prism and cylinder being equal to a rectangular parallelopipedon of an equal base and altitude.

THEOREM CX.

RECTANGULAR parallelopipedons, of equal bases, are to each other as their altitudes.



Let AB, CD, be two rectangular parallelopipedons, standing on the equal bases AE, CF; then will the solid AB be to the solid CD, as the altitude EB is to the altitude FD.

For, let AG be a rectangular parallelopipedon on the base AE, and its altitude EG equal to the altitude FD of the solid CD.

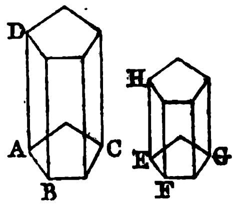
Then AG and CD are equal, being prisms of equal bases and altitudes. But if HB, HG, be considered as bases, the solids AB, AG, of equal altitude AH, will be to each other as those bases HB, HG. But these bases HB, HG, being parallelograms of equal altitude HE, are to each other as their bases EB, EG; therefore the two prisms, AB, AG, are to each other as the lines EB, EG. But AG is equal to CD, and EG equal to FD; consequently the prisms AB, CD, are to each other as their altitudes, EB, FD; that is, AB : CD :: EB : FD. Q. E. D.

Corol. 1. From this theorem, and the corollary to theorem 108, it appears, that all prisms and cylinders, of equal bases, are to one another as their altitudes.

Corol. 2. Because, by corollary 1, prisms and cylinders are as their altitudes, when their bases are equal. And by the corollary to the last theorem, they are as their bases, when their altitudes are equal. Therefore, universally, when neither are equal, they are to one another as the product of their bases and altitudes. And hence also these products are the proper numeral measures of their quantities or magnitudes.

THEOREM CXI.

SIMILAR prisms and cylinders are to each other, as the cubes of their altitudes, or of any other like linear dimensions.

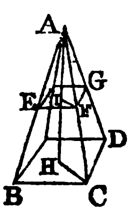


Let ABCD, EFGH, be two similar prisms; then will the prism CD be to the prism GH, as AB^3 to EF^3 or AD^3 to EH^3.

For the solids are to each other as the product of their bases and altitudes (th. 110, cor. 2), that is, as AC . AD to EG . EH. But the bases, being similar planes, are to each other as the squares of their like sides, that is, AC to EG as AB^2 to EF^2; therefore the solid CD is to the solid GH, as AB^2 . AD to EF^2 . EH. But BD and FH, being similar planes, have their like sides proportional, that is, AB : EF : : AD : EH, - - - - - - or AB^2 : EF^2 : : AD^2 : EH^2: therefore AB^2 . AD : EF^2 . EH : : AB^3 : EF^3, or : : AD^3 : EH^3; conseq. the solid CD : solid GH : : AB^3 : EF^3 : : AD^3 : EH^3. Q. E. D.

THEOREM CXII.

IN any pyramid, a section parallel to the base is similar to the base; and these two planes are to each other as the squares of their distances from the vertex.



Let ABCD be a pyramid, and EFG a section parallel to the base BCD, also AIH a line perpendicular to the two planes at H and I: then will BD, EG, be two similar planes, and the plane BD will be to the plane EG, as AH^2 to AI^2.

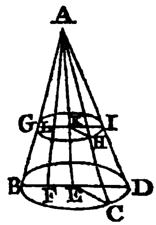
For, join CH, FI. Then because a plane cutting two parallel planes, makes parallel sections (th. 105), therefore the plane ABC, meeting the two parallel planes BD, EG, makes the sections BC, EF, parallel: In like manner, the plane ACD makes the sections CD, FG parallel. Again, because two pair of parallel lines make equal angles (th. 104), the two EF, FG, which are parallel to BC, CD, make the angle EFG equal the angle BCD. And in like manner it is shown, that each angle in the plane EG is equal to each angle in the plane BD, and consequently those two planes are equiangular.

Again, the three lines AB, AC, AD, making with the parallels BC, EF, and CD, FG, equal angles (th. 14), and the angles at A being common, the two triangles ABC, AEF, are equiangular, as also the two triangles ACD, AFG, and have therefore their like sides proportional, namely, - - - - AC : AF : : BC : EF : : CD : FG. And in like manner it may be shown, that all the lines in the plane FG, are proportional to all the corresponding lines in the base BD. Hence these two planes, having their angles equal, and their sides proportional, are similar, by def. 68.

But, similar planes being to each other as the squares of their like sides, the plane BD : EG : : BC^2 : EF^2, or : : AC^2 : AF^2, by what is shown above. Also, the two triangles AHC, AIF, having the angles H and I right ones (th. 98), and the angle A common, are equiangular, and have therefore their like sides proportional, namely, AC : AF : : AH : AI, or AC^2 : AF^2 : : AH^2 : AI^2. Consequently the two planes BD, EG, which are as the former squares AC^2, AF^2, will be also as the latter squares AH^2, AI^2, that is - - - - - - BD : EG : : AH^2 : AI^2. Q. E. D.

THEOREM CXIII.

In a cone, any section parallel to the base is a circle; and this section is to the base, as the squares of their distances from the vertex.



Let ABCD be a cone, and GHI a section parallel to the base BCD; then will GHI be a circle, and BCD, GHI, will be to each other, as the squares of their distances from the vertex.

For, draw ALF perpendicular to the two parallel planes; and let the planes ACE, ADE, pass through the axis of the cone AKE, meeting the section in the three points H, I, K.

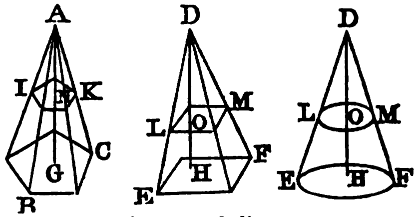
Then, since the section GHI is parallel to the base BCD, and the planes CK, DK, meet them, HK is parallel to CE, and IK to DE (th. 105). And because the triangles formed by these lines are equiangular, KH : EC : : AK : AE : : KI : ED. But EC is equal to ED, being radii of the same circle; therefore KI is also equal to KH. And the same may be shown of any other lines drawn from the point K to the perimeter of the section GHI, which is therefore a circle (def. 44).

Again, by similar triangles, AL : AF : : AK : AE, or : : KI : ED, hence AL^2 : AF^2 : : KI^2 : ED^2; but KI^2 : ED^2 : :

circle GHI : circle BCD (th. 93); therefore $AL^2 : AF^2 ::$ circle GHI : circle BCD. Q. E. D.

THEOREM CXIV.

All pyramids, and cones, of equal bases and altitudes, are equal to one another.



Let ABC, DEF, be any pyramids and cone, of equal bases BC, EF, and equal altitudes AG, DH: then will the pyramids and cone ABC and DEF, be equal.

For, parallel to the bases and at equal distances AN, DO, from the vertices, suppose the planes IK, LM, to be drawn.

Then, by the two preceding theorems, - - - - - -

$DO^2 : DH^2 :: LM : EF$, and

$AN^2 : AG^2 :: IK : BC$.

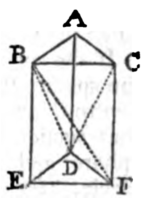
But since AN^2, AG^2, are equal to DO^2, DH^2, respectively, therefore IK : BC :: LM : EF. But BC is equal to EF, by hypothesis: therefore IK is also equal to LM.

In like manner it is shown, that any other sections, at equal distance from the vertex, are equal to each other.

Since then, every section in the cone, is equal to the corresponding section in the pyramids, and the heights are equal, the solids ABC, DEF, composed of all those sections, must be equal also. Q. E. D.

THEOREM CXV.

Every pyramid is the third part of a prism of the same base and altitude.



Let ABCDEF be a prism, and BDEF a pyramid, on the same triangular base DEF: then will the pyramid BDEF be a third part of the prism ABCDEF.

For, in the planes of the three sides of the prism, draw the diagonals BF, BD, CD. Then the two planes BDF, BCD, divide the whole prism into the three pyramids BDEF, DABC, DBCF, which are proved to be all equal to one another, as follows.

Since the opposite ends of the prism are equal to each other, the pyramid whose base is ABC and vertex D, is equal to the pyramid whose base is DEF and vertex B (th. 114), being pyramids of equal base and altitude.

But the latter pyramid, whose base is DEF and vertex B, is the same solid as the pyramid whose base is BEF and vertex D, and this is equal to the third pyramid whose base is BCF and vertex D, being pyramids of the same altitude and equal bases BEF, BCF.

Consequently all the three pyramids, which compose the prism, are equal to each other, and each pyramid is the third part of the prism, or the prism is triple of the pyramid. Q. E. D.

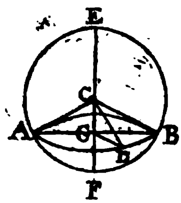
Hence also, every pyramid, whatever its figure may be, is the third part of a prism of the same base and altitude; since the base of the prism, whatever be its figure, may be divided into triangles, and the whole solid into triangular prisms and pyramids.

Corol. Any cone is the third part of a cylinder, or of a prism, of equal base and altitude; since it has been proved that a cylinder is equal to a prism, and a cone equal to a pyramid, of equal base and altitude.

Scholium. Whatever has been demonstrated of the proportionality of prisms, or cylinders, holds equally true of pyramids, or cones; the former being always triple the latter; viz. that similar pyramids or cones are as the cubes of their like linear sides, or diameters, or altitudes, &c. And the same for all similar solids whatever, viz. that they are in proportion to each other, as the cubes of their like linear dimensions, since they are composed of pyramids every way similar.

THEOREM CXVI.

IF a sphere be cut by a plane, the section will be a circle.



Let the sphere AEBF be cut by the plane ADB; then will the section ADB be a circle.

If the section pass through the centre of the sphere, then will the distance from the centre to every point in the periphery of that section be equal to the radius of the sphere, and consequently such section is a circle. Such, in truth, is the circle EAFB in the figure.

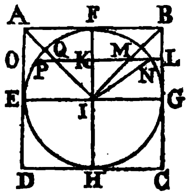
Draw the chord AB, or diameter of the section ADB; perpendicular to which, or to the said section, draw the axis of the sphere ECGF, through the centre C, which will bisect the chord AB in the point G (th. 41). Also, join CA, CB; and draw CD, GD, to any point D in the perimeter of the section ADB.

Then, because CG is perpendicular to the plane ADB, it is perpendicular both to GA and GD (def. 90). So that CGA, CGD are two right-angled triangles, having the perpendicular CG common, and the two hypothenuses CA, CD, equal, being both radii of the sphere; therefore the third sides GA, GD, are also equal (cor. 2, th. 34). In like manner it is shown, that any other line, drawn from the centre G to the circumference of the section ADB, is equal to GA or GB; consequently that section is a circle.

Scholium. The section through the centre, having the same centre and diameter as the sphere, is called a great circle of the sphere; the other plane sections being little circles.

THEOREM CXVII.

EVERY sphere is two-thirds of its circumscribing cylinder.



Let ABCD be a cylinder, circumscribing the sphere EFGH; then will the sphere EFGH be two-thirds of the cylinder ABCD.

For, let the plane AC be a section of the sphere and cylinder through the centre I. Join AI, BI. Also, let FIH be parallel to AD or BC, and EIG and KL parallel to AB or DC, the base of the cylinder; the latter line KL meeting BI in M, and the circular section of the sphere in N.

Then, if the whole plane HFBC be conceived to revolve about the line HF as an axis, the square FG will describe a cylinder AG, and the quadrant IFG will describe a hemisphere EFG, and the triangle IFB will describe a cone IAB. Also, in the rotation, the three lines or parts KL, KN, KM, as radii, will describe corresponding circular sections of those solids, namely, KL a section of the cylinder, KN a section of the sphere, and KM a section of the cone.

Now, FB being equal to FI or IG, and KL parallel to FB, then by similar triangles IK is equal to KM (th. 82). And since, in the right-angled triangle IKN, IN^2 is equal to $IK^2 + KN^2$ (th. 34); and because KL is equal to the radius IG or IN, and $KM = IK$, therefore KL^2 is equal to $KM^2 + KN^2$, or the square of the longest radius, of the said circular sections, is equal to the sum of the squares of the two others. And because circles are to each other as the squares of their diameters, or of their radii, therefore the circle described by KL is equal to both the circles described by KM and KN ; or the section of the cylinder, is equal to both the corresponding sections of the sphere and cone. And as this is always the case in every parallel position of KL, it follows, that the cylinder EB, which is composed of all the former sections, is equal to the hemisphere EFG and cone IAB, which are composed of all the latter sections.

But the cone IAB is a third part of the cylinder EB (cor. 2, th. 115) ; consequently the hemisphere EFG is equal to the remaining two-thirds ; or the whole sphere EFGH equal to two-thirds of the whole cylinder ABCD. Q. E. D.

Corol. 1. A cone, hemisphere, and cylinder of the same base and altitude, are to each other as the numbers 1, 2, 3.

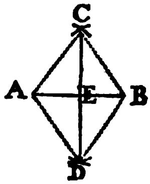
Corol. 2. All spheres are to each other as the cubes of their diameters ; all these being like parts of their circumscribing cylinders.

Corol. 3. From the foregoing demonstration it also appears, that the spherical zone or frustum EGNP, is equal to the difference between the cylinder EGLO and the cone INQ, all of the same common height IK. And that the spherical segment PFN, is equal to the difference between the cylinder ABLO and the conic frustum AQMB, all of the same common altitude FK.

PROBLEMS.

PROBLEM I.

To bisect a line AB; that is, to divide it into two equal parts.



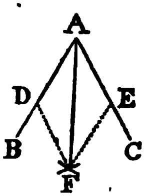
From the two centres A and B, with any equal radii, describe arcs of circles, intersecting each other in C and D; and draw the line CD, which will bisect the given line AB in the point E.

For, draw the radii AC, BC, AD, BD. Then, because all these four radii are equal, and the side CD common, the two triangles ACD, BCD, are mutually equilateral: consequently they are also mutually equiangular (th. 5), and have the angle ACE equal to the angle BCE.

Hence, the two triangles ACE, BCE, having the two sides AC, CE, equal to the two sides BC, CE, and their contained angles equal, are identical (th. 1), and therefore have the side AE equal to EB. Q. E. D.

PROBLEM II.

To bisect an angle BAC*.



From the centre A, with any radius, describe an arc, cutting off the equal lines AD, AE; and from the two centres D, E, with the same radius, describe arcs intersecting in F; then draw AF, which will bisect the angle A as required.

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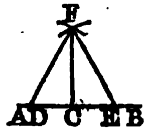
* A very ingenious instrument for *trisecting* an angle, is described in the Mechanic's Magazine, No. 22, p. 344.

For, join DF, EF. Then the two triangles ADF, AEF, having the two sides AD, DF, equal to the two AE, EF (being equal radii), and the side AF common, they are mutually equilateral; consequently they are also mutually equiangular (th. 5), and have the angle BAF equal to the angle CAF.

Scholium. In the same manner is an arc of a circle bisected.

PROBLEM III.

At a given point C, in a line AB, to erect a perpendicular.

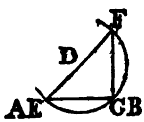


From the given point C, with any radius, cut off any equal parts CD, CE, of the given line; and, from the two centres D and E, with any one radius, describe arcs intersecting in F; then join CF, which will be perpendicular as required.

For, draw the two equal radii DF, EF. Then the two triangles CDF, CEF, having the two sides CD, DF, equal to the two CE, EF, and CF common, are mutually equilateral; consequently they are also mutually equiangular (th. 5), and have the two adjacent angles at C equal to each other; therefore the line CF is perpendicular to AB (def. 11).

Otherwise.

When the given point C is near the end of the line.

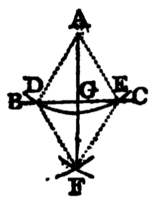


From any point D assumed above the line, as a centre, through the given point C describe a circle, cutting the given line at E; and through E and the centre D, draw the diameter EDF; then join CF, which will be the perpendicular required.

For the angle at C, being an angle in a semicircle, is a right angle, and therefore the line CF is a perpendicular (by def. 15).

PROBLEM IV.

From a given point A, to let fall a perpendicular on a given line BC.

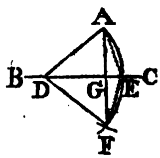


From the given point A as a centre, with any convenient radius, describe an arc, cutting the given line at the two points D and E; and from the two centres D, E, with any radius, describe two arcs, intersecting at F; then draw AGF, which will be perpendicular to BC as required.

For, draw the equal radii AD, AE, and DF, EF. Then the two triangles ADF, AEF, having the two sides AD DF, equal to the two AE, EF, and AF common, are mutually equilateral; consequently they are also mutually equiangular (th. 5), and have the angle DAG equal the angle EAG. Hence then, the two triangles ADG, AEG, having the two sides AD, AG, equal to the two AE, AG, and their included angles equal, are therefore equiangular (th. 1), and have the angles at G equal; consequently AG is perpendicular to BC (def. 11).

Otherwise.

WHEN the given point is nearly opposite the end of the line.

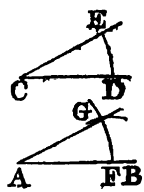


From any point D, in the given line BC, as a centre, describe the arc of a circle through the given point A, cutting BC in E; and from the centre E, with the radius EA, describe another arc, cutting the former in F; then draw AGF, which will be perpendicular to BC as required.

For, draw the equal radii DA, DF, and EA, EF. Then the two triangles DAE, DFE, will be mutually equilateral; consequently they are also mutually equiangular (th. 5), and have the angles at D equal. Hence, the two triangles DAG, DFG, having the two sides DA, DG, equal to the two DF, DG, and the included angles at D equal, have also the angles at G equal (th. 1); consequently those angles at G are right angles, and the line AG is perpendicular to DG.

PROBLEM V.

AT a given point A, in a line AB, to make an angle equal to a given angle C.

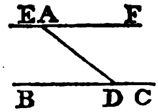


From the centres A and C, with any one radius, describe the arcs DE, FG. Then, with radius DE, and centre F, describe an arc, cutting FG in G. Through G draw the line AG, and it will form the angle required.

For, conceive the equal lines or radii, DE, FG, to be drawn. Then the two triangles CDE, AFG, being mutually equilateral, are mutually equiangular (th. 5), and have the angle at A equal to the angle C.

PROBLEM VI.

Through a given point A, to draw a line parallel to a given line BC.

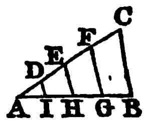


From the given point A draw a line AD to any point in the given line BC. Then draw the line EAF making the angle at A equal to the angle at D (by prob. 5); so shall EF be parallel to BC as required.

For, the angle D being equal to the alternate angle A, the lines BC, EF, are parallel, by th. 13.

PROBLEM VII.

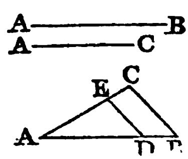
To divide a line AB into any proposed number of equal parts.



Draw any other line AC, forming any angle with the given line AB; on which set off as many of any equal parts AD, DE, EF, FC, as the line AB is to be divided into. Join BC; parallel to which draw the other lines FG, EH, DI: then these will divide AB in the manner as required.—For those parallel lines divide both the sides AB, AC, proportionally, by th. 82.

PROBLEM VIII.

To find a third proportional to two given lines AB, AC.



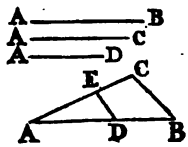
Place the two given lines AB, AC, forming any angle at A; and in AB take also AD equal to AC. Join BC, and draw DE parallel to it; so will AE be the third proportional sought.

For, because of the parallels, BC, DE, the two lines AB, AC, are cut proportionally (th. 82); so that AB : AC : : AD or AC : AE; therefore AE is the third proportional to AB, AC.

PROBLEM IX.

To find a fourth proportional to three lines AB, AC, AD.

Place two of the given lines AB, AC, making any angle at A; also place AD on AB. Join BC; and parallel to it draw

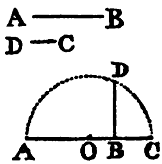


DE: so shall AE be the fourth proportional as required.

For, because of the parallels BC, DE, the two sides AB, AC, are cut proportionally (th. 82); so that . . . . AB : AC :: AD : AE.

PROBLEM X.

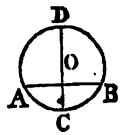
To find a mean proportional between two lines AB, BC.



Place AB, BC, joined in one straight line AC: on which, as a diameter, describe the semicircle ADC; to meet which erect the perpendicular BD; and it will be the mean proportional sought, between AB and BC (by cor. th. 87).

PROBLEM XI.

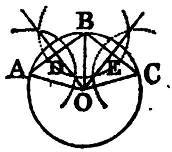
To find the centre of a circle.



Draw any chord AB; and bisect it perpendicularly with the line CD, which will be a diameter (th. 41, cor.). Therefore CD bisected in O, will give the centre, as required.

PROBLEM XII.

To describe the circumference of a circle through three given points A, B, C.

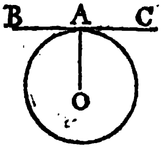


From the middle point B draw chords BA, BC, to the two other points, and bisect these chords perpendicularly by lines meeting in O, which will be the centre. Then from the centre O, at the distance of any one of the points, as OA, describe a circle, and it will pass through the two other points B, C, as required.

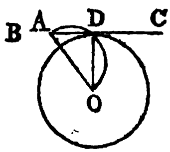
For the two right-angled triangles OAD, OBD, having the sides AD, DB, equal (by constr.), and OD common, with the included right angles at D equal, have their third sides OA, OB, also equal (th. 1). And in like manner it is shown that OC is equal to OB or OA. So that all the three OA, OB, OC, being equal, will be radii of the same circle.

PROBLEM XIII.

To draw a tangent to a circle, through a given point A.



When the given point A is in the circumference of the circle: Join A and the centre O; perpendicular to which draw BAC, and it will be the tangent, by th. 46.

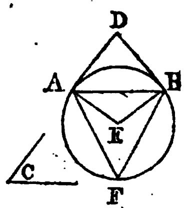


But when the given point A is out of the circle: Draw AO to the centre O; on which as a diameter describe a semicircle, cutting the given circumference in D; through which draw BADC, which will be the tangent as required.

For, join DO. Then the angle ADO, in a semicircle, is a right angle, and consequently AD is perpendicular to the radius DO, or is a tangent to the circle (th. 46).

PROBLEM XIV.

On a given line B to describe a segment of a circle, to contain a given angle C.

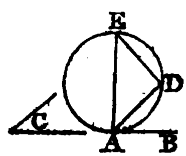


At the ends of the given line make angles DAB, DBA, each equal to the given angle C. Then draw AE, BE, perpendicular to AD, BD; and with the centre E, and radius EA or EB, describe a circle; so shall AFB be the segment required, as any angle F made in it will be equal to the given angle C.

For, the two lines AD, BD, being perpendicular to the radii EA, EB (by constr.), are tangents to the circle (th. 46); and the angle A or B, which is equal to the given angle C by construction, is equal to the angle F in the alternate segment AEB (th. 53).

PROBLEM XV.

To cut off a segment from a circle, that shall contain a given angle C.

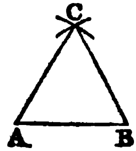


Draw any tangent AB to the given circle; and a chord AD to make the angle DAB equal to the given angle C; then DEA will be the segment required, any angle E made in it being equal to the given angle C.

For the angle A, made by the tangent and chord, which is equal to the given angle C by construction, is also equal to any angle E in the alternate segment (th. 53).

PROBLEM XVI.

To make an equilateral triangle on a given line AB.

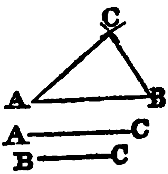


From the centres A and B, with the distance AB, describe arcs, intersecting in C. Draw AC, BC, and ABC will be the equilateral triangle.

For the equal radii, AC, BC, are, each of them, equal to AB.

PROBLEM XVII.

To make a triangle with three given lines AB, AC, BC.

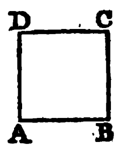


With the centre A, and distance AC, describe an arc. With the centre B, and distance BC, describe another arc, cutting the former in C. Draw AB, BC, and ABC will be the triangle required.

For the radii, or sides of the triangle, AC, BC, are equal to the given lines AC, BC, by construction.

PROBLEM XVIII.

To make a square on a given line AB.

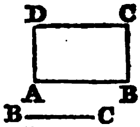


Raise AD, BC, each perpendicular and equal to AB; and join DC; so shall ABCD be the square sought.

For all the three sides AB, AD, BC, are equal, by the construction, and DC is equal and parallel to AB (by th. 24); so that all the four sides are equal, and the opposite ones are parallel. Again, the angle A or B, of the parallelogram, being a right angle, the angles are all right ones (cor. 1, th. 22). Hence, then, the figure, having all its sides equal, and all its angles right, is a square (def. 34).

PROBLEM XIX.

To make a rectangle, or a parallelogram, of a given length and breadth, AB, BC.



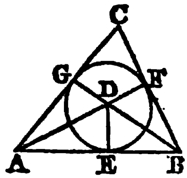
Erect AD, BC, perpendicular to AB, and each equal to BC; then join DC, and it is done.

The demonstration is the same as the last problem.

And in the same manner is described any oblique parallelogram, only drawing AD and BC to make the given oblique angle with AB, instead of perpendicular to it.

PROBLEM XX.

To inscribe a circle in a given triangle ABC.



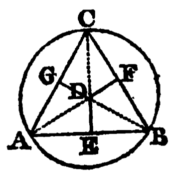
Bisect any two angles A and B, with the two lines AD, BD. From the intersection D, which will be the centre of the circle, draw the perpendiculars DE, DF, DG, and they will be the radii of the circle required.

For, since the angle DAE is equal to the angle DAG, and the angles at E, G, right angles (by constr.), the two triangles, ADE, ADG, are equiangular; and, having also the side AD common, they are identical, and have the sides DE, DG, equal (th. 2). In like manner it is shown, that DF is equal to DE or DG.

Therefore, if with the centre D, and distance DE, a circle be described, it will pass through all the three points, E, F, G, in which points also it will touch the three sides of the triangle (th. 46), because the radii DE, DF, DG, are perpendicular to them.

PROBLEM XXI.

To describe a circle about a given triangle ABC.

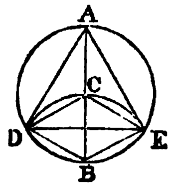


Bisect any two sides with two of the perpendiculars DE, DF, DG, and D will be the centre.

For, join DA, DB, DC. Then the two right-angled triangles DAE, DBE, have the two sides, DE, EA, equal to the two DE, EB, and the included angles at E equal: those two triangles are therefore identical (th. 1), and have the side DA equal to DB. In like manner it is shown, that DC is also equal to DA or DB. So that all the three DA, DB, DC, being equal, they are radii of a circle passing through A, B, and C.

PROBLEM XXII.

To inscribe an equilateral triangle in a given circle.

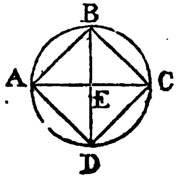


Through the centre C draw any diameter AB. From the point B as a centre, with the radius BC of the given circle, describe an arc DCE. Join AD, AE, DE, and ADE is the equilateral triangle sought.

For, join DB, DC, EB, EC. Then DCB is an equilateral triangle, having each side equal to the radius of the given circle. In like manner, BCE is an equilateral triangle. But the angle ADE is equal to the angle ABE or CBE, standing on the same arc AE; also the angle AED is equal to the angle CBD, on the same arc AD; hence the triangle DAE has two of its angles, ADE, AED, equal to the angles of an equilateral triangle, and therefore the third angle at A is also equal to the same; so that the triangle is equiangular, and therefore equilateral.

PROBLEM XXIII.

To inscribe a square in a given circle.

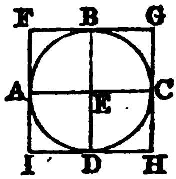


Draw two diameters AC, BD, crossing at right angles in the centre E. Then join the four extremities A, B, C, D, with right lines, and these will form the inscribed square ABCD.

For the four right-angled triangles AEB, BEC, CED, DEA, are identical because they have the sides EA, EB, EC, ED, all equal, being radii of the circle, and the four included angles at E all equal, being right angles, by the construction. Therefore all their third sides AB, BC, CD, DA, are equal to one another, and the figure ABCD is equilateral. Also, all its four angles, A, B, C, D, are right ones, being angles in a semicircle. Consequently the figure is a square.

PROBLEM XXIV.

To describe a square about a given circle.



Draw two diameters AC, BD, crossing at right angles in the centre E. Then through their four extremities draw FG, IH, parallel to AC, and FI, GH, parallel to BD, and they will form the square FGHI.

For, the opposite sides of parallelograms being equal, FG and IH are each equal to the diameter AC, and FI and GH each equal to the diameter BD; so that the figure is equilateral. Again, because the opposite angles of parallelograms are equal, all the four angles F, G, H, I, are right angles, being equal to the opposite angles at E. So that the figure FGHI, having its sides equal, and its angles right ones, is a square, and its sides touch the circle at the four points A, B, C, D, being perpendicular to the radii drawn to those points.

PROBLEM XXV.

To inscribe a circle in a given square.

Bisect the two sides FG, FI, in the points A and B (last fig.). Then through these two points draw AC parallel to FC or IH, and BD parallel to FI or GH. Then the point of intersection E will be the centre, and the four lines EA, EB, EC, ED, radii of the inscribed circle.

For, because the four parallelograms EF, EG, EH, EI, have their opposite sides and angles equal. therefore all the four lines EA, EB, EC, ED, are equal, being each equal to half a side of the square. So that a circle described from the centre E, with the distance EA, will pass through all the points A, B, C, D, and will be inscribed in the square, or will touch its four sides in those points, because the angles there are right ones.

PROBLEM XXVI.

To describe a circle about a given square.

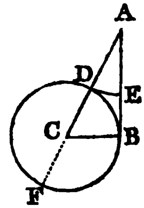
(See fig. Prob. xxiii.)

Draw the diagonals AC, BD, and their intersection E will be the centre.

For the diagonals of a square bisect each other (th. 40), making EA, EB, EC, ED, all equal, and consequently these are radii of a circle passing through the four points A, B, C, D.

PROBLEM XXVII.

To cut a given line in extreme and mean ratio.



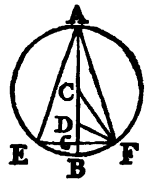
Let AB be the given line to be divided in extreme and mean ratio, that is, so as that the whole line may be to the greater part, as the greater part is to the less part.

Draw BC perpendicular to AB, and equal to half AB. Join AC; and with centre C and distance CB, describe the circle BD; then with centre A and distance AD, describe the arc DE; so shall AB be divided in E in extreme and mean ratio, or so that AB : AE : : AE : EB.

Produce AC to the circumference at F. Then, ADF being a secant, and AB a tangent, because B is a right angle: therefore the rectangle AF . AD is equal to AB^2 (cor. 1, th. 61); consequently the means and extremes of these are proportional (th. 77), viz. AB : AF or AD + DF : : AD : AB. But AE is equal to AD by construction, and AB = 2BC = DF; therefore, AB : AE + AB : : AE : AB; and by division, AB : AE : : AE : EB.

PROBLEM XXVIII.

To inscribe an isosceles triangle in a given circle, that shall have each of the angles at the base double the angle at the vertex.



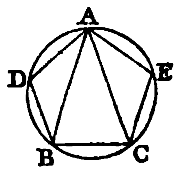
DRAW any diameter AB of the given circle; and divide the radius CB, in the point D, in extreme and mean ratio, by the last problem. From the point B apply the chords BE, BF, each equal to the greater part CD. Then join AE, AF, EF; and AEF will be the triangle required.

For, the chords BE, BF, being equal, their arcs are equal; therefore the supplemental arcs and chords AE, AF, are also equal; consequently the triangle AEF is isosceles, and has the angle E equal to the angle F; also the angles at G are right angles.

Draw CF and DF. Then, BC : CD : : CD : BD, or BC : BF : : BF : BD by constr. And BA : BF : : BF : BG (by th. 87). But BC = ½BA; therefore BG = ½BD = GD; therefore the two triangles GBF, GDF, are identical (th. 1), and each equiangular to ABF and AGF (th. 87). Therefore their doubles, BFD, AFE, are isosceles and equiangular, as well as the triangle BCF; having the two sides BC, CF, equal, and the angle B common with the triangle BFD. But CD is = DF or BF; therefore the angle C = the angle DFC (th. 4); consequently the angle BDF, which is equal to the sum of these two equal angles (th. 16), is double of one of them C; or the equal angle B or CBF double the angle C. So that CBF is an isosceles triangle, having each of its two equal angles double of the third angle C. Consequently the triangle ABF (which it has been shown is equiangular to the triangle C F) has also each of its angles at the base double the angle A at the vertex.

PROBLEM XXIX.

To inscribe a regular pentagon in a given circle.



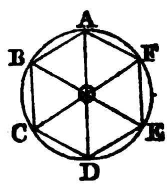
INSCRIBE the isosceles triangle ABC, having each of the angles ABC, ACB, double the angle BAC (prob. 28). Then bisect the two arcs ADB, AEC, in the points D, E; and draw the chords AD, DB, AE, EC, so shall ADBCE be the inscribed equilateral pentagon required.

For, because equal angles stand on equal arcs, and double angles on double arcs, also the angles ABC, ACB, being each double the angle BAC, therefore the arcs ADB, AEC, subtending the two former angles, are each double the arc BC subtending the latter. And since the two former arcs are bisected in D and E it follows that all the five arcs AD, DB, BC, CE, EA, are equal to each other, and consequently the chords also which subtend them, or the five sides of the pentagon, are all equal.

Note. In the construction, the points D and E are most easily found, by applying BD and CE each equal to BC.

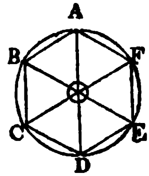
PROBLEM XXX.

To inscribe a regular hexagon in a circle.



APPLY the radius AO of the given circle as a chord, AB, BC, CD, &c. quite round the circumference, and it will complete the regular hexagon ABCDEF.

Draw the radii AO, BO, CO, DO, EO, FO, completing six equal triangles; of which any one, as ABO, being equilateral (by constr.) its three angles are all equal (cor. 2, th. 3), and any one of them, as AOB, is one-third of the whole, or of two right angles (th. 17), or one-sixth of four right angles. But the whole circumference is the measure of four right angles (cor. 4, th. 6). Therefore the arc AB is one-sixth of the circumference of the circle, and consequently its chord AB one side of an equilateral hexagon inscribed in the circle. And the same of the other chords.

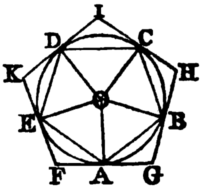


Corol. The side of a regular hexagon is equal to the radius of the circumscribing circle, or to the chord of one-sixth part of the circumference*.

PROBLEM XXXI.

To describe a regular pentagon or hexagon about a circle.

In the given circle inscribe a regular polygon of the same name or number of sides, as ABCDE, by one of the foregoing problems. Then to all its angular points draw tangents (by prob. 13), and these will form the circumscribing polygon required.



For all the chords, or sides of the inscribed figure, AB, BC, &c. being equal, and all the radii OA, OB, &c. being equal, all the vertical angles about the point O are equal. But the angles OEF, OAF, OAG, OBG, made by the tangents and radii, are right angles; therefore OEF + OAF = two right angles, and OAG + OBG = two right angles; consequently, also, AOE + AFE = two right angles, and OAB + AGB = two right angles (cor. 2, th. 18). Hence, then, the angles AOE + AFE being = AOB + AGB, of which AOB is = AOE; consequently the remaining angles F and G are also equal. In the same manner it is shown, that all the angles F, G, H, I, K, are equal.

Again, the tangents from the same point FE, FA, are equal, as also the tangents AG, GB, (cor. 2, th. 61); and the angles F and G of the isosceles triangles AFE, AGB, are equal, as well as their opposite sides AE, AB; consequently those two triangles are identical (th. 1), and have their other sides EF, FA, AG, GB, all equal, and FG equal to the double of any one of them. In like manner it is shown, that all the other sides GH, HI, IK, KI, are equal to FG, or double of the tangents GB, BH, &c.

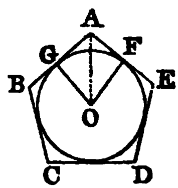
* The best way to describe a polygon of any number of sides, the length of one side being given, is to find the radius of the circumscribing circle by means of the table, at pa. 412, and the rule at pa. 413.

Hence, then, the circumscribed figure is both equilateral and equiangular, which was to be shown.

Corol. The inscribed circle touches the middle of the sides of the polygon.

PROBLEM XXXII.

To inscribe a circle in a regular polygon.

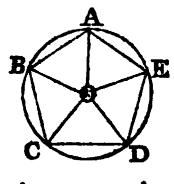


Bisect any two sides of the polygon by the perpendiculars GO, FO, and their intersection O will be the centre of the inscribed circle, and OG or OF will be the radius.

For the perpendiculars to the tangents AF, AG, pass through the centre (cor. th. 47); and the inscribed circle touches the middle points F, G, by the last corollary. Also, the two sides, AG, AO, of the right-angled triangle AOG, being equal to the two sides AF, AO, of the right-angled triangle AOF, the third sides OF, OG, will also be equal (cor. th. 45). Therefore the circle described with the centre O and radius OG, will pass through F, and will touch the sides in the points G and F. And the same for all the other sides of the figure.

PROBLEM XXXIII.

To describe a circle about a regular polygon.

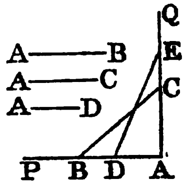


Bisect any two of the angles, C and D, with the lines CO, DO; then their intersection O will be the centre of the circumscribing circle; and OC, or OD, will be the radius.

For, draw OB, OA, OE, &c. to the angular points of the given polygon. Then the triangle OCD is isosceles, having the angles at C and D equal, being the halves of the equal angles of the polygon BCD, CDE; therefore their opposite sides CO, DO, are equal, (th. 4). But the two triangles OCD, OCB, having the two sides OC, CD, equal to the two OC, CB, and the included angles OCD, OCB, also equal, will be identical (th. 1), and have their third sides BO, OD, equal. In like manner it is shown, that all the lines OA, OB, OC, OD, OE, are equal. Consequently a circle described with the centre O and radius OA, will pass through all the other angular points, B, C, D, &c. and will circumscribe the polygon.

PROBLEM XXXIV.

To make a square equal to the sum of two or more given squares.

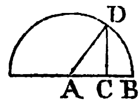


Let AB and AC be the sides of two given squares. Draw two indefinite lines AP, AQ, at right angles to each other; in which place the sides AB, AC, of the given squares; join BC; then a square described on BC will be equal to the sum of the two squares described on AB and AC (th. 34).

In the same manner, a square may be made equal to the sum of three or more given squares. For, if AB, AC, AD, be taken as the sides of the given squares, then, making AE = BC, AD = AD, and drawing DE, it is evident that the square on DE will be equal to the sum of the three squares on AB, AC, AD. And so on for more squares.

PROBLEM XXXV.

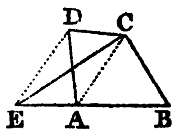
To make a square equal to the difference of two given squares.



Let AB and AC, taken in the same straight line, be equal to the sides of the two given squares.—From the centre A, with the distance AB, describe a circle; and make CD perpendicular to AB, meeting the circumference in D: so shall a square described on CD be equal to $AD^2 - AC^2$, or $AB^2 - AC^2$, as required (cor. th. 34).

PROBLEM XXXVI.

To make a triangle equal to a given quadrangle ABCD.



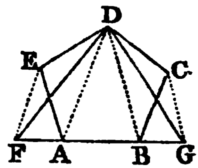
Draw the diagonal AC, and parallel to it DE, meeting BA produced at E, and join CE; then will the triangle CEB be equal to the given quadrilateral ABCD.

For, the two triangles ACE, ACD, being on the same base AC, and between the same parallels AC, DE, are equal (th. 25); therefore, if ABC be added to each, it will make BCE equal to ABCD (ax. 2).

PROBLEM XXXVII.

To make a triangle equal to a given pentagon ABCDE.

DRAW DA and DB, and also EF, CG, parallel to them, meeting AB produced at F and G; then draw DF and DG; so shall the triangle DFG be equal to the given pentagon ABCDE.

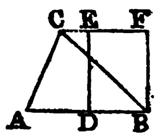


For the triangle DFA = DEA, and the triangle DGB = DCB (th. 25); therefore, by adding DAB to the equals, the sums are equal (ax. 2), that is, DAB + DAF + DBG = DAB + DAE + DBC, or the triangle DFG = to the pentagon ABCDE.

PROBLEM XXXVIII.

To make a rectangle equal to a given triangle ABC.

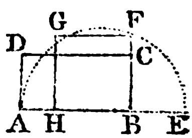
BISECT the base AB in D: then raise DE and BF perpendicular to AB, and meeting CF parallel to AB, at E and F: so shall DF be the rectangle equal to the given triangle ABC (by cor. 2, th. 26).



PROBLEM XXXIX.

To make a square equal to a given rectangle ABCD.

PRODUCE one side AB, till BE be equal to the other side BC. On AE as a diameter describe a circle, meeting BC produced at F: then will BF be the side of the square BFGH, equal to the given rectangle BD, as required; as appears by cor. th. 87, and th. 77.



APPLICATION OF ALGEBRA

TO

GEOMETRY.

When it is proposed to resolve a geometrical problem algebraically, or by algebra, it is proper, in the first place, to draw a figure that shall represent the several parts or conditions of the problem, and to suppose that figure to be the true one. Then having considered attentively the nature of the problem, the figure is next to be prepared for a solution, if necessary, by producing or drawing such lines in it as appear most conducive to that end. This done, the usual symbols or letters, for known and unknown quantities, are employed to denote the several parts of the figure, both the known and unknown parts, or as many of them as necessary, as also such unknown line or lines as may be easiest found, whether required or not. Then proceed to the operation, by observing the relations that the several parts of the figure have to each other ; from which, and the proper theorems in the foregoing elements of geometry, make out as many equations independent of each other, as there are unknown quantities employed in them : the resolution of which equations, in the same manner as in arithmetical problems, will determine the unknown quantities, and resolve the problem proposed.

As no general rule can be given for drawing the lines, and selecting the fittest quantities to substitute for, so as always to bring out the most simple conclusions, because different problems require different modes of solution ; the best way to gain experience, is to try the solution of the same problem in different ways, and then apply that which succeeds best, to other cases of the same kind, when they afterwards occur. The following particular directions, however, may be of some use.

1st, In preparing the figure, by drawing lines, let them be either parallel or perpendicular to other lines in the figure, or so as to form similar triangles. And if an angle be given, it will be proper to let the perpendicular be opposite to that angle, and to fall from one end of a given line, if possible.

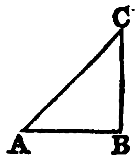
2d, In selecting the quantities proper to substitute for, those are to be chosen, whether required or not, which lie nearest the known or given parts of the figure, and by means of which the next adjacent parts may be expressed by addition and subtraction only, without using surds.

3d, When two lines or quantities are alike related to other parts of the figure or problem, the best way is, not to make use of either of them separately, but to substitute for their sum, or difference, or rectangle, or the sum of their alternate quotients, or for some line or lines, in the figure, to which they have both the same relation.

4th, When the area, or the perimeter, of a figure is given, or such parts of it as have only a remote relation to the parts required: it is sometimes of use to assume another figure similar to the proposed one, having one side equal to unity, or some other known quantity. For, hence the other parts of the figure may be found, by the known proportions of the like sides, or parts, and so an equation be obtained. For examples, take the following problems.

PROBLEM I.

In a right-angled triangle, having given the base (3), *and the sum of the hypothenuse and perpendicular* (9); *to find both these two sides.*



LET ABC represent the proposed triangle right-angled at B. Put the base AB $= 3 = b$, and the sum AC + BC of the hypothenuse and perpendicular $= 9 = s$; also, let x denote the hypothenuse AC, and y the perpendicular BC.

Then by the question . . . $x + y = s$,
and by theorem 34, ... $x^2 = y^2 + b^2$,
By transpos. y in the 1st equ. gives $x = s - y$,
This value of x substi. in the 2d, gives . . . . $s^2 - 2sy + y^2 = y^2 + b^2$,
Taking away y^2 on both sides leaves $s^2 - 2sy = b^2$,
By transpos. $2sy$ and b^2, gives . $s^2 - b^2 = 2sy$,
And dividing by $2s$, gives . . $\frac{s^2 - b^2}{2s} = y = 4 =$ BC.
Hence $x = s - y = 5 =$ AC.

N. B. In this solution, and the following ones, the notation is made by using as many unknown letters, x and y, as there are unknown sides of the triangle, a separate letter for each; in preference to using only one unknown letter for one side, and expressing the other unknown side in terms of that letter and the given sum or difference of the sides; though this latter way would render this solution shorter and sooner; because the former way gives occasion for more and better practice in reducing equations; which is the very end and reason for which these problems are given at all.

PROBLEM II.

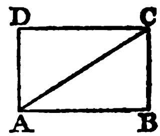
In a right-angled triangle, having given the hypothenuse (5); *and the sum of the base and perpendicular* (7); *to find both these two sides.*

LET ABC represent the proposed triangle, right-angled at B. Put the given hypothenuse AC $= 5 = a$, and the sum AB + BC of the base and perpendicular $= 7 = s$; also let x denote the base AB, and y the perpendicular BC.

Then by the question - - - $x + y = s$,
and by theorem 34 . . . $x^2 + y^2 = a^2$,
By transpos. y in the 1st, gives $x = s - y$,
By substitu. this value for x, gives $s^2 - 2sy + 2y^2 = a^2$,
By transposing s^2, gives - . $2y^2 - 2sy = a^2 - s^2$,
By dividing by 2, gives - - $y^2 - sy = \frac{1}{2}a^2 - \frac{1}{2}s^2$,
By completing the square, gives $y^2 - sy + \frac{1}{4}s^2 = \frac{1}{2}a^2 - \frac{1}{4}s^2$,
By extracting the root, gives - $y - \frac{1}{2}s = \sqrt{(\frac{1}{2}a^2 - \frac{1}{4}s^2)}$
By transposing $\frac{1}{2}s$, gives - . $y = \frac{1}{2}s \pm \sqrt{(\frac{1}{2}a^2 - \frac{1}{4}s^2)} =$
4 and 3, the values of x and y.

PROBLEM III.

In a rectangle, having given the diagonal (10), *and the perimeter, or sum of all the four sides* (28); *to find each of the sides severally.*

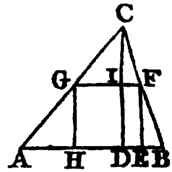


LET ABCD be the proposed rectangle; and put the diagonal AC $= 10 = d$, and half the perimeter AB + BC or AD + DC $= 14 = a$; also put one side AB $= x$, and the other side BC $= y$. Hence, by right-angled triangles, - - - - - - $x^2 + y^2 = d^2$,
And by the question - - - - - - $x + y = a$,
Then by transposing y in the 2d, gives $x = a - y$,
This value substituted in the 1st, gives $a^2 - 2ay + 2y^2 = d^2$,

Transposing a^2, gives . . $2y^2 - 2ay = d^2 - a^2$,
And dividing by 2, gives - $y^2 - ay = \frac{1}{2}d^2 - \frac{1}{2}a^2$,
By completing the square, it is $y^2 - ay + \frac{1}{4}a^2 = \frac{1}{2}d^2 - \frac{1}{4}a^2$,
And extracting the root, gives $y - \frac{1}{2}a = \sqrt{(\frac{1}{2}d^2 - \frac{1}{4}a^2)}$,
And transposing $\frac{1}{2}a$, gives - $y = \frac{1}{2}a \pm \sqrt{(\frac{1}{2}d^2 - \frac{1}{4}a^2)} = 8$, or 6, the values of x and y.

PROBLEM IV.

Having given the base any perpendicular of any triangle; to find the side of a square inscribed in the same.

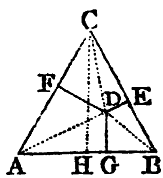


LET ABC represent the given triangle, and EFGH its inscribed square. Put the base AB $= b$, the perpendicular CD $= a$, and the side of the square GF or GH $=$ DI $= x$; then will CI $=$ CD $-$ DI $= a - x$.

Then, because the like lines in the similar triangles ABC, GFC, are proportional (by theor. 84, Geom.), AB : CD : : GF : CI, that is, $b : a :: x : a - x$. Hence, by multiplying extremes and means, $ab - bx = ax$, and transposing bx, gives $ab = ax + bx$; then dividing by $a + b$, gives $x = \frac{ab}{a + b} =$ GF or GH the side of the inscribed square: which therefore is of the same magnitude, whatever the species or the angles of the triangles may be.

PROBLEM V.

In an equilateral triangle, having given the lengths of the three perpendiculars, drawn from a certain point within, on the three sides; to determine the sides.



LET ABC represent the equilateral triangle, and DE, DF, DG, the given perpendiculars from the point D. Draw the lines DA, DB, DC, to the three angular points; and let fall the perpendicular CH on the base AB. Put the three given perpendiculars, DE $= a$, DF $= b$, DG $= c$, and put $x =$ AH or BH, half the side of the equilateral triangle. Then is AC or BC $= 2x$, and by right-angled triangles the perpendicular CH $= \sqrt{(AC^2 - AH^2)} = \sqrt{(4x^2 - x^2)} = \sqrt{3x^2} = x\sqrt{3}$.

Now, since the area or space of a rectangle, is expressed by the product of the base and height (cor. 2, th 81, Geom.), and that a triangle is equal to half a rectangle of equal base and height (cor. 1, th. 26), it follows that,

the whole triangle ABC is $= \frac{1}{2}$AB $\times$ CH $= x \times x\sqrt{3} = x^2\sqrt{3}$,
the triangle ABD $= \frac{1}{2}$AB $\times$ DG $= x \times c = cx$,
the triangle BCD $= \frac{1}{2}$BC $\times$ DE $= x \times a = ax$,
the triangle ACD $= \frac{1}{2}$AC $\times$ DF $= x \times b = bx$.

But the three last triangles make up, or are equal to, the whole former, or great triangle;

that is, $x^2\sqrt{3} = ax + bx + cx$; hence, dividing by x, gives $x\sqrt{3} = a + b + c$, and dividing by $\sqrt{3}$, gives $x = \frac{a+b+c}{\sqrt{3}}$, half the side of the triangle sought.

Also, since the whole perpendicular CH is $= x\sqrt{3}$, it is therefore $= a + b + c$. That is, the whole perpendicular CH, is just equal to the sum of all the three smaller perpendiculars DE + DF + DG taken together, wherever the point D is situated.

PROBLEM VI.

In a right-angled triangle, having given the base (3), and the difference between the hypothenuse and perpendicular (1); to find both these two sides.

PROBLEM VII.

In a right-angled triangle, having given the hypothenuse (5), and the difference between the base and perpendicular (1); to determine both these two sides.

PROBLEM VIII.

Having given the area, or measure of the space, of a rectangle, inscribed in a given triangle; to determine the sides of the rectangle.

PROBLEM IX.

In a triangle, having given the ratio of the two sides, together with both the segments of the base, made by a perpendicular from the vertical angle; to determine the sides of the triangle.

PROBLEM X.

In a triangle, having given the base, the sum of the other two sides, and the length of a line drawn from the vertical angle to the middle of the base; to find the sides of the triangle.

PROBLEM XI.

In a triangle, having given the two sides about the vertical angle, with the line bisecting that angle, and terminating in the base; to find the base.

PROBLEM XII.

To determine a right-angled triangle; having given the lengths of two lines drawn from the acute angles, to the middle of the opposite sides.

PROBLEM XIII.

To determine a right-angled triangle; having given the perimeter, and the radius of its inscribed circle.

PROBLEM XIV.

To determine a triangle; having given the base, the perpendicular, and the ratio of the two sides.

PROBLEM XV.

To determine a right-angled triangle; having given the hypothenuse, and the side of the inscribed square.

PROBLEM XVI.

To determine the radii of three equal circles, described in a given circle, to touch each other and also the circumference of the given circle.

PROBLEM XVII.

In a right-angled triangle, having given the perimeter, or sum of all the sides, and the perpendicular let fall from the right angle on the hypothenuse; to determine the triangle, that is, its sides.

PROBLEM XVIII.

To determine a right-angled triangle; having given the hypothenuse, and the difference of two lines drawn from the two acute angles to the centre of the inscribed circle.

PROBLEM XIX.

To determine a triangle; having given the base, the perpendicular, and the difference of the two other sides.

PROBLEM XX.

To determine a triangle; having given the base, the perpendicular, and the rectangle or product of the two sides.

PROBLEM XXI.

To determine a triangle; having given the lengths of three lines drawn from the three angles, to the middle of the opposite sides.

PROBLEM XXII.

In a triangle, having given all the three sides; to find the radius of the inscribed circle.

PROBLEM XXIII.

To determine a right-angled triangle; having given the side of the inscribed square, and the radius of the inscribed circle.

PROBLEM XXIV.

To determine a triangle, and the radius of the inscribed circle; having given the lengths of three lines drawn from the three angles, to the centre of that circle.

PROBLEM XXV.

To determine a right-angled triangle; having given the hypothenuse, and the radius of the inscribed circle.

PROBLEM XXVI.

To determine a triangle; having given the base, the line bisecting the vertical angle, and the diameter of the circumscribing circle.

PLANE TRIGONOMETRY.

DEFINITIONS.

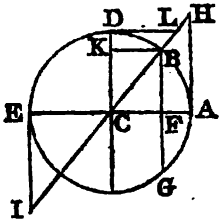
1. PLANE TRIGONOMETRY treats of the relations and calculations of the sides and angles of plane triangles.

2. The circumference of every circle (as before observed in Geom. Def. 56) is supposed to be divided into 360 equal parts, called Degrees; also each degree into 60 Minutes, and each minute into 60 Seconds, and so on. Hence a semicircle contains 180 degrees, and a quadrant 90 degrees.

3. The Measure of an angle (Def. 57, Geom.) is an arc of any circle contained between the two lines which form that angle, the angular point being the centre; and it is estimated by the number of degrees contained in that arc.

Hence, a right angle, being measured by a quadrant, or quarter of the circle, is an angle of 90 degrees; and the sum of the three angles of every triangle, or two right angles, is equal to 180 degrees. Therefore, in a right-angled triangle, taking one of the acute angles from 90 degrees, leaves the other acute angle; and the sum of the two angles, in any triangle, taken from 180 degrees, leaves the third angle; or one angle being taken from 180 degrees, leaves the sum of the other two angles.

4. Degrees are marked at the top of the figure with a small °, minutes with ′, seconds with ″, and so on. Thus, 57° 30′ 12″, denote 57 degrees 30 minutes and 12 seconds.



5. The Complement of an arc, is what it wants of a quadrant or 90°. Thus, if AD be a quadrant, then BD is the complement of the arc AB; and, reciprocally, AB is the complement of BD. So that, if AB be an arc of 50°, then its complement BD will be 40°.

6. The Supplement of an arc, is what it wants of a semicircle, or 180°. Thus, if ADE be a semicircle, then BDE is the supplement of the arc AB; and, reciprocally, AB

is the supplement of the arc BDE. So that, if AB be an arc of 50°, then its supplement BDE will be 130°.

7. The Sine, or Right Sine, of an arc, is the line drawn from one extremity of the arc, perpendicular to the diameter which passes through the other extremity. Thus, BF is the sine of the arc AB, or of the supplemental arc BDE. Hence the sine (BF) is half the chord (BG) of the double arc (BAG).

8. The Versed Sine of an arc, is the part of the diameter intercepted between the arc and its sine. So, AF is the versed sine of the arc AB, and EF the versed sine of the arc EDB.

9. The Tangent of an arc, is a line touching the circle in one extremity of that arc, continued from thence to meet a line drawn from the centre through the other extremity; which last line is called the Secant of the same arc. Thus, AH is the tangent, and CH the secant, of the arc AB. Also, EI is the tangent, and CI the secant, of the supplemental arc BDE. And this latter tangent and secant are equal to the former, but are accounted negative, as being drawn in an opposite or contrary direction to the former.

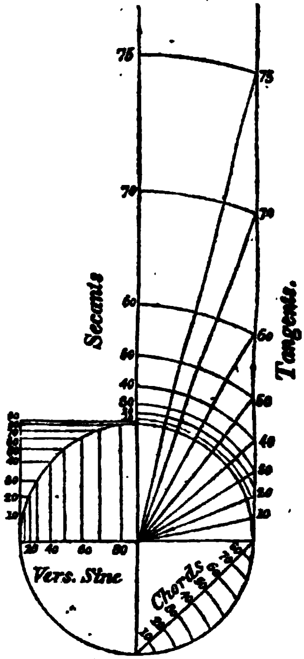
10. The Cosine, Cotangent, and Cosecant, of an arc, are the sine, tangent, and secant of the complement of that arc, the Co being only a contraction of the word complement. Thus, the arcs AB, BD, being the complements of each other, the sine, tangent, or secant of the one of these, is the cosine, cotangent, or cosecant of the other. So, BF, the sine of AB, is the cosine of BD; and BK, the sine of BD, is the cosine of AB: in like manner, AH, the tangent of AB, is the cotangent of BD; and DL, the tangent of DB, is the cotangent of AB; also, CH, the secant of AB, is the cosecant of BD; and CL, the secant of BD, is the cosecant of AB.

Corol. Hence several important properties easily follow from these definitions; as,

1*st*, That an arc and its supplement have the same sine, tangent, and secant; but the two latter, the tangent and secant, are accounted negative when the arc is greater than a quadrant or 90 degrees.

2*d*, When the arc is 0, or nothing, the sine and tangent are nothing, but the secant is then the radius CA, the least it can be. As the arc increases from 0, the sines, tangents, and secants, all proceed increasing, till the arc becomes a whole quadrant AD, and then the sine is the greatest it can be, being the radius CD of the circle; and both the tangent and secant are infinite.

3*d*, Of any arc AB, the versed sine AF, and cosine BK, or CF, together make up the radius CA of the circle.—The adius CA, the tangent AH, and the secant CH, form a right-angled triangle CAH. So also do the radius, sine, and cosine, form another right-angled triangle CBF or CBK. As also the radius, cotangent, and cosecant, another right-angled triangle CDL. And all these right-angled triangles are similar to each other.



11. The sine, tangent, or secant of an angle, is the sine, tangent, or secant of the arc by which the angle is measured, or of the degrees, &c. in the same arc or angle.

12. The method of constructing the scales of chords, sines, tangents, and secants, usually engraven on instruments, for practice, is exhibited in the annexed figure.

13. A Trigonometrical Canon, is a table showing the length of the sine, tangent, and secant, to every degree and minute of the quadrant, with respect to the radius, which is expressed by unity or 1, with any number of ciphers. The logarithms of these sines, tangents, and secants, are also ranged in the tables; and these are most commonly used, as they perform the calculations by only addition and subtraction, instead of the multiplication and division by the natural sines, &c. according to the nature of logarithms. Such tables of log. sines and tangents, as well as the logs of common numbers, greatly facilitate trigonometrical computations, and are now very common. Among the most correct are those published by the author of this Course.

PROBLEM I.

To compute the Natural Sine and Cosine of a Given Arc.

This problem is resolved after various ways. One of these is as follows, viz. by means of the ratio between the diameter and circumference of a circle, together with the known series for the sine and cosine, hereafter demonstrated. Thus, the semicircumference of the circle, whose radius is 1, being 3·141592653589793 &c, the proportion will therefore be,

as the number of degrees or minutes in the semicircle,
is to the degrees or minutes in the proposed arc,
so is 3·14159265 &c, to the length of the said arc.

This length of the arc being denoted by the letter a; and its sine and cosine by s and c; then will these two be expressed by the two following series, viz.

$$s = a - \frac{a^3}{2.3} + \frac{a^5}{2.3.4.5} - \frac{a^7}{2.3.4.5.6.7} + \&c.$$

$$= a - \frac{a^3}{6} + \frac{a^5}{120} - \frac{a^7}{5040} + \&c.$$

$$c = 1 - \frac{a^2}{2} + \frac{a^4}{2.3.4} - \frac{a^6}{2.3.4.5.6} + \&c.$$

$$= 1 - \frac{a^2}{2} + \frac{a^4}{24} - \frac{a^6}{720} + \&c.$$

Exam. 1. If it be required to find the sine and cosine of 1 minute. Then, the number of minutes in 180° being 10800, it will be first, as 10800 : 1 : : 3·14159265 &c. : ·000290888208665 = the length of an arc of one minute. Therefore, in this case,

$a =$ ·0002908882
and $\frac{1}{6}a^3 =$ ·000000000004 &c.
the diff. is $s =$ ·0002908882 the sine of 1 minute.
Also, from 1·
take $\frac{1}{2}a^2 =$ 0 0000000423079 &c.
leaves $c =$ ·9999999577 the cosine of 1 minute.

Exam. 2. For the sine and cosine of 5 degrees.
Here as 180° : 5° : : 3·14159265 &c. : ·08726646 = a the length of 5 degrees. Hence $a =$ ·08726646
$- \frac{1}{6}a^3 = -$ ·00011076
$+ \frac{1}{120}a^5 =$ ·00000004

these collected give $s =$ ·08715574 the sine of 5°.

And, for the cosine, 1 = 1·
$- \frac{1}{2}a^2 = -$ ·00380771
$+ \frac{1}{24}a^4 =$ ·00000241

these collected give $c =$ ·99619470 the cosine of 5°.

After the same manner, the sine and cosine of any other arc may be computed. But the greater the arc is, the slower the series will converge, in which case a greater number of terms must be taken, to bring out the conclusion to the same degree of exactness.

Or, having found the sine, the cosine will be found from it, by the property of the right-angled triangle CBF, viz. the cosine $CF = \surd(CB^2 - BF^2)$, or $c = \surd(1 - s^2)$.

There are also other methods of constructing the canon of sines and cosines, which, for brevity's sake, are here omitted: some of them, however, are explained under the analytical trigonometry in the second volume of this Course.

PROBLEM II.

To compute the Tangents and Secants.

The sines and cosines being known, or found by the foregoing problem; the tangents and secants will be easily found, from the principle of similar triangles, in the following manner:

In the first figure, where, of the arc AB, BF is the sine, CF or BK the cosine, AH the tangent, CH the secant, DL the cotangent, and CL the cosecant, the radius being CA or CB or CD; the three similar triangles CFB, CAH, CDL, give the following proportions:

1*st*, CF : FB : : CA : AH; whence the tangent is known, being a fourth proportional to the cosine, sine, and radius.

2*d*, CF : CB : : CA : CH; whence the secant is known, being a third proportional to the cosine and radius: or, being, indeed, the reciprocal of the cosine when the radius is unity.

3*d*, BF : FC : : CD : DL; whence the cotangent is known, being a fourth proportional to the sine, cosine, and radius.

Or, AH : AC : : CD : DL; whence it appears that the cotangent is a third proportional to the tangent and radius; or the reciprocal of the tangent to radius 1.

4*th* BF : BC : : CD : CL; whence the cosecant is known, being a third proportional to the sine and radius; or the reciprocal of the sine to radius 1.

As for the log. sines, tangents, and secants, in the tables, they are only the logarithms of the natural sines, tangents, and secants, calculated as above.

Having given an idea of the calculation and use of sines, tangents, and secants, we may now proceed to resolve the several cases of Trigonometry; previous to which, however, it may be proper to add a few preparatory notes and observations, as below.

Note 1. There are three methods of resolving triangles, or the cases of trigonometry; namely, Geometrical Construction, Arithmetical Computation, and Instrumental Operation; of which the first two will here be treated.

In the First Method, The triangle is constructed, by making the parts of the given magnitudes, namely, the sides from a scale of equal parts, and the angles from a scale of chords, or by some other instrument. Then measuring the unknown parts by the same scales or instruments, the solution will be obtained near the truth.

In the Second Method, Having stated the terms of the proportion according to the proper rule or theorem, resolve it like any other proportion, in which a fourth term is to be found from three given terms. by multiplying the second and third together, and dividing the product by the first, in working with the natural numbers; or, in working with the logarithms, add the logs. of the second and third terms together, and from the sum take the log. of the first term; then the natural number answering to the remainder is the fourth term sought.

Note 2. Every triangle has six parts, viz. three sides and three angles. And in every triangle proposed, there must be given three of these parts, to find the other three. Also, of the three parts that are given, one of them at least must be a side; because, with the same angles, the sides may be greater or less in any proportion.

Note 3. All the cases in trigonometry, may be comprised in three varieties only; viz.

1*st*, When a side and its opposite angle are given.

2*d*, When two sides and the contained angle are given.

3*d*, When the three sides are given.

For there cannot possibly be more than these three varieties of cases; for each of which it will therefore be proper to give a separate theorem, as follows:

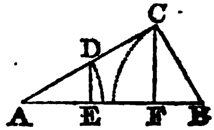
THEOREM I.

When a Side and its Opposite Angle are two of the Given Parts.

Then the unknown parts will be found by this theorem: viz. The sides of the triangle have the same proportion to each other, as the sines of their opposite angles have.

That is, As any one side,
Is to the sine of its opposite angle ;
So is any other side,
To the sine of its opposite angle.

Demonstr. For, let ABC be the proposed triangle, having AB the greatest side, and BC the least. Take AD = BC, considering it as a radius ; and let fall the perpendiculars DE, CF, which will evidently be the sines of the angles A and B, to the radius AD or BC.



Now the triangles ADE, ACF, are equiangular ; they therefore have their like sides proportional, namely, AC : CF : : AD or BC : DE ; that is, the side AC is to the sine of its opposite angle B, as the side BC is to the sine of its opposite angle A.

Note 1. In practice, to find an angle, begin the proportion with a side opposite to a given angle. And to find a side, begin with an angle opposite to a given side.

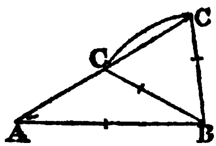
Note 2. An angle found by this rule is ambiguous, or uncertain whether it be acute or obtuse, unless it be a right angle, or unless its magnitude be such as to prevent the ambiguity ; because the sine answers to two angles, which are supplements to each other ; and accordingly the geometrical construction forms two triangles with the same parts that are given, as in the example below ; and when there is no restriction or limitation included in the question, either of them may be taken. The number of degrees in the table, answering to the sine, measure the acute angle ; but if the angle be obtuse, subtract those degrees from 180°, and the remainder will be the obtuse angle. When a given angle is obtuse, or a right one, there can be no ambiguity ; for then neither of the other angles can be obtuse, and the geometrical construction will form only one triangle.

EXAMPLE I.

In the plane triangle ABC,

Given { AB 345 yards, BC 232 yards, ∠ A 37° 20′ }

Required the other parts.



1. *Geometrically.*

Draw an indefinite line ; on which set off AB = 345, from some convenient scale of equal parts.—Make the angle

A = 37°½.—With a radius of 232, taken from the same scale of equal parts, and centre B, cross AC in the two points, C, C.—Lastly, join BC, BC, and the figure is constructed, which gives two triangles, and shows that the case is ambiguous.

Then, the sides AC measured by the scale of equal parts, and the angles B and C measured by the line of chords, or other instrument, will be found to be nearly as below; viz.

AC 174	∠ B 27°	∠ C 115°½.
or 374½	or 78¼	or 64½.

2. *Arithmetically.*

First, to find the angles at C.

As side	BC 232		.	.	log. 2·3654880
To sin. op.	∠ A 37° 20′		.	.	9·7827958
So side	AB 345		.	.	2·5378191
To sin. op.	∠ C 115° 36′	or 64° 24′			9·9551269
add	∠ A 37 20	37 20			
the sum	152 56	or 101 44			
taken from	180 00	180 00			
leaves	∠ B 27 04	or 78 16			

Then, to find the side AC.

As sine ∠ A	37° 20′	.	.	log. 9·7827958
To op. side BC	232	.	.	2·3654880
So sin. ∠ B	27° 04′	.	.	9·6580371
	78 16	.	.	9·9908291
To op. side AC	174·07	.	.	2·2407293
or	374·56	.	.	2·5735213

EXAMPLE II.

In the plane triangle ABC,

Given { AB 365 poles, ∠A 57° 12′, ∠B 24 45 } Ans. { ∠C 98° 3′, AC 154·33, BC 309·86 }

Required the other parts.

EXAMPLE III.

In the plane triangle ABC,

Given { AC 120 feet, BC 112 feet, ∠A 57° 27′ }

Ans. { ∠B 64° 34′ 21″, or 115 25 39, ∠C 57 58 39, or 7 7 21, AB 112·65 feet, or 10·47 feet }

Required the other parts.

THEOREM II.

When two Sides and their Contained Angle are given.

First find the sum and the difference of the given sides. Next subtract the given angle from 180°, and the remainder will be the sum of the two other angles; then divide that by 2, which will give the half sum of the said unknown angles. Then say,

As the sum of the two given sides,
Is to the difference of the same sides;
So is the tang. of half the sum of their op. angles,
To the tang. of half the diff. of the same angles.

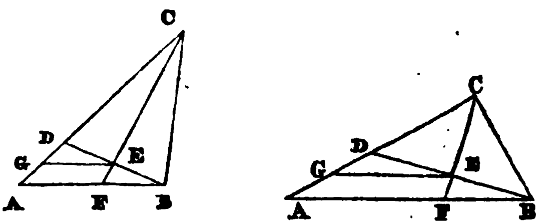
Add the half difference of the angles, so found, to their half sum, and it will give the greater angle, and subtracting the same will leave the less angle: because the half sum of any two quantities, increased by their half difference, gives the greater, and diminished by it gives the less.

All the angles being thus known, the unknown side will be found by the former theorem.

Note. Instead of the tangent of the half sum of the unknown angles, in the third term of the proportion, may be used the cotangent of half the given angle, which is the same thing.

Demon. Let ABC be a plane triangle of which AC, CB, and the included angle C are given: C being *acute* in the first figure, *obtuse* in the second.

On AC, the longer side, set off CD = CB the shorter; join



BD, and bisect it in E; also, bisect AD in G, and join GE, CE, producing the latter to F.

$$\text{Now } \tfrac{1}{2}(AC + CB) = \tfrac{1}{2}(2GD + 2DC) = CG$$
$$\text{and } \tfrac{1}{2}(AC - CB) = \tfrac{1}{2}(2AG) = AG$$
$$\text{also } \tfrac{1}{2}(A + B) = \tfrac{1}{2}(CDB + CBD) = CBD$$
$$\text{and } \tfrac{1}{2}(B - A) = ABC - \tfrac{1}{2}\text{ sum} = ABD:$$

also, because CE bisects the base of the isosceles triangle CBD, it is perpendicular to it:

$$\text{Therefore } EC = \text{tangent of } CBD \\ EF = \text{tangent of } ABD \Big\} \text{ to radius } BE.$$

Lastly, because in the triangle ACF, GE is parallel to AF (Geom. th. 82) we have

CG : GA : : CE : EF; that is,

$\frac{1}{2}(AC + CB) : \frac{1}{2}(AC - CB) :: \tan. \frac{1}{2}(B + A) : \tan. \frac{1}{2}(B - A)$; or, since doubling both the antecedent and consequent of the first ratio does not change the mutual relation of its terms, we have

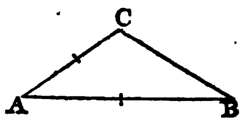
$AC + CB : AC - CB :: \tan. \frac{1}{2}(B + A) : \tan. \frac{1}{2}(B - A)$. Q. E. D.

EXAMPLE I.

In the plane triangle ABC,

Given { AB 345 yards; AC 174·07 yards; ∠A 37° 20′ }

Required the other parts.



1. *Geometrically.*

Draw AB = 345 from a scale of equal parts. Make the angle A = 37° 20′. Set off AC = 174 by the scale of equal parts. Join BC, and it is done.

Then the other parts being measured, they are found to be nearly as follow; viz. the side BC 232 yards, the angle B 27°, and the angle C 115°½.

2. *Arithmetically.*

The side AB	345	From	180° 00′
the side AC	174·07	take ∠A	37 20
their sum	519·07	sum of C and B	142 40
their differ.	170·93	half sum of do.	71 20

As sum of sides AB, AC, - -	519·07 log.	2·7152259
To diff. of sides AB, AC, - -	170·93 -	2·2328183
So tang. half sum ∠s C and B	71° 20′ -	10·4712979
To tang. half diff. ∠s C and B	44 16 -	9·9888903
these added give ∠C	115 36	
and subtr. give ∠B	27 4	

Then, by the former theorem,

As sin. ∠ C 115° 36′ or 64° 24′ -	log. 9·9551259
To its op. side AB 345 - - -	2·5378191
So sin. of ∠ A 37° 20′ - - -	9·7827458
To its op. side BC 232 - - -	2·3654890

EXAMPLE II.

In the plane triangle ABC,

Given { AB 365 poles, AC 154·33, ∠A 57° 12′ } Ans. { BC 309·86, ∠B 24° 45′, ∠C 98 3 }

Required the other parts.

EXAMPLE III.

In the plane triangle ABC,

Given { AC 120 yards, BC 112 yards, ∠C 57° 58′ 30″ } Ans. { AB 112·65, ∠A 57° 27′ 0″, ∠B 65 34 21 }

Required the other parts.

THEOREM III.

When the Three Sides of a Triangle are given.

First, let fall a perpendicular from the greatest angle on the opposite side, or base, dividing it into two segments, and the whole triangle into two right-angled triangles : then the proportion will be,

As the base, or sum of the segments,
Is to the sum of the other two sides ;
So is the difference of those sides,
To the diff. of the segments of the base.

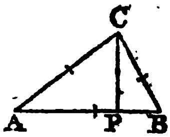
Then take half this difference of the segments, and add it to the half sum, or the half base, for the greater segment, and subtract the same for the less segment.

Hence, in each of the two right-angled triangles, there will be known two sides, and the right angle opposite to one of them ; consequently the other angles will be found by the first theorem.

Demonstr. By theor. 35, Geom. the rectangle of the sum and difference of the two sides, is equal to the rectangle of the sum and difference of the two segments. Therefore, by forming the sides of these rectangles into a proportion by theor. 76, Geometry, it will appear that the sums and differences are proportional as in this theorem.

N. B. Before you commence a solution of an example to this case, ascertain whether the triangle be right-angled or not, by determining whether the square of the longest side be equal or unequal to the sums of the squares of the other two. If equal, the example may be referred to the notes to theorem IV.

EXAMPLE I.



In the plane triangle ABC,

Given the sides $\begin{cases} \text{AB } 345 \text{ yards} \\ \text{AC } 232 \\ \text{BC } 174{\cdot}07 \end{cases}$

To find the angles.

1. *Geometrically.*

Draw the base AB = 345 by a scale of equal parts. With radius 232, and centre A, describe an arc; and with radius, 174, and centre B, describe another arc, cutting the former in C. Join AC, BC, and it is done.

Then, by measuring the angles, they will be found to be nearly as follows, viz.

$\angle$ A 27°, $\angle$ B 37°½, and $\angle$ C 115°½.

2. *Arithmetically.*

Having let fall the perpendicular CP, it will be,
As the base AB : AC + BC : : AC — BC : AP — BP,
that is, as 345 : 406·07 : : 57·93 : 68·18 = AP — BP,

its half is	34·09
the half base is	172·50
the sum of these is	206·59 = AP.
and their diff. is	138·41 = BP.

Then, in the triangle APC, right-angled at P,

As the side	AC	232	log.	2·3654880
To sin. op.	$\angle$ P	90°		10·0000000
So is the side	AP	206·59		2·3151093
To sin. op. $\angle$ ACP		62° 56′		9·9496213
which taken from		90 00		
leaves the $\angle$ A		27 04		

Again, in the triangle BPC, right-angled at P,

As the side	BC	174·07	log.	2·2407239
To sin. op.	∠P	90°		10·0000000
So is side	BP	138·41		2·1411675
To sin op.	∠BCP	52° 40′		9·9004436
which taken from		90 00		
leaves the	∠B	37 20		

Also the	∠ACP	62° 56′
added to	∠BCP	52 40
gives the whole	∠ACB	115 36

So that all the three angles are as follow, viz.
the ∠A 27° 4′; the ∠B 37° 20′; the ∠C 115° 36′.

The angles A and B may also easily be found by the expressions sec. $A = \frac{AC}{AP}$, sec. $B = \frac{BC}{BP}$, or the equivalent logs.

EXAMPLE II.

In the plane triangle ABC,

Given the sides { AB 365 poles, AC 154·33, BC 309·86 } Ans. { ∠A 57° 12′, ∠B 24 45, ∠C 98 3 }

To find the angles

EXAMPLE III.

Given the sides { AB 120, AC 112·65, BC 112 } Ans. { ∠A 57° 27′ 0″, ∠B 57 58 39, ∠C 64 34 21 }

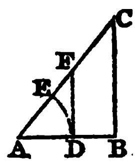
To find the angles.

The three foregoing theorems include all the cases of plane triangles, both right-angled and oblique. But there are other theorems suited to some particular forms of triangles (see vol. ii.), which are sometimes more expeditious in their use than the general ones; one of which, as the case for which it serves so frequently occurs, may be here explained.

THEOREM IV.

When a Triangle is Right-angled; any of the unknown parts may be found by the following proportions: viz.

As radius
Is to either leg of the triangle;
So is tang. of its adjacent angle,
To its opposite leg;
And so is secant of the same angle,
To the hypothenuse.



Demonstr. AB being the given leg, in the right-angled triangle ABC: with the centre A, and any assumed radius AD, describe an arc DE, and draw DF perpendicular to AB, or parallel to BC. Then it is evident, from the definitions, that DF is the tangent, and AF the secant of the arc DE, or of the angle A which is measured by that arc, to the radius AD. Then, because of the parallels BC, DF, it will be . . . . . . as AD : AB :: DF : BC and :: AF : AC, which is the same as the theorem is in words.

Note. The radius is equal, either to the sine of 90°, or the tangent of 45°; and is expressed by 1, in a table of natural sines, or by 10 in the log. sines.

EXAMPLE I.

In the right-angled triangle ABC,

Given $\left\{ \begin{array}{l} \text{the leg AB } 162 \\ \angle \text{A } 53^\circ\, 7'\, 48'' \end{array} \right\}$ To find AC and BC.

1. *Geometrically.*

Make AB = 162 equal parts, and the angle A = 53° 7′ 48″; then raise the perpendicular BC, meeting AC in C. So shall AC measure 270, and BC 216.

2. *Arithmetically.*

As radius	-		-	log. 10·0000000
To leg AB	-	162	-	2·2095150
So tang. ∠A	-	53° 7′ 48″	-	10·1249371
To leg BC	-	216	-	2·3344521

So secant ∠A	-	53° 7′ 48″	- 10·2218477
To hyp. AC	-	270 -	- 2·4313627

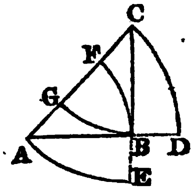
EXAMPLE II.

In the right-angled triangle ABC,

Given { the leg AB 180, the ∠A 62 40′ } Ans. { AC 392·0146, BC 348·2464 }

To find the other two sides.

Note. There is sometimes given another method for right-angled triangles, which is this:



ABC being such a triangle, make one leg AB radius; that is, with centre A, and distance AB, describe an arc BF. Then it is evident that the other leg BC represents the tangent, and the hypothenuse AC the secant, of the arc BF, or of the angle A.

In like manner, if the leg BC be made radius; then the other leg AB will represent the tangent, and the hypothenuse AC the secant, of the arc BG or angle C.

But if the hypothenuse be made radius; then each leg will represent the sine of its opposite angle; namely, the leg AB the sine of the arc AE or angle C, and the leg BC the sine of the arc CD or angle A.

Then the general rule for all these cases is this, namely, that the sides of the triangle bear to each other the same proportion as the parts which they represent.

And this is called, Making every side radius.

Note 2. When there are given two sides of a right-angled triangle, to find the third side; this is to be found by the property of the squares of the sides, in theorem 34, Geom. viz. that the square of the hypothenuse, or longest side, is equal to both the squares of the two other sides together. Therefore, to find the longest side, add the squares of the two shorter sides together, and extract the square root of that sum; but to find one of the shorter sides, subtract the one square from the other, and extract the root of the remainder. Or, when the hypothenuse, H, and either the base, B, or the perpendicular, P, are given: then half the sum of log. (H + P) and log. (H − P) = log. B; and half the sum of log. (H + B) and log. (H − B) = log. P.

When B and P are given, the following logarithmic operation may sometimes be advantageously employed; viz. Find N the number answering to the log. diff., 2 log. P — log. B; and make $B + N = M$: then, $\frac{1}{2}$ (log. M + log. B) = log. H, the hypothenuse.

The truth of this rule is evident: for, from the nature of logarithms. $\frac{P^2}{B} = N$; whence $B + N = B + \frac{P^2}{B} = \frac{B^2+P^2}{B} = M$; and $\frac{1}{2}$ (log. M + log. B) = $\frac{1}{2}$ log. MB = $\frac{1}{2}$ log. $(B^2 + P^2)$ = log. $\sqrt{(B^2 + P^2)}$ = log. H.

Or, still more simply, find 10 + the diff. (log. P — log. B) in the log. tangents. The corresponding log. secant added to log. B = log. H.

Note, also, as many right-angled triangles in integer numbers as we please may be found by making

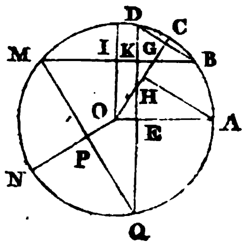
$$m^2 + n^2 = \text{hypothenuse}$$
$$m^2 - n^2 = \text{perpendicular}$$
$$2mn = \text{base}$$

m and n being taken at pleasure, m greater than n.

Before we proceed to the subject of Heights and Distances we shall give,

A CONCISE INVESTIGATION OF SOME OF THE MOST USEFUL TRIGONOMETRICAL FORMULÆ.

Let AB, AC, AD, be three arches, such that BC = CD, and O the centre. Join AO, OC, BD. Draw DEQ and OI perpendicular, and BIM ∥ to OA. Join MQ and bisect it by the radius ON; and draw AH ∥ to BD.



Then is AH = sin. AC
OH = cos. AC;
also DE = EQ = sin. AD
EK = OI = sin. AB
QK = sin. AD + sin. AB
DK = sin. AD — sin. AB
BI = IM = cos. AB
OE = KI = cos. AD
MK = cos. AB + cos. AD
BK = cos. AB — cos. AD

Because the angles at K are right angles:

arc BD + arc MQ = 180°, and arc DC + arc MN = 90°
∴ MP = PQ = OG = cos. DC = cos. BC;

also, because AC $= \frac{1}{2}$(AB + AD) $= \frac{1}{2}$BAQ = angle AOC (at centre) = BDQ (at circumf.) = BMQ (on same arc)
∴ triangles AOH, BDK, QMK, are equiangular.

Hence—

I. OA : AH : : MQ : QK ;
that is, rad. : sin. AC : : 2 cos. BC : sin. AD + sin. AB
II. AO : OH : : BD : DK ;
or, rad. : cos. AC : : 2 sin. BC : sin. AD − sin. AB
III. AO : OH : : QM : MK ;
or, rad. : cos. AC : : 2 cos. BC : cos. AB + cos. AD
IV. AO : AH : : DB : DK ;
or, rad. : sin. AC : : 2 sin. BC : cos. AB − cos AD ;

also,
V. BK . KM = DK . KQ, that is (cos. AB − cos. AD) (cos. AB + cos. AD) = (sin. AD − sin. AB) (sin. AD + sin. AB).

By reducing the above four proportions into equations, making rad. = 1, we obtain two distinct classes of formulæ, thus :—

First Class. AC $= a$, CB $= b$; then AD $= a + b$, AB $= a - b$,

1. $\sin. (a + b) + \sin. (a - b) = 2 \sin. a \cos. b$
2. $\sin. (a + b) - \sin. (a - b) = 2 \cos. a \sin. b$
3. $\cos. (a - b) + \cos. (a + b) = 2 \cos. a \cos. b$
4. $\cos. (a - b) - \cos. (a + b) = 2 \sin. a \sin. b$

Second Class. AD $= a$, AB $= b$; then AC $= \frac{1}{2}(a + b)$, BC $= \frac{1}{2}(a - b)$.

5. $\sin. a + \sin. b = 2 \sin. \frac{1}{2}(a + b) \cos. \frac{1}{2}(a - b)$
6. $\sin. a - \sin. b = 2 \cos. \frac{1}{2}(a + b) \sin. \frac{1}{2}(a - b)$
7. $\cos. b + \cos. a = 2 \cos. \frac{1}{2}(a + b) \cos. \frac{1}{2}(a - b)$
8. $\cos. b - \cos. a = 2 \sin. \frac{1}{2}(a + b) \sin. \frac{1}{2}(a - b)$

The first class is useful in transforming the products of sines into simple sines, and the contrary.

The second facilitates the substitution of sums or differences of sines for the products, and the contrary.

Taking the sum and the difference of equations 1 and 2, also of 3 and 4, remembering that sin. = cos. tan. we obtain the following :

Third Class.

9. $\sin. (a + b) = \sin. a \cos. b + \sin. b \cos. a$
$= \cos. a \cos. b (\tan. a + \tan. b)$

10. $\sin. (a - b) = \sin. a \cos. b - \sin. b \cos. a$
$= \cos. a \cos. b (\tan. a - \tan. b)$

11. $\cos. (a + b) = \cos. a \cos. b - \sin. a \sin. b$
$= \cos. a \cos. b (1 - \tan. a \tan. b)$

12. $\cos. (a - b) = \cos. a \cos. b + \sin. a \sin. b.$
$= \cos. a \cos. b (1 + \tan. a \tan. b).$

From these, making $a = b$, we readily obtain the expressions for sines and cosines of double arcs; also dividing equation 9 by 11, and equation 10 by 12, we obtain expressions for the tangents of $a + b$ and $a - b$. Thus we have :—

Fourth Class.

13. $\sin. 2a = 2 \sin. a \cos. a = 2 \cos.^2 a \tan. a$

14. $\cos. 2a = \cos.^2 a - \sin.^2 a = \cos.^2 a (1 - \tan.^2 a)$

15. $\frac{\sin.}{\cos.} (a + b) = \tan. (a + b) = \frac{\tan. a + \tan. b}{1 - \tan. a \tan. b}$

16. $\frac{\sin.}{\cos.} (a - b) = \tan. (a - b) = \frac{\tan. a - \tan. b}{1 + \tan. a \tan. b}$

17. $\tan. 2a = \frac{2 \tan. a}{1 - \tan.^2 a}$

18. $\cot. 2a = \frac{1 - \tan.^2 a}{2 \tan. a}.$

Substituting in the second class,
for $\sin. \frac{1}{2}(a+b)$, $\cos. \frac{1}{2}(a+b) \tan. \frac{1}{2}(a+b)$,
and for $\sin. \frac{1}{2}(a-b)$, $\cos. \frac{1}{2}(a-b) \tan. \frac{1}{2}(a-b)$, we have :—

Fifth Class.

19. $\cos. b + \cos. a = 2 \cos. \frac{1}{2}(a+b) \cos. \frac{1}{2}(a-b)$.—See equa. 7.

20. $\cos. b - \cos. a = \tan. \frac{1}{2}(a+b) \tan. \frac{1}{2}(a-b)\, 2 \cos. \frac{1}{2}(a+b)$
$\cos. \frac{1}{2}(a-b) = \tan. \frac{1}{2}(a+b) \tan. \frac{1}{2}(a-b) (\cos. b + \cos. a)$

21. $\sin. a + \sin. b = \tan. \frac{1}{2}(a+b)\, 2 \cos. \frac{1}{2}(a+b) \cos. \frac{1}{2}(a-b)$
$= \tan. \frac{1}{2}(a+b) (\cos. a + \cos. b)$

22. $\sin. a - \sin. b = \tan. \frac{1}{2}(a-b)\, 2 \cos. \frac{1}{2}(a+b) \cos. \frac{1}{2}(a-b)$
$= \tan. \frac{1}{2}(a-b) (\cos. a + \cos. b)$

23. $\frac{\sin. a + \sin. b}{\sin. a - \sin. b} = \frac{\tan. \frac{1}{2}(a+b)}{\tan. \frac{1}{2}(a-b)}$: from 21 and 22.

24. $\frac{\sin. a + \sin. b}{\cos. a + \cos. b} = \tan. \frac{1}{2}(a + b)$: from 21.

25. $\frac{\sin. a - \sin. b}{\cos. a + \cos. b} = \tan. \frac{1}{2}(a - b)$: from 22.

Examples for Exercise.

1. Demonstrate that in any right-angled plane triangle the following properties obtain: viz.

(1.) $\frac{\text{perp.}}{\text{base}}$=tan. ang. at base. (2.) $\frac{\text{base}}{\text{perp.}}$ =tan. ang. at vertex.

(3.) $\frac{\text{perp.}}{\text{hyp.}}$=sin. ang. at base. (4.) $\frac{\text{base}}{\text{hyp.}}$=sin. ang. at vertex.

(5.) $\frac{\text{hyp.}}{\text{base}}$=sec. ang. at base. (6.) $\frac{\text{hyp.}}{\text{perp.}}$=sec. ang. at vertex.

2. Demonstrate that tan. A + sec. A = tan. $(45^\circ + \frac{1}{2}A)$.

3. Demonstrate that sec. $2A = \frac{1+\tan.^2 A}{1-\tan.^2 A}$, and that cosec. $2A = \frac{1+\tan.^2 A}{2\tan. A} = \frac{\sec.^2 A}{2\tan. A}$.

4. Given $Axy = By^2 + Dx^2$; to find x and y the sine and cosine of an arc.

5. Demonstrate that of any arc, $\tan.^2 - \sin.^2 = \tan.^2 \sin.^2$.

6. Demonstrate that if the tan. of an arc be $= \sqrt{n}$, the sine of the same arc is $= \sqrt{\frac{n}{n+1}}$.

OF HEIGHTS AND DISTANCES, &c.

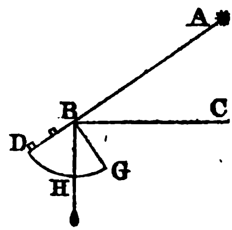
By the mensuration and protraction of lines and angles, are determined the lengths, heights, depths, and distances of bodies or objects.

Accessible lines are measured by applying to them some certain measure a number of times, as an inch, or a foot, or yard. But inaccessible lines must be measured by taking angles, or by such-like method, drawn from the principles of geometry.

When instruments are used for taking the magnitude of the angles in degrees, the lines are then calculated by trigonometry: in the other methods, the lines are calculated from the principle of similar triangles, or some other geometrical property, without regard to the measure of the angles.

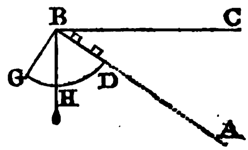
Angles of elevation, or of depression, are usually taken either with a theodolite, or with a quadrant, divided into degrees and minutes, and furnished with a plummet suspended from the centre, and two open sights fixed on one of the radii, or else with telescopic sights.

To take an Angle of Altitude and Depression with the Quadrant.



Let A be any object, as the sun, moon, or a star, or the top of a tower, or hill, or other eminence: and let it be required to find the measure of the angle ABC, which a line drawn from the object makes above the horizontal line BC.

Place the centre of the quadrant in the angular point, and move it round there as a centre, till with one eye at D, the other being shut, you perceive the object A through the sights; then will the arc GH of the quadrant, cut off by the plumb-line, BH, be the measure of the angle ABC as required.



The angle ABC of depression of any object A, below the horizontal line BC, is taken in the same manner; except that here the eye is applied to the centre, and the measure of the angle is the arc GH, on the other side of the plumb-line.

The following examples are to be constructed and calculated by the rules of Trigonometry.

EXAMPLE I.

Having measured a distance of 200 feet, in a direct horizontal line, from the bottom of a steeple, the angle of elevation of its top, taken at that distance, was found to be 47° 30′; hence it is required to find the height of the steeple.

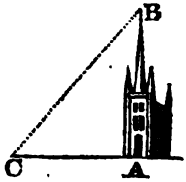
Construction.

Draw an indefinite line; on which set off AC = 200 equal parts, for the measured distance. Erect the indefinite perpendicular AB; and draw CB so as to make the angle C =

47° 30′, the angle of elevation; and it is done. Then AB, measured on the scale of equal parts, is nearly 218¼.

Calculation.

As radius - - -	10·0000000
To AC 200 - -	2·3010300
So tang. ∠C 47° 30′	10·0379475
To AB 218·26 required	2·3389775



Or, by the nat. tangents, we have AC × tan. BCA = 200 × 1·091308 = 218·2616 = AB.

EXAMPLE II.

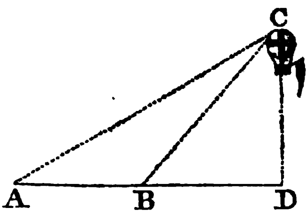
What was the perpendicular height of a cloud, or of a balloon, when its angles of elevation were 35° and 64°, as taken by two observers, at the same time, both on the same side of it, and in the same vertical plane; the distance between them being half a mile or 880 yards? And what was its distance from the said two observers?

Construction.

Draw an indefinite ground line, on which set off the given distance AB = 880; then A and B are the places of the observers. Make the angle A = 35°, and the angle B = 64°; then the intersection of the lines at C will be the place of the balloon: whence the perpendicular CD, being let fall, will be its perpendicular height. Then, by measurement are found the distances and height nearly as follow, viz. AC 1631, BC 1041, DC 936.

Calculation.

First, from	∠ B	64°
take	∠ A	35
leaves	∠ ACB	29



Then in the triangle ABC,

As sin. ∠ACB	29° - - -	9·6855712
To op. side AB	880 - - -	2·9444827
So sin. ∠A	35° - - -	9·7585913
To op. side BC	1041·125 - -	3·0175028

As sin. ∠ACB	29°	9·6855712
To op. side AB	880	2·9444827
So sin. ∠B 116° or 64°		9·9536602
To op. side AC	1631·442	3·2125717

And in the triangle BCD,

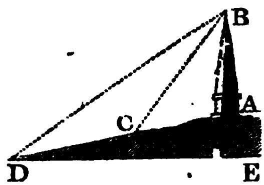
As sin. ∠D	90°	10·0000000
To op. side BC	1041·125	3·0175028
So sin. ∠B	64°	9·9536602
To op. side CD	935·757	2·9711630

EXAMPLE III.

Having to find the height of an obelisk standing on the top of a declivity, I first measured from its bottom a distance of 40 feet, and there found the angle, formed by the oblique plane and a line imagined to go to the top of the obelisk, 41°; but after measuring on in the same direction 60 feet farther, the like angle was only 23° 45′. What then was the height of the obelisk?

Construction.

Draw an indefinite line for the sloping plane or declivity, in which assume any point A for the bottom of the obelisk, from which set off the distance AC = 40, and again CD = 60 equal parts. Then make the angle C = 41°, and the angle D = 23° 45′; and the point B where the two lines meet will be the top of the obelisk. Therefore AB joined, will be its height.—Draw also the horizontal line DE perp. to AB.



Calculation.

From the ∠C	41° 00′
take the ∠D	23 45
leaves the ∠DBC	17 15

Then in the triangle DBC,

As sin. ∠DBC	17° 15′	9·4720856
To op. side DC	60	1·7781513
So sin. ∠D	23 45	9·6050320
To op. side CB	81·488	1·9110977

And in the triangle ABC,

As sum of sides CB, CA	121·488	-	2·0845333
To diff. of sides CB, CA	41·488	-	1·6179225
So tang. $\frac{1}{2}$(A + B) -	69° 30′	-	10·4272623
To tang. $\frac{1}{2}$(A − B) -	42 24$\frac{1}{2}$	-	9·9606516

the diff. of these is ∠CBA 27 5$\frac{1}{2}$
the sum is ∠CAB 111 54$\frac{1}{2}$

Lastly, as sin. ∠CBA	27° 5′$\frac{1}{2}$	- -	9·6582842
To op. side CA	40	- -	1·6020600
So sin. ∠C -	41° 0′	- -	9·8169429
To op. side AB	57·623	- -	1·7607187

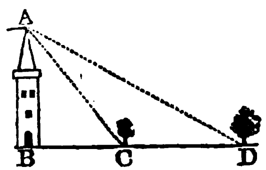
Also the ∠ADE = BAC — 90° = 21° 54′$\frac{1}{2}$.

EXAMPLE IV.

Wanting to know the distance between two inaccessible trees, or other objects, from the top of a tower 120 feet high, which lay in the same right line with the two objects, I took the angles formed by the perpendicular wall and lines conceived to be drawn from the top of the tower to the bottom of each tree, and found them to be 33° and 64°$\frac{1}{2}$. What is the distance between the two objects?

Construction.

Draw the indefinite ground line BD, and perpendicular to it BA = 120 equal parts. Then draw the two lines AC, AD, making the two angles BAC, BAD, equal to the given measures 33° and 64°$\frac{1}{2}$. So shall C and D be the places of the two objects.



Calculation.

First, in the right-angled triangle ABC,

As radius		- - - - -	10·0000000
To AB -	120	- - -	2·0791812
So tang. ∠BAC	33°	- - -	9·8125174
To BC -	77·920	- - -	1·8916986

Then in the right-angled triangle ABD,

As radius	- - -	- -	10·0000000
To AB	- -	120 - -	2·0791812
So tang. ∠BAD -		64° 30′ - -	10·3215039
To BD	-	251·585 - -	2·4006851

From which take BC 77·929
leaves the dist. CD 173·656, as required.

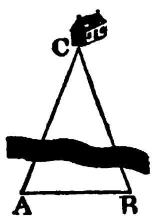
Or thus, by the natural tangents,

From nat. tan. DAB	- -	64° 30′	= 2·0965436
Take nat. tan. CAB	- -	33 0	= 0 6494076
Difference	- - -		1·4471360
If drawn into AB -	- -		120
The result gives CD	- -		= 173·65632

EXAMPLE V.

Being on the side of a river, and wanting to know the distance to a house which was seen on the other side, I measured 200 yards in a straight line by the side of the river; and then, at each end of this line of distance, took the horizontal angle formed between the house and the other end of the line; which angles were, the one of them 68° 2′, and the other 73° 15′. What were the distances from each end to the house?

Construction.



Draw the line AB = 200 equal parts. Then draw AC so as to make the angle A = 68° 2′, and BC to make the angle B = 73° 15′. So shall the point C be the place of the house required.

The calculation, which is left for the student's exercise, gives AC = 306·19, BC = 296·54.

EXAM. VI. From the edge of a ditch, of 36 feet wide, surrounding a fort, having taken the angle of elevation of the top of the wall, it was found to be 62° 40′: required the height of the wall, and the length of a ladder to reach from my station to the top of it? Ans. { height of wall 69·64, ladder, 78·4 feet.

Exam. vii. Required the length of a shoar, which being to strut 11 feet from the upright of a building, will support a jamb 23 feet 10 inches from the ground?

Ans. 26 feet 3 inches.

Exam. viii. A ladder, 40 feet long, can be so placed, that it shall reach a window 33 feet from the ground, on one side of the street; and by turning it over, without moving the foot out of its place, it will do the same by a window 21 feet high, on the other side: required the breadth of the street?

Ans. 56·649 feet.

Exam. ix. A maypole, whose top was broken off by a blast of wind, struck the ground at 15 feet distance from the foot of the pole: what was the height of the whole maypole, supposing the broken piece to measure 39 feet in length?

Ans. 75 feet.

Exam. x. At 170 feet distance from the bottom of a tower, the angle of its elevation was found to be 52° 30′: required the altitude of the tower? Ans. 221·55 feet.

Exam. xi. From the top of a tower, by the sea-side, of 143 feet high, it was observed that the angle of depression of a ship's bottom, then at anchor, measured 35°; what was the ship's distance from the bottom of the wall?

Ans. 204·22 feet.

Exam. xii. What is the perpendicular height of a hill; its angle of elevation, taken at the bottom of it, being 46°, and 200 yards farther off, on a level with the bottom, the angle was 31°? Ans. 286·28 yards.

Exam. xiii. Wanting to know the height of an inaccessible tower; at the least distance from it, on the same horizontal plane, I took its angle of elevation equal to 58°; then going 300 feet directly from it, found the angle there to be only 32°: required its height, and my distance from it at the first station?

Ans. { height 307·53, distance 192·15 }

Exam. xiv. Being on a horizontal plane, and wanting to know the height of a tower placed on the top of an inaccessible hill; I took the angle of elevation of the top of the hill 40°, and of the top of the tower 51°; the measuring in a line directly from it to the distance of 200 feet farther, I found the angle to the top of the tower to be 33° 45′. What is the height of the tower? Ans. 93·33148 feet.

Exam. xv. From a window near the bottom of a house, which seemed to be on a level with the bottom of a steeple, I took the angle of elevation of the top of the steeple equal 40°; then from another window, 18 feet directly above the former, the like angle was 37° 30′: required the height and distance of the steeple.

Ans. { height 210·44; distance 250·79 }

EXAM. XVI. Wanting to know the height of, and my distance from, an object on the other side of a river, which appeared to be on a level with the place where I stood, close by the side of the river; and not having room to measure backward, in the same line, because of the immediate rise of the bank, I placed a mark where I stood, and measured in a direction from the object, up the ascending ground, to the distance of 264 feet, where it was evident that I was above the level of the top of the object; there the angles of depression were found to be, viz. of the mark left at the river's side 42°, of the bottom of the object 27°, and of its top 19°. Required the height of the object, and the distance of the mark from its bottom?

Ans. { height 57·26; distance 150·50 }

EXAM. XVII. If the height of the mountain called the Peak of Teneriffe be 2½ miles, as it is very nearly, and the angle taken at the top of it, as formed between a plumb-line and a line conceived to touch the earth in the horizon, or farthest visible point, be 88° 2′; it is required from these measures to determine the magnitude of the whole earth, and the utmost distance that can be seen on its surface from the top of the mountain, supposing the form of the earth to be perfectly globular?

Ans. { dist. 135·943; diam. 7918 } miles.

EXAM. XVIII. Two ships of war, intending to cannonade a fort, are, by the shallowness of the water, kept so far from it, that they suspect their guns cannot reach it with effect. In order therefore to measure the distance, they separate from each other a quarter of a mile, or 440 yards; then each ship observes and measuses the angle which the other ship and the fort subtends, which angles are 83° 45′ and 85° 15′. What is the distance between each ship and the fort?

Ans. { 2292·26; 2298·05 } yards.

EXAM. XIX. Wanting to know the breadth of a river, I measured a base of 500 yards in a straight line close by one side of it; and at each end of this line I found the angles subtended by the other end and a tree, close to the bank on the other side of the river, to be 53° and 79° 12'. What was the perpendicular breadth of the river?

Ans. 529·48 yards.

Exam. xx. Wanting to know the extent of a piece of water, or distance between two headlands; I measured from each of them to a certain point inland, and found the two distances to be 735 yards and 840 yards; also the horizontal angle subtended between these two lines was 55° 40'. What was the distance required? Ans. 741·2 yards.

Exam. xxi. A point of land was observed, by a ship at sea, to bear east-by-south; and after sailing north-east 12 miles, it was found to bear south-east-by-east. It is required to determine the place of that headland, and the ship's distance from it at the last observation? Ans. 26·0728 miles.

Exam. xxii. Wanting to know the distance between a house and a mill, which were seen at a distance on the other side of a river, I measured a base line along the side where I was, of 600 yards, and at each end of it took the angles subtended by the other end and the house and mill, which were as follow, viz. at one end the angles were 58° 20' and 95° 20', and at the other end the like angles were 53° 30' and 98° 45'. What then was the distance between the house and mill? Ans. 959·5866 yards.

Exam. xxiii. Wanting to know my distance from an inaccessible object o, on the other side of a river; and having no instrument for taking angles, but only a chain or cord for measuring distances; from each of two stations, A and B, which were taken at 500 yards asunder, I measured in a direct like from the object o 100 yards, viz. AC and BD each equal to 100 yards; also the diagonal AD measured 550 yards, and the diagonal BC 560. What was the distance of the object o from each station A and B?

Ans. { AO 536·81, BO 500·47 }

Exam. xxiv. In a garrison besieged are three remarkable objects, A, B, C, the distances of which from each other are discovered by means of a map of the place, and are as follow, viz. AB 266¼, AC 530, BC 327¼ yards. Now, having to erect a battery against it, at a certain spot without the place, and being desirous to know whether my distances from the three objects be such, as that they may from thence be battered with effect, I took, with an instrument, the horizontal angles subtended by these objects from the station S, and found them to be as follow, viz. the angle ASB 13° 30', and the angle BSC 29° 50'. Required the three distances, SA, SB, SC; the object B being situated nearest me, and between the two others A and C.

Ans. { SA 757·14
SB 537·10
SC 655·30 }

EXAM. XXV. Required the same as in the last example, when the object B is the farthest from my station, but still seen between the two others as to angular position, and those angles being thus, the angle ASB 33° 45′, and BSC 22° 30′, also the three distances, AB 600, AC 800, BC 400 yards?

Ans. { SA 710·3
SB 1041·85
SC 934·14 }

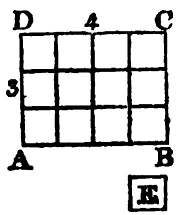
EXAM. XXVI. If DB in the figure at pa. 378 represent a portion of the earth's surface, and D the point where the levelling instrument is placed, then LB will be the difference between the true and the apparent level; and you are required to demonstrate that, for distances not exceeding 5 or 6 miles measured on the earth's surface, BL, estimated in feet, is equal to $\frac{2}{3}$ of the square of BD, taken in miles.

MENSURATION OF PLANES.

THE Area of any plane figure, is the measure of the space contained within its extremes or bounds; without any regard to thickness.

This area, or the content of the plane figure, is estimated by the number of little squares that may be contained in it; the side of those little measuring squares being an inch, or a foot, or a yard, or any other fixed quantity. And hence the area or content is said to be so many square inches, or square feet, or square yards, &c.

Thus, if the figure to be measured be the rectangle ABCD, and the little square E, whose side is one inch, be the measuring unit proposed: then as often as the said little square is contained in the rectangle, so many square inches the rectangle is said to contain, which in the present case is 12.



PROBLEM I.

To find the Area of any Parallelogram; whether it be a Square, a Rectangle, a Rhombus, or a Rhomboid.

MULTIPLY the length by the perpendicular breadth, or height, and the product will be the area*.

EXAMPLES.

Ex. 1. To find the area of a parallelogram, the length being 12·25, and breadth or height 8·5.

```
12·25 length
  8·5 breadth
-------
 6125
9800
-------
104·125 area.
```

Ex. 2. To find the area of a square, whose side is 35·25 chains. Ans. 124 acres, 1 rood, 1 perch.

Ex. 3. To find the area of a rectangular board, whose length is 12½ feet, and breadth 9 inches. Ans. 9⅜ feet.

Ex 4. To find the content of a piece of land, in form of a rhombus, its length being 6·20 chains, and perpendicular breadth 5·45. Ans. 3 acres, 1 rood, 20 perches.

Ex. 5. To find the number of square yards of painting in a rhomboid, whose length is 37 feet, and height 5 feet 3 inches. Ans. 21$\frac{7}{12}$ square yards.

---

* The truth of this rule is proved in the Geom. theor. 81, cor. 2.

The same is otherwise proved thus: Let the foregoing rectangle be the figure proposed; and let the length and breadth be divided into several parts, each equal to the linear measuring unit, being here 4 for the length, and 3 for the breadth; and let the opposite points of division be connected by right lines.—Then it is evident that these lines divide the rectangle into a number of little squares, each equal to the square measuring unit E; and further, that the number of these little squares, or the area of the figure, is equal to the number of linear measuring units in the length, repeated as often as there are linear measuring units in the breadth, or height; that is, equal to the length drawn into the height; which here is 4 × 3 or 12.

And it is proved (Geom theor. 25, cor. 2), that any oblique parallelogram is equal to a rectangle, of equal length and perpendicular breadth. Therefore the rule is general for all parallelograms whatever.

PROBLEM II.

To find the Area of a Triangle.

RULE I. MULTIPLY the base by the perpendicular height, and take half the product for the area*. Or, multiply the one of these dimensions by half the other.

EXAMPLES.

Ex. 1. To find the area of a triangle, whose base is 625, and perpendicular height 520 links?

Here $625 \times 260 = 162500$ square links,

or equal 1 acre, 2 roods, 20 perches, the answer.

Ex. 2. How many square yards contains the triangle, whose base is 40, and perpendicular 30 feet?

Ans. $66\frac{2}{3}$ square yards.

Ex. 3. To find the number of square yards in a triangle, whose base is 49 feet, and height $25\frac{1}{4}$ feet.

Ans. $68\frac{53}{72}$, or 68·7361.

Ex. 4. To find the area of a triangle, whose base is 18 feet 4 inches, and height 11 feet 10 inches?

Ans. 108 feet, $5\frac{2}{3}$ inches.

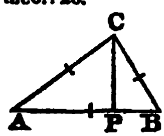
RULE II. When two sides and their contained angle are given: Multiply the two given sides together, and take half their product: Then say, as radius is to the sine of the given angle, so is that half product, to the area of the triangle.

Or, multiply that half product by the natural sine of the said angle, for the area †.

---

* The truth of this rule is evident, because any triangle is the half of a parallelogram of equal base and altitude, by Geom. theor. 26.

† For, let AB, AC, be the two given sides, including the given angle A. Now $\frac{1}{2}$ AB $\times$ CP is the area, by the first rule, CP being the perpendicular. But by trigonometry, as sin. $\angle$ P, or radius : AC : : sin. $\angle$ A : CP, which is therefore = AC $\times$ sin. $\angle$ A, taking radius = 1. Therefore the area $\frac{1}{2}$AB $\times$ CP is = $\frac{1}{2}$AB $\times$ AC $\times$ sin. $\angle$ A, to radius 1; or, as radius : sin. $\angle$ A : : $\frac{1}{2}$AB $\times$ AC : the area.



Ex. 1. What is the area of a triangle, whose two sides are 30 and 40, and their contained angle 28° 57′ ?

By Natural Numbers.	*By Logarithms.*
First, $\frac{1}{2} \times 40 \times 30 = 600$,	
then, 1 : 600 : : ·484046 sin. 28° 57′	log. 9·684887
600	2·778151
Answer 290·4276 the area answ. to	2·463038

Ex. 2. How many square yards contains the triangle of which one angle is 45°, and its containing sides 25 and 21$\frac{1}{4}$ feet ? Ans. 20·86947.

Rule III. When the three sides are given : Add all the three sides together, and take half that sum. Next, subtract each side severally from the said half sum, obtaining three remainders. Then multiply the said half sum and those three remainders all together, and extract the square root of the last product, for the area of the triangle †.

---

† For, let b denote the base AB of the triangle ABC (see the last fig.), also a the side AC, and c the side BC. Then, by theor. 3,

Trigon. as $b : a + c :: a - c : \frac{aa - cc}{b} =$ AP — PB the diff. of the segments ;

theref. $\frac{1}{2}b + \frac{aa - cc}{2b} = \frac{bb + aa - cc}{2b} =$ the segment AP ;

hence $\sqrt{(AC^2 - AP^2)} =$ the perp. CP, that is,

$\sqrt{(aa - (\frac{bb + aa - cc}{2b})^2)} =$ . . . .

$\sqrt{\frac{2a^2b^2 - a^4 + 2b^2c^2 - b^4 + 2a^2c^2 - c^4}{4bb}} =$ CP.

But $\frac{1}{2}$AB × CP is the area, that is,

$\frac{1}{2}b \times CP = \sqrt{\frac{2a^2b^2 - a^4 + 2b^2c^2 - b^4 + 2a^2c^2 - c^4}{16}}$

$= \sqrt{(\frac{aa - bb - cc + 2bc}{4} \times \frac{-aa + bb + cc + 2bc}{4})}$ (A)

$= \sqrt{(\frac{a + b + c}{2} \times \frac{-a + b + c}{2} \times \frac{a - b + c}{2} \times \frac{a + b - c}{2})}$

$= \sqrt{\{s \times (s - a) \times (s - b) \times (s - c)\}}$, which is the rule, where s denotes half the sum of the three sides.

The expression marked (A), if we put $s = b + c$, and d for $b - c$, is equivalent to $\frac{1}{4}\sqrt{\{(a^2 - d^2)(s^2 - a^2)\}}$; which, in most cases, furnishes a more commodious rule for practice than rule III. here given ; especially if the computer have a table of squares at hand, If the sides of the triangle be large, then add the logs. of the half sum, and of the three remainders together, and half their sum will be the log. of the area.

Ex. 1. To find the area of the triangle whose three sides are 20, 30, 40.

20	45	45	45
30	20	30	40
40	—	—	—
——	25 1st rem.	15 2d rem.	5 3d rem.
2) 90	—	—	—
45 half sum			

Then $45 \times 25 \times 15 \times 5 = 84375$,
The root of which is 290·4737, the area.

Ex. 2. How many square yards of plastering are in a triangle, whose sides are 30, 40, 50 feet? Ans. $66\frac{2}{3}$.

Ex. 3. How many acres, &c. contains the triangle, whose sides are 2569, 4900, 5025 links?

Ans. 61 acres, 1 rood, 39 perches.

PROBLEM III.

To find the Area of a Trapezoid.

Add together the two parallel sides; then multiply their sum by the perpendicular breadth, or the distance between them; and take half the product for the area. By Geom. theor. 29.

Ex. 1. In a trapezoid, the parallel sides are 750 and 1225, and the perpendicular distance between them 1540 links; to find the area.

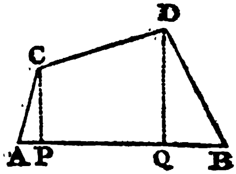
1225
750
——
$1975 \times 770 = 152075$ square links = 15 acr. 33 perc.

Ex. 2. How many square feet are contained in the plank, whose length is 12 feet 6 inches, the breadth at the greater end 15 inches, and at the less end 11 inches?

Ans. $13\frac{13}{24}$ feet.

Ex. 3. In measuring along one side AB of a quadrangular field, that side, and the two perpendiculars let fall on it from the two opposite corners, measured as follow: required the content.

AP = 110 links
AQ = 745
AB = 1110
CP = 352
DQ = 595
Ans. 4 acres, 1 rood, 5·792 perches.



PROBLEM IV.

To find the Area of any Trapezium.

DIVIDE the trapezium into two triangles by a diagonal; then find the areas of these triangles, and add them together.

Or thus, let fall two perpendiculars on the diagonal from the other two opposite angles; then add these two perpendiculars together, and multiply that sum by the diagonal, taking half the product for the area of the trapezium.

Ex. 1. To find the area of the trapezium, whose diagonal is 42, and the two perpendiculars on it 16 and 18.

Here $16 + 18 = 34$, its half is 17.
Then $42 \times 17 = 714$ the area.

Ex. 2. How many square yards of paving are in the trapezium, whose diagonal is 65 feet, and the two perpendiculars let fall on it 28 and $33\frac{1}{2}$ feet? Ans. $222\frac{1}{12}$ yards.

Ex. 3. In the quadrangular field ABCD, on account of obstructions there could only be taken the following measures, viz. the two sides BC 265 and AD 220 yards, the diagonal AC 378, and the two distances of the perpendiculars from the ends of the diagonal, namely, AE 100, and CF 70 yards. Required the construction of the figure, and the area in acres, when 4840 square yards make an acre?

Ans. 17 acres, 2 roods, 21 perches.

PROBLEM V.

To find the Area of an Irregular Polygon.

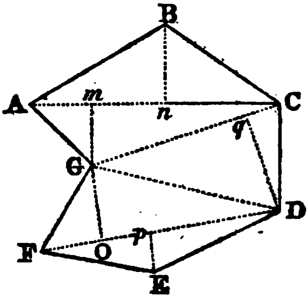
DRAW diagonals dividing the proposed polygon into trapeziums and triangles. Then find the areas of all these separately, and add them together for the content of the whole polygon.

EXAMPLE. To find the content of the irregular figure

ABCDEFGA, in which are given the following diagonals and perpendiculars: namely,

AC 55
FD 52
GC 44
Gm 13
Bn 18
Go 12
Ep 8
Dq 23

Ans. 1878½.



PROBLEM VI.

To find the Area of a Regular Polygon.

RULE I. MULTIPLY the perimeter of the polygon, or sum of its sides, by the perpendicular drawn from its centre on one of its sides, and take half the product for the area*.

Ex. 1. To find the area of a regular pentagon, each side being 25 feet, and the perpendicular from the centre on each side 17·2047737.

Here 25 × 5 = 125 is the perimeter.
And 17·2047737 × 125 = 2150·5967125.
Its half 1075·298356 is the area sought.

RULE II. Square the side of the polygon; then multiply that square by the tabular area, or multiplier set against its name in the following table, and the product will be the area†.

---

* This is only in effect resolving the polygon into as many equal triangles as it has sides, by drawing lines from the centre to all the angles; then finding their areas, and adding them all together.

† This rule is founded on the property, that like polygons, being similar figures, are to one another as the squares of their like sides; which is proved in the Geom. theor. 89. Now, the multipliers in the table, are the areas of the respective polygons to the side 1. Whence the rule is manifest.

No. of Sides.	Names.	Areas, or Multipliers.	Radius of circum. circle.
3	Trigon or triangle	0·4330127	0·5773503
4	Tetragon or square	1·0000000	0·7071068
5	Pentagon	1·7204774	0·8506508
6	Hexagon	2·5980762	1·0000000
7	Heptagon	3·6339124	1·1523824
8	Octagon	4·8284271	1·3065628
9	Nonagon	6·1818242	1·4619022
10	Decagon	7·6942088	1·6180340
11	Undecagon	9·3656399	1·7747324
12	Dodecagon	11·1961524	1·9318517

EXAM. Taking here the same example as before, namely, a pentagon, whose side is 25 feet.

Then 25^2 being = 625,

And the tabular area 1·7204774;

Theref. 1·7204774 × 625 = 1075·298375, as before.

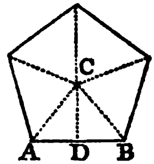
Ex. 2. To find the area of the trigon or equilateral triangle, whose side is 20. Ans. 173·20508.

Ex. 3. To find the area of the hexagon whose side is 20. Ans. 1039·23048.

Ex. 4. To find the area of an octagon whose side is 20. Ans. 1931·37084.

Ex. 5. To find the area of a decagon whose side is 20. Ans. 3077·68352.

Note. If AB = 1, and n the number of sides of the polygon, then area of polygon = n times area of the triangle ABC = n AD . DC = n AD tan. CAD (to rad. AD) = $\frac{1}{4}n$ tan. CAD = $\frac{1}{4}n$ cot. ACD = $\frac{1}{4}n$ cot. $\frac{180^\circ}{n}$. The radius of the circumscribing circle, to side 1, is evidently equal to $\frac{1}{2}$ sec. CAD. Multiplying, therefore, the radius of the table by the numeral value of any proposed side, the product is the radius of a circle in which that polygon may be inscribed; and from which it may readily be constructed.



PROBLEM VII.

To find the Diameter and Circumference of any Circle, the one from the other.

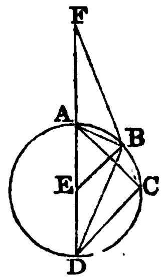
This may be done nearly, by either of the four following proportions,

viz. As 7 is to 22, so is the diameter to the circumference.

Or, As 1 is to 3·1416, so is the diam. to the circumf.

Or, As 113 to 355, so is the diam. to the circumf.*

And, as 1 : ·318309 : : the circumf. : the diameter.



* For let ABCD be any circle, whose centre is E, and let AB, BC, be any two equal arcs. Draw the several chords as in the figure, and join BE; also draw the diameter DA, which produce to F, till BF be equal to the chord BD.

Then the two isosceles triangles DEB, DBF, are equiangular, because they have the angle at D common; consequently DE : DB : : DB : DF. But the two triangles AFB, DCB, are identical, or equal in all respects, because they have the angle F = the angle BDC, being each equal to the angle ADB, these being subtended by the equal arcs AB, BC; also the exterior angle FAB of the quadrangle ABCD, is equal to the opposite interior angle at C; and the two triangles have also the side BF = the side BD; therefore the side AF is also equal to the side DC. Hence the proportion above, viz. DE : DB :: DB : DF = DA + AF, becomes DE : DB :: DB : 2DE + DC. Then, by taking the rectangles of the extremes and means, it is $DB^2 = 2DE^2 + DE . DC$.

Now, if the radius DE be taken = 1, this expression becomes $DB^2 = 2 + DC$, and hence the root $DB = \sqrt{(2 + DC)}$. That is, if the measure of the supplemental chord of any arc be increased by the number 2, the square root of the sum will be the supplemental chord of half that arc.

Now, to apply this to the calculation of the circumference of the circle, let the arc AC be taken equal to $\frac{1}{6}$ of the circumference, and be successively bisected by the above theorem: thus the chord AC of $\frac{1}{6}$ of the circumference, is the side of the inscribed regular hexagon, and is therefore equal to the radius AE or 1: hence, in the right-angled triangle ACD, it will be $DC = \sqrt{(AD^2 - AC^2)} = \sqrt{(2^2 - 1^2)} = \sqrt{3} = 1·7320508076$, the supplemental chord of $\frac{1}{6}$ of the periphery.

Then, by always bisecting the arcs, and adding 2 to the last square root, there will be found the supplemental chords of the 12th, the 24th, the 48th, the 96th, &c., parts of the periphery; thus,

$\sqrt{3·7320508076} = 1·9318516525$	for the supplemental chord of	$\frac{1}{12}$ the periphery.
$\sqrt{3·9318516525} = 1·9828897227$		$\frac{1}{24}$
$\sqrt{3·9828897227} = 1·9957178465$		$\frac{1}{48}$
$\sqrt{3·9957178465} = 1·9989291743$		$\frac{1}{96}$
$\sqrt{3·9989291743} = 1·9997322757$		$\frac{1}{192}$
$\sqrt{3·9997322757} = 1·9999330678$		$\frac{1}{384}$
$\sqrt{3·9999330678} = 1·9999832669$		$\frac{1}{768}$
$\sqrt{3·9999832669}$ = - - - - - - -		$\frac{1}{1536}$

Ex. 1. To find the circumference of the circle whose diameter is 20.

By the first rule, as 7 : 22 : : 20 : 62$\frac{6}{7}$, the answer.

Ex. 2. If the circumference of the earth be 24877·4 miles, what is its diameter?

By the 2d rule, as 3·1416 : 1 : : 24877·4 : 7918·7 nearly the diameter.

By the 3d rule, as 355 : 113 : : 24877·4 : 7918·7 nearly.

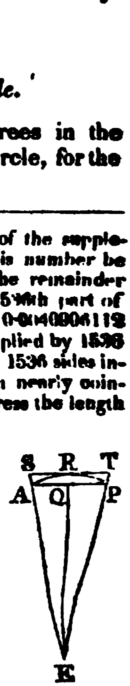
By the 4th rule, as 1 : ·318309 : : 24877·4 : 7918·7 nearly.

PROBLEM VIII.

To find the Length of any Arc of a Circle.

MULTIPLY the decimal ·017453 by the degrees in the given arc, and that product by the radius of the circle, for the length of the arc*.

Since then it is found that 3·9999832669 is the square of the supplemental chord of the 1536th part of the periphery, let this number be taken from 4, which is the square of the diameter, and the remainder 0·0000167331 will be the square of the chord of the said 1536th part of the periphery, and consequently the root $\sqrt{0·0000167331} = 0·0040906112$ is the length of that chord; this number then being multiplied by 1536 gives 6·2831788 for the perimeter of a regular polygon of 1536 sides inscribed in the circle; which, as the sides of the polygon nearly coincide with the circumference of the circle, must also express the length of the circumference itself, very nearly.



But now, to show how near this determination is to the truth, let AQP = 0·0040906112 represent one side of such a regular polygon of 1536 sides, and SRT a side of another similar polygon described about the circle; and from the centre E let the perpendicular EQR be drawn, bisecting AP and ST in Q and R. Then since AQ is = $\frac{1}{2}$ AP = 0·0020453056, and EA = 1, therefore $EQ^2 = EA^2 - AQ^2$ = ·9999958167, and consequently its root gives EQ = ·9999979084; then because of the parallels AP, ST, it is EQ : ER : : AP : ST : : as the whole inscribed perimeter : to the circumscribed one, that is, as ·9999979084 : 1 : : 6·2831788 : 6·2831920 the perimeter of the circumscribed polygon. Now, the circumference of the circle being greater than the perimeter of the inner polygon, but less than that of the outer, it must consequently be greater than 6·2831788,

but less than 6·2831920,

and must therefore be nearly equal to $\frac{1}{2}$ their sum, or 6·2831854, which is in fact is true to the last figure, which should be a 3, instead of the 4.

Hence the circumference being 6·2831854 when the diameter is 2, it will be the half of that, or 3·1415927, when the diameter is 1, to which the ratio in the rule, viz. 1 to 3·1416, is very near. Also the first ratio in the rule, 7 to 22 or 1 to 3$\frac{1}{7}$ = 3·1428 &c. is another near approximation. But the third ratio, 113 to 355, = 1 to 3·1415929, is the nearest.

* It having been found, in the demonstration of the foregoing problem, that when the radius of a circle is 1, the length of the whole circumfe-

Ex. 1. To find the length of an arc of 30 degrees, the radius being 9 feet. Ans. 4·71231.

Ex. 2. To find the length of an arc of 12° 10′, or 12$\frac{1}{6}$°, the radius being 10 feet. Ans. 2·1234.

PROBLEM IX.

To find the Area of a Circle.*

Rule i. Multiply half the circumference by half the diameter. Or multiply the whole circumference by the whole diameter, and take $\frac{1}{4}$ of the product.

Rule ii. Square the diameter, and multiply that square by the decimal ·7854, for the area.

Rule iii. Square the circumference, and multiply that square by the decimal ·07958.

Ex. 1. To find the area of a circle whose diameter is 10, and its circumference 31·416.

By Rule 1.	By Rule 2.	By Rule 3.
31·416	·7854	31·416
10	10^2 = 100	31·416
4)314·16	78·54	986·965
78·54		·07958
		78·54

So that the area is 78·54 by all the three rules.

---

rence is 6·2831854. which consists of 360 degrees; therefore as 360° : 6·2831854 : : 1 : ·017453, &c. the length of the arc of 1 degree. Hence the decimal ·017453 multiplied by any number of degrees, will give the length of the arc of those degrees. And because the circumferences and arcs are in proportion as the diameters, or as the radii of the circles, therefore as the radius 1 is to any other radius *r*, so is the length of the arc above mentioned, to ·017453 × degrees in the arc × *r*, which is the length of that arc, as in the rule.

* The first rule is proved in the Geom. theor. 94.

And the 2d and 3d rules are deduced from the first rule, in this manner.—By that rule, $dc \div 4$ is the area, when d denotes the diameter, and c the circumference. But, by prob. 7, c is $= 3{\cdot}1416d$; therefore the said area $dc \div 4$, becomes $d \times 3{\cdot}1416d \div 4 = {\cdot}7854d^2$, which gives the 2d rule.—Also, by the same prob. 7, d is $= c \div 3{\cdot}1416$; therefore again the same first area $dc \div 4$, becomes $(c \div 3{\cdot}1416) \times (c \div 4) = c^2 \div 12{\cdot}5664$, which is $= c^2 \times {\cdot}07958$, by taking the reciprocal of 12·5664, or changing that divisor into the multiplier ·07958: which gives the 3d rule.

Corol. Hence the areas of different circles are in proportion to one another, as the square of their diameters or as the square of their circumferences; as before proved in the Geom. theor. 93.

Ex. 2. To find the area of a circle, whose diameter is 7, and circumference 22. Ans. $38\frac{1}{2}$.

Ex. 3. How many square yards are in a circle, whose diameter is $3\frac{1}{2}$ feet? Ans. 1·069.

Ex. 4. To find the area of a circle, whose circumference is 12 feet. Ans. 11·4595.

PROBLEM X.

To find the Area of a Circular Ring, or of the Space included between the Circumferences of two Circles; the one being contained within the other.

Take the difference between the areas of the two circles, as found by the last problem, for the area of the ring.—Or, which is the same thing, subtract the square of the less diameter from the square of the greater, and multiply their difference by ·7854.—Or, lastly, multiply the sum of the diameters by the difference of the same, and that product by ·7854; which is still the same thing, because the product of the sum and difference of any two quantities, is equal to the difference of their squares.

Ex. 1. The diameters of two concentric circles being 10 and 6, required the area of the ring contained between their circumferences.

Here $10 + 6 = 16$ the sum, and $10 - 6 = 4$ the diff.
Therefore $·7854 \times 16 \times 4 = ·7854 \times 64 = 50·2656$, the area.

Ex. 2. What is the area of the ring, the diameters of whose bounding circles are 10 and 20? Ans. 235·62.

PROBLEM XI.

To find the Area of the Sector of a Circle.

Rule i. Multiply the radius, or half the diameter, by half the arc of the sector, for the area. Or, multiply the whole diameter by the whole arc of the sector, and take $\frac{1}{4}$ of the product. The reason of which is the same as for the first rule to problem 9, for the whole circle.

Rule ii. Compute the area of the whole circle: then say, as 360 is to the degrees in the arc of the sector, so is the area of the whole circle, to the area of the sector.

This is evident, because the sector is proportional to the length of the arc, or to the degrees contained in it.

Ex. 1. To find the area of a circular sector, whose arc contains 18 degrees; the diameter being 3 feet.

1. By the first Rule.

First, $3{\cdot}1416 \times 3 = 9{\cdot}4248$, the circumference.

And $360 : 18 :: 9{\cdot}4248 : {\cdot}47124$, the length of the arc.

Then ${\cdot}47124 \times 3 \div 4 = 1{\cdot}41372 \div 4 = {\cdot}35343$, the area.

2. By the 2d Rule.

First, ${\cdot}7854 \times 3^2 = 7{\cdot}0686$, the area of the whole circle.

Then, as $360 : 18 :: 7{\cdot}0686 : {\cdot}35343$, the area of the sector.

Ex. 2. To find the area of a sector, whose radius is 10, and arc 20. Ans. 100.

Ex. 3. Required the area of a sector, whose radius is 25, and its arc containing $147^\circ 29'$. Ans. 804·3986.

PROBLEM XII.

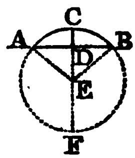
To find the Area of a Segment of a Circle.

Rule i. Find the area of the sector having the same arc with the segment, by the last problem.

Find also the area of the triangle, formed by the chord of the segment and the two radii of the sector.

Then add these two together for the answer, when the segment is greater than a semicircle: or subtract them when it is less than a semicircle.—As is evident by inspection.

Ex. 1. To find the area of the segment ACBDA, its chord AB being 12, and the radius AE or CE 10.



First, As AE : sin. $\angle$ D 90° :: AD : sin. $36^\circ 52'\frac{1}{5} = 36{\cdot}87$ degrees, the degrees in the $\angle$ AEC or arc AC. Their double, 73·74, are the degrees in the whole arc ACB.

Now ${\cdot}7854 \times 400 = 314{\cdot}16$, the area of the whole circle.

Therefore $360^\circ : 73{\cdot}74 :: 314{\cdot}16 : 64{\cdot}3504$, area of the sector ACBE.

Again, $\surd(\text{AE}^2 - \text{AD}^2) = \surd(100 - 36) = \surd 64 = 8 = \text{DE}$.

Theref. AD $\times$ DE $= 6 \times 8 = 48$, the area of the triangle AEB.

Hence sector ACBE — triangle AEB $= 16{\cdot}3504$, area of seg. ACBDA.

Rule ii. Multiply the square of the radius of the circle by either half the difference of the arc ACB and its sine (both to the radius 1), or half the sum of the arc and its sine, according as the segment is less or greater than a semicircle; the product will be the area.

The reason of this rule, also, is evident from an inspection of the diagram.

Exam. the same as before, in which AB = 12, AE = 10; and from the former computation arc ACB = 73° 44′¾.

Then, by Hutton's Mathematical Tables, pp. 340, &c.

arc 73° 44′¾, to radius 1 = 1·2870059
sin. 73° 44′¾, to radius 1 = ·9600010

2) ·3270069

·1635034

whence, ·1635034 × 10^2 = 16·35034, the area of the segment; very nearly as before.

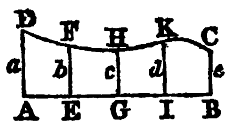
Ex. 2. What is the area of the segment, whose height is 18, and diameter of the circle 50? Ans. 636·375.

Ex. 3. Required the area of the segment whose chord is 16, the diameter being 20? Ans. 44·728, or 269·432.

PROBLEM XIII.

To measure long Irregular Figures.

Take or measure the breadth at both ends, and at several places, at equal distances. Then add together all these intermediate breadths and half the two extremes, which sum multiply by the length, and divide by the number of parts, for the area*.



* This rule is made out as follows:—Let ABCD be the irregular piece; having the several breadths AD, EF, GH, IK, BC, at the equal distances AE, EG, GI, IB. Let the several breadths in order be denoted by the corresponding letters a, b, c, d, e, and the whole length AB by l; then compute the areas of the parts into which the figure is divided by the perpendiculars, as so many trapezoids, by prob. 3, and add them all together. Thus, the sum of the parts is,

$$\frac{a+b}{2}\times \text{AE}+\frac{b+c}{2}\times \text{EG}+\frac{c+d}{2}\times \text{GI}+\frac{d+e}{2}\times \text{IB}$$

$$=\frac{a+b}{2}\times \tfrac{1}{4}l+\frac{b+c}{2}\times \tfrac{1}{4}l+\frac{c+d}{2}\times \tfrac{1}{4}l+\frac{d+e}{2}\times \tfrac{1}{4}l$$

$$=(\tfrac{1}{2}a+b+c+d+\tfrac{1}{2}e)\times \tfrac{1}{4}l=(m+b+c+d)\,\tfrac{1}{4}l,$$

Note. If the perpendiculars or breadths be not at equal distances, compute all the parts separately, as so many trapezoids, and add them all together for the whole area.

Or else, add all the perpendicular breadths together, and divide their sum by the number of them for the mean breadth, to multiply by the length; which will give the whole area, not far from the truth.

Ex. 1. The breadths of an irregular figure, at five equidistant places, being 8·2, 7·4, 9·2, 10·2, 8·6; and the whole length 39; required the area.

8·2	35·2 sum.
8·6	39
2) 16·8 sum of the extremes.	3168
8·4 mean of the extremes.	1056
7·4	4) 1372·8
9·2	343·2 Ans.
10·2	
35·2 sum.	

Ex. 2. The length of an irregular figure being 84, and the breadths at six equidistant places 17·4, 20·6, 14·2, 16·5, 20·1, 24·4; what is the area? Ans. 1550·64.

PROBLEM XIV.

To find the Area of an Ellipsis or Oval.

Multiply the longest diameter, or axis, by the shortest; then multiply the product by the decimal ·7854, for the area. As appears from cor. 2, theor. 3, of the Ellipse, in the Conic Sections.

Ex. 1. Required the area of an ellipse whose two axes are 70 and 50. Ans. 2748·9.

Ex. 2. To find the area of the oval whose two axes are 24 and 18. Ans. 339·2928.

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which is the whole area, agreeing with the rule: *m* being the arithmetical mean between the extremes, or half the sum of them both, and 4 the number of the parts. And the same for any other number of parts whatever.

PROBLEM XV.

To find the Area of an Elliptic Segment.

Find the area of a corresponding circular segment, having the same height and the same vertical axis or diameter. Then say, as the said vertical axis is to the other axis, parallel to the segment's base, so is the area of the circular segment before found, to the area of the elliptic segment sought. This rule also comes from cor. 2, theor. 3, of the Ellipse.

Ex. 1. To find the area of the elliptic segment, whose height is 20, the vertical axis being 70, and the parallel axis 50. Ans. 648·13.

Ex. 2. Required the area of an elliptic segment, cut off parallel to the shorter axis; the height being 10, and the two axes 25 and 35. Ans. 162·03.

Ex. 3. To find the area of the elliptic segment, cut off parallel to the longer axis; the height being 5, and the axes 25 and 35. Ans. 97·8425.

PROBLEM XVI.

To find the Area of a Parabola, or its Segment.

Multiply the base by the perpendicular height; then take two-thirds of the product for the area. As is proved in theorem 17 of the Parabola, in the Conic Sections.

Ex. 1. To find the area of a parabola; the height being 2, and the base 12.

Here $2 \times 12 = 24$. Then $\frac{2}{3}$ of $24 = 16$, is the area.

Ex. 2. Required the area of the parabola, whose height is 10, and its base 16. Ans. $106\frac{2}{3}$.

MENSURATION OF SOLIDS.

By the Mensuration of Solids are determined the spaces included by contiguous surfaces; and the sum of the measures of these including surfaces is the whole surface or superficies of the body.

The measure of a solid, is called its solidity, capacity, or content.

Solids are measured by cubes, whose sides are inches, or feet, or yards, &c. And hence the solidity of a body is said to be so many cubic inches, feet, yards, &c. as will fill its capacity or space, or another of an equal magnitude.

The least solid measure is the cubic inch, other cubes being taken from it according to the proportion in the following table, which is formed by cubing the linear proportions.

Table of Cubic or Solid Measures.

1728	cubic inches make	1 cubic foot
27	cubic feet	1 cubic yard
166⅜	cubic yards	1 cubic pole
64000	cubic poles	1 cubic furlong
512	cubic furlongs	1 cubic mile.

PROBLEM I.

To find the Superficies of a Prism or Cylinder.

MULTIPLY the perimeter of one end of the prism by the length or height of the solid, and the product will be the surface of all its sides. To which add also the area of the two ends of the prism, when required*.

Or, compute the areas of all the sides and ends separately, and add them all together.

Ex. 1. To find the surface of a cube, the length of each side being 20 feet. Ans. 2400 feet.

Ex. 2. To find the whole surface of a triangular prism, whose length is 20 feet, and each side of its end or base 18 inches. Ans. 91·948 feet.

Ex. 3. To find the convex surface of a round prism, or cylinder, whose length is 20 feet, and the diameter of its base is 2 feet. Ans. 125·664.

Ex. 4. What must be paid for lining a rectangular cistern with lead, at 3*d*. a pound weight, the thickness of the lead being such as to weigh 7lb. for each square foot of surface; the inside dimensions of the cistern being as follow, viz. the length 3 feet 2 inches, the breadth 2 feet 8 inches, and depth 2 feet 6 inches? Ans. 3*l*. 5*s*. 9¾*d*.

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* The truth of this will easily appear, by considering that the sides of any prism are parallelograms, whose common length is the same as the length of the solid, and their breadths taken all together make up the perimeter of the ends of the same.

And the rule is evidently the same for the surface of a cylinder.

PROBLEM II.

To find the Surface of a Pyramid or Cone.

Multiply the perimeter of the base by the slant height, or length of the side, and half the product will evidently be the surface of the sides, or the sum of the areas of all the triangles which form it. To which add the area of the end or base, if requisite.

Ex. 1. What is the upright surface of a triangular pyramid, the slant height being 20 feet, and each side of the base 3 feet? Ans. 90 feet.

Ex. 2. Required the convex surface of a cone, or circular pyramid, the slant height being 50 feet, and the diameter of its base $8\frac{1}{2}$ feet. Ans. 667·59.

PROBLEM III.

To find the Surface of the Frustum of a Pyramid or Cone, being the lower part, when the top is cut off by a plane parallel to the base.

Add together the perimeters of the two ends, and multiply their sum by the slant height, taking half the product for the answer.—As is evident, because the sides of the solid are trapezoids, having the opposite sides parallel.

Ex. 1. How many square feet are in the surface of the frustum of a square pyramid, whose slant height is 10 feet; also each side of the base or greater end being 3 feet 4 inches, and each side of the less end 2 feet 2 inches? Ans. 110 feet.

Ex. 2. To find the convex surface of the frustum of a cone, the slant height of the frustum being $12\frac{1}{2}$ feet, and the circumferences of the two ends 6 and 8·4 feet.

Ans. 90 feet.

PROBLEM IV.

To find the Solid Content of any Prism or Cylinder.

Find the area of the base, or end, whatever the figure of it may be; and multiply it by the length of the prism or cylinder, for the solid content*.

* This rule appears from the Geom. theor. 110, cor. 2. The same is more particularly shown as follows: Let the annexed rectangular paral-

Note. For a cube, take the cube of its side by multiplying this twice by itself; and for a parallelopipedon, multiply the length, breadth, and depth all together, for the content.

Ex. 1. To find the solid content of a cube, whose side is 24 inches. Ans. 13824.

Ex. 2. How many cubic feet are in a block of marble, its length being 3 feet 2 inches, breadth 2 feet 8 inches, and thickness 2 feet 6 inches? Ans. $21\frac{1}{9}$.

Ex. 3. How many gallons of water will the cistern contain, whose dimensions are the same as in the last example, when $277\frac{1}{4}$ cubic inches are contained in one gallon? Ans. 131·53.

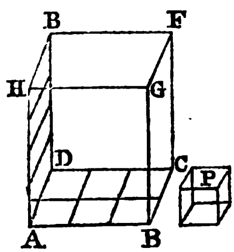
Ex. 4. Required the solidity of a triangular prism, whose length is 10 feet, and the three sides of its triangular end or base are 3, 4, 5 feet. Ans. 60.

Ex. 5. Required the content of a round pillar, or cylinder, whose length is 20 feet, and circumference 5 feet 6 inches. Ans. 48·1459 feet.

PROBLEM V.

To find the Content of any Pyramid or Cone.

FIND the area of the base, and multiply that area by the perpendicular height; then take $\frac{1}{3}$ of the product for the content*.



lelopipedon be the solid to be measured, and the cube P the solid measuring unit, its side being 1 inch, or 1 foot, &c.; also, let the length and breadth of the base ABCD, and also the height AH, be each divided into spaces equal to the length of the base of the cube P, namely, here 3 in the length and 2 in the breadth, making 3 times 2 or 6 squares in the base AC, each equal to the base of the cube P. Hence it is manifest that the parallelopipedon will contain the cube P, as many times as the base AC contains the base of the cube, repeated as often as the height AH contains the height of the cube. That is, the content of any parallelopipedon is found, by multiplying the area of the base by the altitude of that solid.

And because all prisms and cylinders are equal to parallelopipedons of equal bases and altitudes, by Geom. theor. 108, it follows that the rule is general for all such solids, whatever the figure of the base may be.

* This rule follows from that of the prism, because any pyramid is $\frac{1}{3}$ of a prism of equal base and altitude; by Geom. theor. 115, cor. 1 and 2.

Ex. 1. Required the solidity of a square pyramid, each side of its base being 30, and its perpendicular height 25.
Ans. 7500.

Ex. 2. To find the content of a triangular pyramid, whose perpendicular height is 30, and each side of the base 3.
Ans. 38·971143.

Ex. 3. To find the content of a triangular pyramid, its height being 14 feet 6 inches, and the three sides of its base 5, 6, 7 feet.
Ans. 71·0352.

Ex. 4. What is the content of a pentagonal pyramid, its height being 12 feet, and each side of its base 2 feet?
Ans. 27·5276.

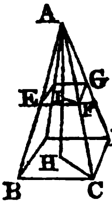
Ex. 5. What is the content of the hexagonal pyramid, whose height is 6·4 feet, and each side of its base 6 inches?
Ans. 1·38564 feet.

Ex. 6. Required the content of a cone, its height being $10\frac{1}{2}$ feet, and the circumference of its base 9 feet.
Ans. 22·56093.

PROBLEM VI.

To find the Solidity of a Frustum of a Cone or Pyramid.

ADD into one sum, the areas of the two ends, and the mean proportional between them: and take $\frac{1}{3}$ of that sum for a mean area; which being multiplied by the perpendicular height, or length of the frustum, will give its content*.



* Let ABCD be any pyramid, of which BCDGFE is a frustum. And put a^2 for the area of the base BCD, b^2 the area of the top, EFG, h the height IH of the frustum, and c the height AI of the top part above it. Then $c + h =$ AH is the height of the whole pyramid.

Hence, by the last prob. $\frac{1}{3}a^2(c+h)$ is the content of the whole pyramid ABCD, and $\frac{1}{3}b^2c$ the content of the top part AEFG; therefore the difference - - - $\frac{1}{3}a^2(c+h) - \frac{1}{3}b^2c$ is the content of the frustum BCDGFE. But the quantity c being no dimension of the frustum, it must be expelled from this formula, by substituting its value, found in the following manner. By Geom. theor. 112, $a^2 : b^2 :: (c+h)^2 : c^2$, or $a : b :: c+h : c$, hence (Geom. th. 69) $a - b : b :: h : c$, and $a - b : a :: h : c+h$; hence therefore $c = \frac{bh}{a-b}$, and $c + h = \frac{ah}{a-b}$;

Note. This general rule may be otherwise expressed, as follows, when the ends of the frustum are circles or regular polygons. In this latter case, square one side of each polygon, and also multiply the one side by the other; add all these three products together; then multiply their sum by the tabular area proper to the polygon, and take one-third of the product for the mean area, to be multiplied by the length, to give the solid content. And in the case of the frustum of a cone, the ends being circles, square the diameter or the circumference of each end, and also multiply the same two dimensions together; then take the sum of the three products, and multiply it by the proper tabular number, viz. by ·7854 when the diameters are used, or by ·07958 in using the circumferences; then taking one-third of the product, to multiply by the length, for the content.

Ex. 1. To find the number of solid feet in a piece of timber, whose bases are squares, each side of the greater end being 15 inches, and each side of the less end 6 inches; also, the length or the perpendicular altitude 24 feet. Ans. 19½.

Ex. 2. Required the content of a pentagonal frustum, whose height is 5 feet, each side of the base 18 inches; and each side of the top or less end 6 inches. Ans. 9·31925 feet.

Ex. 3. To find the content of a conic frustum, the altitude being 18, the greatest diameter 8, and the least diameter 4.
Ans. 527·7888.

Ex. 4. What is the solidity of the frustum of a cone, the altitude being 25, also the circumference at the greater end being 20, and at the less end 10? Ans. 464·216.

Ex. 5. If a cask, which is two equal conic frustums joined together at the bases, have its bung diameter 28 inches, the head diameter 20 inches, and length 40 inches; how many gallons of wine will it hold? Ans. 79·0613.

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then these values of c and $c+h$ being substituted for them in the expression for the content of the frustum gives that content

$$= \tfrac{1}{3}a^2 \times \frac{ah}{a-b} - \tfrac{1}{3}b^2 \times \frac{bh}{a-b} = \tfrac{1}{3}h \times \frac{a^3-b^3}{a-b} = \tfrac{1}{3}h \times (a^2+ab+b^2);$$ which is the rule above given; ab being the mean between a^2 and b^2.

Note. If D, d be the corresponding linear dimensions of the ends, δ their difference, m the appropriate multiplier, h the height of the frustum, then is the content $= \tfrac{1}{3}mh\,(3Dd + \delta)$; which is a convenient practical expression.

PROBLEM VII.

To find the Surface of a Sphere, or any Segment.

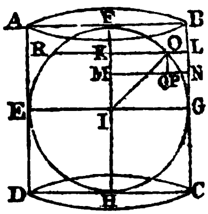
RULE I. MULTIPLY the circumference of the sphere by its diameter, and the product will be the whole surface of it*.

RULE II. Square the diameter and multiply that square by 3·1416, for the surface.

RULE III. Square the circumference ; then either multiply that square by the decimal ·3183, or divide it by 3·1416, for the surface.

Note. For the surface of a segment or frustum, multiply

* These rules come from the following theorems for the surface of a sphere, viz. That the said surface is equal to the curve surface of its circumscribing cylinder ; or that it is equal to 4 great circles of the same sphere, or of the same diameter ; which are thus proved.



Let ABCD be a cylinder, circumscribing the sphere EFGH ; the former generated by the rotation of the rectangle FBCH about the axis or diameter FH ; and the latter by the rotation of the semicircle FGH about the same diameter FH. Draw two lines KL, MN, perpendicular to the axis, intercepting the parts LN, OP, of the cylinder and sphere ; then will the ring or cylindric surface generated by the rotation of LN, be equal to the ring or spherical surface generated by the arc OP. For, first, suppose the parallels KL and MN to be indefinitely near together; drawing IO, and also OQ parallel to LN. Then the two triangles IKO, OQP, being equiangular, it is, as OP : OQ or LN :: IO or KL : KO :: circumference described by KL : circumf. described by KO; therefore the rectangle OP × circumf. of KO is equal to the rectangle LN × circumf. of KL; that is, the ring described by OP on the sphere, is equal to the ring described by LN on the cylinder.

And as this is every where the case, therefore the sums of any corresponding number of these are also equal ; that is, the whole surface of the sphere, described by the whole semicircle FGH, is equal to the whole curve surface of the cylinder, described by the height BC ; as well as the surface of any segment described by FO, equal to the surface of the corresponding segment described by BL.

Corol. 1. Hence the surface of the sphere is equal to 4 of its great circles, or equal to the circumference EFGH, or of DC, multiplied by the height BC, or by the diameter FH.

Corol. 2. Hence also, the surface of any such part, as a segment or frustum, or zone, is equal to the same circumference of the sphere, multiplied by the height of the said part. And consequently such spherical curve surfaces are to one another in the same proportion as their altitudes.

the whole circumference of the sphere by the height of the part required.

Ex. 1. Required the convex superficies of a sphere, whose diameter is 7, and circumference 22. Ans. 154.

Ex. 2. Required the superficies of a globe, whose diameter is 24 inches. Ans. 1809·5616.

Ex. 3. Required the area of the whole surface of the earth, its diameter being 7957¾ miles, and its circumference 25000 miles. Ans. 198943750 sq. miles.

Ex. 4. The axis of a sphere being 42 inches, what is the convex superficies of the segment whose height is 9 inches? Ans. 1187·5248 inches.

Ex. 5. Required the convex surface of a spherical zone, whose breadth or height is 2 feet, and cut from a sphere of 12½ feet diameter. Ans. 78·54 feet.

PROBLEM VIII.

To find the Solidity of a Sphere or Globe.

Rule I. Multiply the surface by the diameter, and take $\frac{1}{6}$ of the product for the content*. Or, which is the same thing, multiply the square of the diameter by the circumference, and take $\frac{1}{6}$ of the product.

Rule II. Take the cube of the diameter, and multiply it by the decimal ·5236, for the content.

Rule III. Cube the circumference, and multiply by ·01688.

Ex. 1. To find the solid content of the globe of the earth, supposing its circumference to be 25000 miles. Ans. 263750000000 miles.

Ex. 2. Supposing that a cubic inch of cast iron weighs ·269 of a lb. avoird. what is the weight of an iron ball of 5·04 inches diameter?

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* For, put d = the diameter, c = the circumference, and s = the surface of the sphere, or of its circumscribing cylinder; also, a = the number 3·1416.

Then, $\frac{1}{4}s$ is = the base of the cylinder, or one great circle of the sphere; and d is the height of the cylinder: therefore $\frac{1}{4}ds$ is the content of the cylinder. But $\frac{2}{3}$ of the cylinder is the sphere, by th. 117, Geom. that is, $\frac{2}{3}$ of $\frac{1}{4}ds$, or $\frac{1}{6}ds$ is the sphere; which is the first rule.

Again, because the surface s is $= ad^2$; therefore $\frac{1}{6}ds = \frac{1}{6}ad^3 = \cdot5236d^3$, is the content, as in the 2d rule. Also, d being $= c \div a$, therefore $\frac{1}{6}ad^3 = \frac{1}{6}c^3 \div a^2 = \cdot01688c^3$, the 3d rule for the content.

PROBLEM IX.

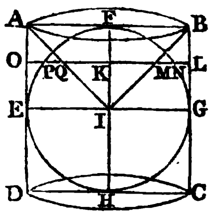
To find the Solid Content of a Spherical Segment.

* **Rule i.** From 3 times the diameter of the sphere take double the height of the segment; then multiply the remainder by the square of the height, and the product by the decimal ·5236, for the content.

Rule ii. To 3 times the square of the radius of the segment's base, add the square of its height; then multiply the sum by the height, and the product by ·5236, for the content.

Ex. 1. To find the content of a spherical segment, of 2 feet in height, cut from a sphere of 8 feet diameter.

Ans. 41·888.



* By corol. 3, of theor. 117, Geom. it appears that the spheric segment PFN, is equal to the difference between the cylinder ABLO, and the conic frustum ABMQ.

But, putting d = AB or FH the diameter of the sphere or cylinder, h = FK the height of the segment, r = PK the radius of its base, and $a = 3·1416$; then the content of the cone ABI is $= \frac{1}{4}ad^2 \times \frac{1}{2}$FI $= \frac{1}{24}ad^3$; and by the similar cones ABI, QMI, as FI^3 : KI^3 :: $\frac{1}{24}ad^3 : \frac{1}{24}ad^3 \times (\frac{\frac{1}{2}d - h}{\frac{1}{2}d})^3$ = the cone QMI; therefore the cone ABI — the cone QMI $= \frac{1}{24}ad^3 - \frac{1}{24}ad^3 \times (\frac{\frac{1}{2}d - h}{\frac{1}{2}d})^3 = \frac{1}{4}ad^2h - \frac{1}{2}adh^2 + \frac{1}{3}ah^3$ is = the conic frustum of ABMQ.

And $\frac{1}{4}ad^2h$ is = the cylinder ABLO.

Then the difference of these two is $\frac{1}{2}adh^2 - \frac{1}{3}ah^3 = \frac{1}{6}ah^2 \times (3d - 2h)$, for the spheric segment PFN; which is the first rule.

Again, because PK^2 = FK × KH (cor. to theor. 87, Geom.) or $r^2 = h(d - h)$, therefore $d = \frac{r^2}{h} + h$, and $3d - 2h = \frac{3r^2}{h} + h = \frac{3r^2 + h^2}{h}$; which being substituted in the former rule, it becomes $\frac{1}{6}ah^2 \times \frac{3r^2 + h^2}{h} = \frac{1}{6}ah \times (3r^2 + h^2)$, which is the 2d rule.

Note. By subtracting a segment from a half sphere, or from another segment, the content of any frustum or zone may be found.

Ex. 2. What is the solidity of the segment of a sphere, its height being 9, and the diameter of its base 20?

Ans. 1795·4244.

Note. The general rules for measuring the most useful figures having been now delivered, we may proceed to apply them to the several practical uses in life, as follows.

LAND SURVEYING.

SECTION I.

DESCRIPTION AND USE OF THE INSTRUMENTS.

1. OF THE CHAIN.

LAND is measured with a chain, called Gunter's Chain, from its inventor, the length of which is 4 poles, or 22 yards, or 66 feet. It consists of 100 equal links; and the length of each link is therefore $\frac{22}{100}$ of a yard, or $\frac{66}{100}$ of a foot, or 7·92 inches.

Land is estimated in acres, roods, and perches. An acre is equal to 10 square chains, or as much as 10 chains in length and 1 chain in breadth. Or, in yards, it is $220 \times 22 = 4840$ square yards. Or, in poles, it is $40 \times 4 = 160$ square poles. Or, in links, it is $1000 \times 100 = 100000$ square links: these being all the same quantity.

Also, an acre is divided into 4 parts called roods, and a rood into 40 parts called perches, which are square poles, or the square of a pole of $5\frac{1}{2}$ yards long, or the square of $\frac{1}{4}$ of a chain, or of 25 links, which is 625 square links. So that the divisions of land measure will be thus:

625 sq. links = 1 pole or perch
40 perches = 1 rood
4 roods = 1 acre.

The lengths of lines measured with a chain, are best set down in links as integers, every chain in length being 100 links; and not in chains and decimals. Therefore, after the content is found, it will be in square links; then cut off five of the figures on the right hand for decimals, and the rest will be acres. These decimals are then multiplied by 4 for roods, and the decimals of these again by 40 for perches.

Exam. Suppose the length of a rectangular piece of ground be 792 links, and its breadth 385; to find the area in acres, roods, and perches.

792	3·04920
385	4
3960	·19680
6336	40
2376	
	7·87200
3·04920	

Ans. 3 acres, 0 roods, 7 perches.

2. OF THE PLAIN TABLE.

This instrument consists of a plain rectangular board, of any convenient size: the centre of which, when used, is fixed by means of screws to a three-legged stand, having a ball and socket, or other joint, at the top, by means of which, when the legs are fixed on the ground, the table is inclined in any direction.

To the table belong various parts, as follow.

1. A frame of wood, made to fit round its edges, and to be taken off, for the convenience of putting a sheet of paper on the table. One side of this frame is usually divided into equal parts, for drawing lines across the table, parallel or perpendicular to the sides; and the other side of the frame is divided into 360 degrees, to a centre in the middle of the table; by means of which the table may be used as a theodolite, &c.

2. A magnetic needle and compass, either screwed into the side of the table, or fixed beneath its centre, to point out the directions, and to be a check on the sights.

3. An index, which is a brass two-foot scale, with either a small telescope, or open sights set perpendicularly on the ends. These sights and one edge of the index are in the same plane, and that is called the fiducial edge of the index.

To use this instrument, take a sheet of paper which will cover it, and wet it to make it expand; then spread it flat on the table, pressing down the frame on the edges, to stretch it and keep it fixed there; and when the paper is become dry, it will, by contracting again, stretch itself smooth and flat from any cramps and unevenness. On this paper is to be drawn the plan or form of the thing measured.

Thus, begin at any proper part of the ground, and make a point on a convenient part of the paper or table, to represent that place on the ground; then fix in that point one leg of the compasses, or a fine steel pin, and apply to it the fiducial edge of the index, moving it round till through the sights you perceive some remarkable object, as the corner of a field, &c.; and from the station-point draw a line with the point of the compasses along the fiducial edge of the index, which is called setting or taking the object: than set another object or corner, and draw its line; do the same by another; and so on, till as many objects are taken as may be thought fit. Then measure from the station towards as many of the objects as may be necessary, but not more, taking the requisite offsets to corners or crooks in the hedges, laying the measures down on their respective lines on the table. Then at any convenient place measured to, fix the table in the same position, and set the objects which appear from that place; and so on, as before. And thus continue till the work is finished, measuring such lines only as are necessary, and determining as many as may be by intersecting lines of direction drawn from different stations.

Of shifting the Paper on the Plain Table.

When one paper is full, and there is occasion for more, draw a line in any manner through the farthest point of the last station line, to which the work can be conveniently laid down; then take the sheet off the table, and fix another on, drawing a line over it, in a part the most convenient for the rest of the work; then fold or cut the old sheet by the line drawn on it, applying the edge to the line on the new sheet, and, as they lie in that position, continue the last station line on the new paper, placing on it the rest of the measure, beginning at where the old sheet left off. And so on from sheet to sheet.

When the work is done, and you would fasten all the sheets together into one piece, or rough plan, the aforesaid lines are to be accurately joined together, in the same manner as when the lines were transferred from the old sheets to the new ones. But it is to be noted, that if the said joining lines, on the old and new sheets, have not the same inclination to the side of the table, the needle will not point to the original degree when the table is rectified; and if the needle be required to respect still the same degree of the compass, the easiest way of drawing the line in the same position, is to draw them both parallel to the same sides of the table, by means of the equal divisions marked on the other two sides.

3. OF THE THEODOLITE.

The theodolite is a brazen circular ring, divided into 360 degrees, &c. and having an index with sights, or a telescope, placed on the centre, about which the index is moveable; also a compass fixed to the centre, to point out courses and check the sights; the whole being fixed by the centre on a stand of a convenient height for use.

In using this instrument, an exact account, or field-book, of all measures and things necessary to be remarked in the plan, must be kept, from which to make out the plan on returning home from the ground.

Begin at such part of the ground, and measure in such directions as are judged most convenient; taking angles or directions to objects, and measuring such distances as appear necessary, under the same restrictions as in the use of the plain table. And it is safest to fix the theodolite in the original position at every station, by means of fore and back objects, and the compass, exactly as in using the plain table; registering the number of degrees cut off by the index when directed to each object; and, at any station, placing the index at the same degree as when the direction towards that station was taken from the last preceding one, to fix the theodolite there in the original position.

The best method of laying down the aforesaid lines of direction, is to describe a pretty large circle; then quarter it, and lay on it the several numbers of degrees cut off by the index in each direction, and drawing lines from the centre to all these marked points in the circle. Then, by means of a parallel ruler, draw from station to station, lines parallel to the aforesaid lines drawn from the centre to the respective points in the circumference.

4. OF THE CROSS.

The cross consists of two pair of sights set at right angles to each other, on a staff having a sharp point at the bottom, to fix in the ground.

The cross is very useful to measure small and crooked pieces of ground. The method is, to measure a base or chief line, usually in the longest direction of the piece, from corner to corner; and while measuring it, finding the places where perpendiculars would fall on this line, from the several corners and bends in the boundary of the piece, with the cross, by fixing it, by trials, on such parts of the line, as that through one pair of the sights both ends of the line may appear, and through the other pair the corresponding bends or corners : and then measuring the lengths of the said per-pendiculars.

REMARKS.

Besides the fore-mentioned instruments, which are most commonly used, there are some others ; as,

The perambulator, used for measuring roads, and other great distances, level ground, and by the sides of rivers. It has a wheel of 8¼ feet, or half a pole, in circumference, by the turning of which the machine goes forward ; and the distance measured is pointed out by an index, which is moved round by clock-work.

Levels, with telescopic or other sights, are used to find the level between place and place, or how much one place is higher or lower than another. And in measuring any sloping or oblique line, either ascending or descending, a small pocket level is useful for showing how many links for each chain are to be deducted, to reduce the line to the horizontal length.

An offset-staff is a very useful instrument, for measuring the offsets and other short distances. It is 10 links in length, being divided and marked at each of the 10 links.

Ten small arrows, or rods of iron or wood, are used to mark the end of every chain length, in measuring lines. And sometimes pickets, or staves with flags, are set up as marks or objects of direction.

Various scales are also used in protracting and measuring on the plan or paper ; such as plane scales, line of chords, protractor, compasses, reducing scale, parallel and perpendicular rules, &c. Of plane scales, there should be several sizes, as a chain in 1 inch, a chain in ¾ of an inch, a chain in ½ an inch, &c. And of these, the best for use are those that are laid on the very edges of the ivory scale, to mark off distances, without compasses.

SECTION II.

THE PRACTICE OF SURVEYING.

This part contains the several works proper to be done in the field, or the ways of measuring by all the instruments, and in all situations.

PROBLEM I.

To measure a Line or Distance.

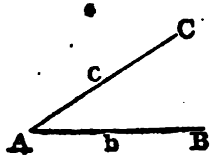
To measure a line on the ground with the chain: Having provided a chain, with 10 small arrows, or rods, to fix one into the ground, as a mark, at the end of every chain; two persons take hold of the chain, one at each end of it; and all the 10 arrows are taken by one of them, who goes foremost, and is called the leader; the other being called the follower, for distinction's sake.

A picket, or station-staff, being set up in the direction of the line to be measured, if there do not appear some marks naturally in that direction, they measure straight towards it, the leader fixing down an arrow at the end of every chain, which the follower always takes up, as he comes at it, till all the ten arrows are used. They are then all returned to the leader, to use over again. And thus the arrows are changed from the one to the other at every 10 chains length, till the whole line is finished; then the number of changes of the arrows shows the number of tens, to which the follower adds the arrows he holds in his hand, and the number of links of another chain over to the mark or end of the line. So, if there have been 3 changes of the arrows, and the follower hold 6 arrows, and the end of the line cut off 45 links more, the whole length of the line is set down in links thus, 3645.

When the ground is not level, but either ascending or descending; at every chain length, lay the offset-staff, or link-staff, down in the slope of the chain, on which lay the small pocket level, to show now many links or parts the slope line is longer than the true level one; then draw the chain forward so many links or parts, which reduces the line to the horizontal direction.

PROBLEM II.

To take Angles and Bearings.



Let B and C be two objects, or two pickets set up perpendicular; and let it be required to take their bearings, or the angles formed between them at any station A.

1. *With the Plain Table.*

The table being covered with a paper, and fixed on its stand; plant it at the station A, and fix a fine pin, or a foot of the compasses, in a proper point of the paper, to represent the place A: Close by the side of this pin lay the fiducial edge of the index, and turn it about, still touching the pin, till one object B can be seen through the sights: then by the fiducial edge of the index draw a line. In the same manner draw another line in the direction of the other object C. And it is done.

2. *With the Theodolite, &c.*

Direct the fixed sights along one of the lines, as AB, by turning the instrument about till the mark B is seen through these sights; and there screw the instrument fast. Then turn the moveable index round, till through its sights the other mark C is seen. Then the degrees cut by the index, on the graduated limb or ring of the instrument, show the quantity of the angle.

3. *With the Magnetic Needle and Compass.*

Turn the instrument or compass so, that the north end of the needle point to the flower-de-luce. Then direct the sights to one mark as B, and note the degrees cut by the needle. Next direct the sights to the other mark C, and note again the degrees cut by the needle. Then their sum or difference, as the case may be, will give the quantity of the angle BAC.

4. *By Measurement with the Chain, &c.*

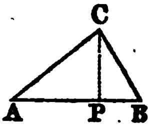
Measure one chain length, or any other length, along both directions, as to *b* and *c*. Then measure the distance *bc*, and it is done.—This is easily transferred to paper, by making a triangle A*bc* with these three lengths, and then measuring the angle A.

PROBLEM III.

To survey a Triangular Field ABC.

1. *By the Chain.*

AP	794
AB	1321
PC	826



Having set up marks at the corners, which is to be done in all cases where there are not marks naturally; measure with the chain from A to P, where a perpendicular would fall from the angle C, and set up a mark at P, noting down the distance AP. Then complete the distance AB, by measuring from P to B. Having set down this measure, return to P, and measure the perpendicular PC. And thus, having the base and perpendicular, the area from them is easily found. Or having the place P of the perpendicular, the triangle is easily constructed.

Or, measure all the three sides with the chain, and note them down. From which the content is easily found, or the figure is constructed.

2. *By taking some of the Angles.*

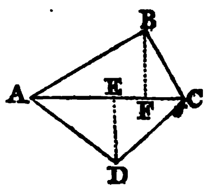
Measure two sides AB, AC, and the angle A between them. Or measure one side AB, and the two adjacent angles A and B. From either of these ways the figure is easily planned; then by measuring the perpendicular CP on the plan, and multiplying it by half AB, the content is found.

PROBLEM IV.

To Measure a Four-sided Field.

1. *By the Chain.*

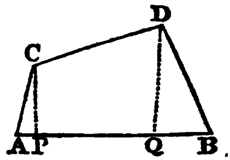
AE	214	210	DE
AF	362	306	BF
AC	592		



Measure along one of the diagonals, as AC; and either the two perpendiculars DE, BF, as in the last problem; or else the sides AB, BC, CD, DA. From either of which the figure may be planned and computed as before directed.

Otherwise, by the Chain.

AP	110	352	PC
AQ	745	595	QD
AB	1110		



Measure, on the longest side, the distances AP, AQ, AB; and the perpendiculars PC, QD.

2. By taking some of the Angles.

Measure the diagonal AC (see the last fig. but one), and the angles CAB, CAD, ACB, ACD.—Or measure the four sides, and any one of the angles, as BAD.

Thus.		Or thus.	
AC	591	AB	486
CAB	37° 20′	BC	394
CAD	41 15	CD	410
ACB	72 25	DA	462
ACD	54 40	BAD	78° 35′.

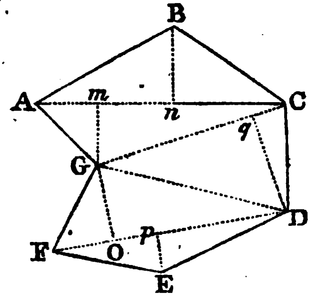
PROBLEM. V.

To survey any Field by the Chain only.

HAVING set up marks at the corners, where necessary, of the proposed field ABCDEFG, walk over the ground, and consider how it can best be divided into triangles and trapeziums; and measure them separately, as in the last two problems. Thus, the following figure is divided into the two trapeziums ABCG, GDEF, and the triangle GCD. Then, in the first trapezium, beginning at A, measure the diagonal AC, and the two perpendiculars G*m*, B*n*. Then the base GC, and the perpendicular D*q*. Lastly, the diagonal DF, and the two perpendiculars *p*E, OG. All which measures write against the corresponding parts of a rough figure drawn to resemble the figure surveyed, or set them down in any other form you choose.

Thus.

AM	135	130	MG
AN	410	180	NB
AC	550		
CQ	152	230	QD
CG	440		
FO	237	120	OG
FP	288	80	PE
FD	520		



Or thus.

Measure all the sides AB, BC, CD, DE, EF, FG, GA; and the diagonals AC, CG, GD, DF.

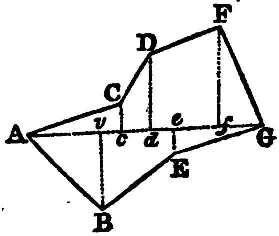
Otherwise.

Many pieces of land may be very well surveyed, by measuring any base line, either within or without them, with the perpendiculars let fall on it from every corner. For they are by those means divided into several triangles and trapezoids, all whose parallel sides are perpendicular to the base line; and the sum of these triangles and trapeziums will be equal to the figure proposed if the base line fall within it; if not, the sum of the parts which are without being taken from the sum of the whole which are both within and without, will leave the area of the figure proposed.

In pieces that are not very large, it will be sufficiently exact to find the points, in the base line, where the several perpendiculars will fall, by means of the *cross*, or even by judging by the eye only, and from thence measuring to the corners for the lengths of the perpendiculars.—And it will be most convenient to draw the line so as that all the perpendiculars may fall within the figure.

Thus, in the following figure, beginning at A, and measuring along the line AG, the distances and perpendiculars on the right and left are as below.

A*b*	315	350	*b*B
A*c*	440	70	*c*C
A*d*	585	320	*d*D
A*e*	610	50	*e*E
A*f*	990	470	*f*F
AG	1020	0	

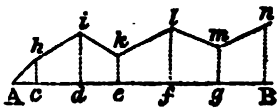


PROBLEM VI.

To measure the Offsets.

A*hiklmn* being a crooked hedge, or brook, &c. From A measure in a straight direction along the side of it to B. And in measuring along this line AB, observe when you are directly opposite any bends or corners of the boundary, as at *c, d, e,* &c.; and from these measure the perpendicular offsets *ch, di,* &c. with the offset-staff, if they are not very large, otherwise with the chain itself; and the work is done. The register, or field-book, may be as follows:

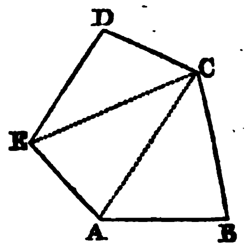
Offs. left.		Base line AB	
	0	⊙	A
ch	62	45	A*c*
di	84	220	A*d*
ek	70	340	A*e*
fl	98	510	A*f*
gm	57	634	A*g*
B*n*	91	785	AB



PROBLEM VII.

To survey any Field with the Plain Table.

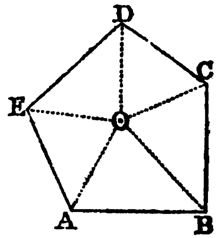
1. *From one Station.*



PLANT the table at any angle as C, from which all the other angles, or marks set up, can be seen; turn the table about till the needle point to the flower-de-luce; and there screw it fast. Make a point for C on the paper on the table, and lay the edge of the index to C, turning it about C till through the sights you see the mark D: and by the edge of the index draw a dry or obscure line: then measure the distance CD, and lay that distance down on the line CD. Then turn the index about the point C, till the mark E be seen through the sights, by which draw a line and measure the distance to E, laying it on the line from C to E. In like manner determine the positions of CA and CB, by turning the sights successively to

A and B; and lay the length of those lines down. Then connect the points, by drawing the black lines CD, DE, EA, AB, BC, for the boundaries of the field.

From a Station within the Field.



When all the other parts cannot be seen from one angle, choose some place O within, or even without, if more convenient, from which the other parts can be seen. Plant the table at O, then fix it with the needle north, and mark the point O on it. Apply the index successively to O, turning it round with the sights to each angle, A, B, C, D, E, drawing dry lines to them by the edge of the index; then measuring the distances OA, OB, &c. and laying them down on those lines. Lastly, draw the boundaries AB, BC, CD, DE, EA.

3. *By going round the Figure.*

When the figure is a wood, or water, or when from some other obstruction you cannot measure lines across it; begin at any point A, and measure around it, either within or without the figure, and draw the directions of all the sides, thus: Plant the table at A; turn it with the needle to the north or flower-de-luce; fix it, and mark the point A. Apply the index to A, turning it till you can see the point E, and there draw a line: then the point B, and there draw a line: then measure these lines, and lay them down from A to E and B. Next move the table to B, lay the index along the line AB, and turn the table about till you can see the mark A, and screw fast the table; in which position also the needle will again point to the flower-de-luce, as it will do indeed at every station when the table is in the right position. Here turn the index about B till through the sights you seek the mark C; there draw a line, measure BC, and lay the distance on that line after you have set down the table at C. Turn it then again into its proper position, and in like manner find the next line CD. And so on quite around by E, to A again. Then the proof of the work will be the joining at A: for if the work be all right, the last direction EA on the ground, will pass exactly through the point A on the paper; and the measured distance will also reach exactly to A. If these do not coincide, or nearly so, some error has been committed, and the work must be examined over again.

PROBLEM VIII.

To survey a Field with the Theodolite, &c.

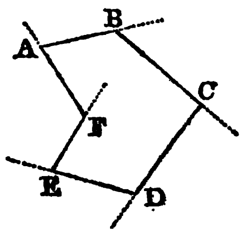
1. From One Point or Station.

When all the angles can be seen from one point, as the angle C (first fig. to last prob.), place the instrument at C, and turn it about, till through the fixed sights you see the mark B, and there fix it. Then turn the moveable index about till the mark A be seen through the sights, and note the degrees cut on the instrument. Next turn the index successively to E and D, noting the degrees cut off at each; which gives all the angles BCA, BCE, BCD. Lastly, measure the lines CB, CA, CE, CD; and enter the measures in a field-book, or rather, against the corresponding parts of a rough figure drawn by guess to resemble the field.

2. From a Point within or without.

Plant the instrument at O (last fig.), and turn it about till the fixed sights point to any object, as A; and there screw it fast. Then turn the moveable index round till the sights point successively to the other points E, D, C, B, noting the degrees cut off at each of them; which gives all the angles round the point O. Lastly, measure the distances OA, OB, OC, OD, OE, noting them down as before, and the work is done.

3. By going round the Field.



By measuring round, either within or without the field, proceed thus. Having set up marks at B, C, &c. near the corners as usual, plant the instrument at any point A, and turn it till the fixed index be in the direction AB, and there screw it fast: then turn the moveable index to the direction AC; and the degrees cut off will be the angle A. Measure the line AB, and plant the instrument at B, and there in the same manner observe the angle A. Then measure BC, and observe the angle C. Then measure the distance CD, and take the angle D. Then measure DE, and take the angle E. Then measure EF, and take the angle F. And lastly, measure the distance FA.

To prove the work; add all the inward angles, A, B, C, &c. together; for when the work is right, their sum will be equal to twice as many right angles as the figure has sides, wanting 4 right angles. But when there is an angle, as F, that bends inwards, and you measure the external angle, which is less than two right angles, subtract it from 4 right angles, or 360 degrees, to give the internal angle greater than a semicircle or 180 degrees.

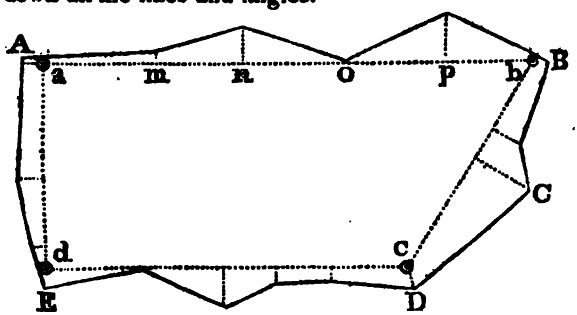
Otherwise.

Instead of observing the internal angles, we may take the external angles, formed without the figure by producing the sides farther out. And in this case, when the work is right, their sum altogether will be equal to 360 degrees. But when one of them, as F, runs inwards, subtract it from the sum of the rest, to leave 360 degrees.

PROBLEM IX.

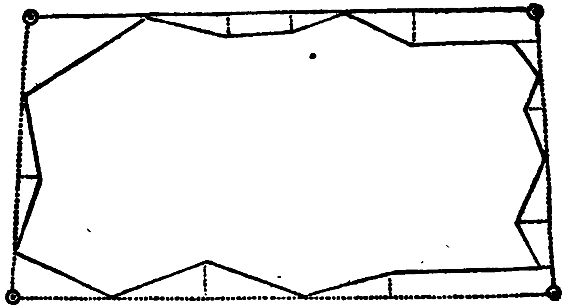
To survey a Field with crooked Hedges, &c.

WITH any of the instruments, measure the lengths and positions of imaginary lines running as near the sides of the field as you can; and in going along them, measure the offsets in the manner before taught; then you will have the plan on the paper in using the plain table, drawing the crooked hedges through the ends of the offsets; but in surveying with the theodolite, or other instrument, set down the measures properly in a field-book, or memorandum-book, and plan them after returning from the field, by laying down all the lines and angles.



So in surveying the piece ABCDE, set up marks, *a*, *b*, *c*, *d*, dividing it so as to have as few sides as may be. Then begin at any station, *a*, and measure the lines *ab*, *bc*, *cd*, *da*, taking their positions, or the angles. *a*, *b*, *c*, *d*; and, in going along the lines, measure all the offsets, as at *m*, *n*, *o*, *p*, &c. along every station-line.

And this is done either within the field, or without, as may be most convenient. When there are obstructions within, as wood, water, hills, &c. then measure without, as in the next following figure.



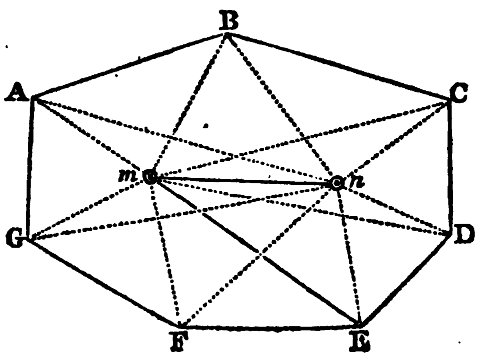
PROBLEM X.

To Survey a Field, or any other Thing, by two Stations.

This is performed by choosing two stations from which all the marks and objects can be seen; then measuring the distance between the stations, and at each station, taking the angles formed by every object from the station line or distance.

The two stations may be taken either within the bounds, or in one of the sides, or in the direction of two of the objects, or quite at a distance and without the bounds of the objects or part to be surveyed.

In this manner, not only grounds may be surveyed, without even entering them, but a map may be taken of the principal parts of a county, or the chief places of a town, or any part of a river or coast surveyed, or any other inaccessible objects; by taking two stations, on two towers, or two hills, or such-like.



PROBLEM XI.

To survey a large Estate.

If the estate be very large, and contain a great number of fields, it cannot well be done by surveying all the fields singly, and then putting them together; nor can it be done by taking all the angles and boundaries that enclose it. For in these cases, any small errors will be so much increased, as to render it very much distorted. But proceed as below.

1. Walk over the estate two or three times, in order to get a perfect idea of it, or till you can keep the figure of it pretty well in mind. And to help your memory, draw an eye-draught of it on paper, at least of the principal parts of it, to guide you; setting the names within the fields in that draught.

2. Choose two or more eminent places in the estate, for stations, from which all the principal parts of it can be seen: selecting these stations as far distant from one another as convenient.

3. Take such angles, between the stations, as you think necessary, and measure the distances from station to station, always in a right line: these things must be done, till you get as many angles and lines as are sufficient for determining all the points of station. And in measuring any of these station-distances, mark accurately where these lines meet with any hedges, ditches, roads, lanes, paths, rivulets, &c.; and where any remarkable object is placed, by measuring its distance from the station-line; and where a perpendicular from it cuts that line. And thus as you go along any main station-line, take offsets to the ends of all hedges, and to any pond, house, mill, bridge, &c. noting every thing down that is remarkable.

4 As to the inner parts of the estate, they must be determined, in like manner, by new station-lines; for, after the main stations are determined, and every thing adjoining to them, then the estate must be subdivided into two or three parts by new station-lines; taking inner stations at proper places, where you can have the best view. Measure these station-lines as you did the first, and all their intersections with hedges, and offsets to such objects as appear. Then proceed to survey the adjoining fields, by taking the angles that the sides make with the station-line, at the intersections, and measuring the distances to each corner, from the intersections. For the station-lines will be the bases to all the future operations; the situation of all parts being entirely

dependent on them; and therefore they should be taken of as great length as possible; and it is best for them to run along some of the hedges or boundaries of one or more fields, or to pass through some of their angles. All things being determined for these stations, you must take more inner stations, and continue to divide and subdivide till at last you come to single fields: repeating the same work for the inner stations as for the outer ones, till all is done; and close the work as often as you can, and in as few lines as possible.

5. An estate may be so situated that the whole cannot be surveyed together; because one part of the estate cannot be seen from another. In this case, you may divide it into three or four parts, and survey the parts separately, as if they were lands belonging to different persons; and at last join them together.

6. As it is necessary to protract or lay down the work as you proceed in it, you must have a scale of a due length to do it by. To get such a scale, measure the whole length of the estate in chains; then consider how many inches long the map is to be; and from these will be known how many chains you must have in an inch; then make the scale accordingly, or choose one already made.

PROBLEM XII.

To survey a County, or large Tract of Land.

1. Choose two, three, or four eminent places, for stations; such as the tops of high hills or mountains, towers, or church steeples, which may be seen from one another; from which most of the towns and other places of note may also be seen; and so as to be as far distant from one another as possible. On these places raise beacons, or long poles, with flags of different colours flying at them, so as to be visible from all the other stations.

2. At all the places which you would set down in the map, plant long poles, with flags at them of several colours, to distinguish the places from one another; fixing them on the tops of church steeples, or the tops of houses; or in the centres of smaller towns and villages.

These marks then being set up at a convenient number of places, and such as may be seen from both stations; go to one of these stations, and, with an instrument to take angles, standing at that station, take all the angles between the other station and each of these marks. Then go to the other station, and take all the angles between the first station and each of the former marks, setting them down with the others, each against its fellow with the same colour. You may, if convenient, also take the angles at some third station, which may serve to prove the work, if the three lines intersect in that point where any mark stands. The marks must stand till the observations are finished at both stations; and then they may be taken down, and set up at new places. The same operations must be performed at both stations, for these new places; and the like for others. The instrument for taking angles must be an exceeding good one, made on purpose with telescopic sights, and of a good length of radius.

3. And though it be not absolutely necessary to measure any distance, because, a stationary line being laid down from any scale, all the other lines will be proportional to it; yet it is better to measure some of the lines, to ascertain the distances of places in miles, and to know how many geometrical miles there are in any length; as also from thence to make a scale to measure any distance in miles. In measuring any distance, it will not be exact enough to go along the high roads; which, by reason of their turnings and windings, hardly ever lie in a right line between the stations; which must cause endless reductions, and require great trouble to make it a right line; for which reason it can never be exact. But a better way is to measure in a straight line with a chain, between station and station, over hills and dales, or level fields and all obstacles. Only in case of water, woods, towns, rocks, banks, &c. where we cannot pass, such parts of the line must be measured by the methods of inaccessible distances; and besides, allowing for ascents and descents, when they are met with. A good compass, that shows the bearing of the two stations, will always direct us to go straight, when the two stations cannot be seen; and in the progress, if we can go straight, offsets may be taken to any remarkable places, likewise noting the intersection of the station-line with all roads, rivers, &c.

4. From all the stations, and in the whole progress, we must be very particular in observing sea-coasts, river-mouths, towns, castles, houses, churches, mills, trees, rocks, sands, roads, bridges, fords, ferries, woods, hills, mountains, rills, brooks, parks, beacons, sluices, floodgates, locks, &c., and in general every thing that is remarkable.

5. After we have done with the first and main station-lines, which command the whole county: we must then take inner stations, at some places already determined; which will divide the whole into several partitions: and from these stations we must determine the places of as many of the

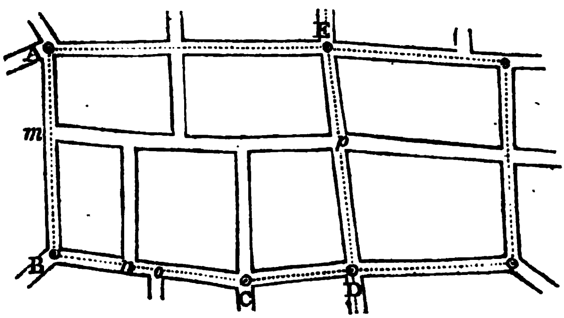
remaining towns as we can. And if any remain in that part, we must take more stations, at some places already determined; from which we may determine the rest. And thus go through all the parts of the county, taking station after station, till we have determined the whole. And in general the station distances must always pass through such remarkable points as have been determined before, by the former stations.

PROBLEM XIII.

To survey a Town or City.

THIS may be done with any of the instruments for taking angles, but best of all with the plain table, where every minute part is drawn while in sight. Instead of the common surveying or Gunter's chain, it will be best, for this purpose, to have a chain 50 feet long, divided into 50 links of one foot each, and an offset-staff of 10 feet long.

Begin at the meeting of two or more of the principal streets, through which you can have the longest prospects, to get the longest station-lines: there having fixed the instrument, draw lines of direction along those streets, using two men as marks, or poles set in wooden pedestals, or perhaps some remarkable places in the houses at the farther ends, as windows, doors, corners, &c. Measure those lines with the chain, taking offsets with the staff, at all corners of streets, bendings, or windings, and to all remarkable things, as churches, markets, halls, colleges, eminent houses, &c. Then remove the instrument to another station, along one of these lines; and there repeat the same process as before. And so on till the whole is finished.



Thus, fix the instrument at A, and draw lines in the direction of all the streets meeting there; then measure AB, noting the street on the left at *m*. At the second station B, draw the directions of the streets meeting there; and measure from B to C, noting the places of the streets at *n* and *o* as you pass by them. At the third station C, take the direction of all the streets meeting there, and measure CD. At D do the same, and measure DE, noting the place of the cross streets at *p*. And in this manner go through all the principal streets. This done, proceed to the smaller and intermediate streets; and lastly to the lanes, alleys, courts, yards, and every part that it may be thought proper to represent in the plan.

THEOREM XIV.

To lay down the Plan of any Survey.

If the survey was taken with the plain table, we have a rough plan of it already on the paper which covered the table. But if the survey was with any other instrument, a plan of it is to be drawn from the measures that were taken in the survey; and first of all a rough plan on paper.

To do this, you must have a set of proper instruments, for laying down both lines and angles, &c.; as scales of various sizes, the more of them, and the more accurate, the better, scales of chords, protractors, perpendicular and parallel rulers, &c. Diagonal scales are best for the lines, because they extend to three figures, or chains, and links, which are 100 parts of chains. But in using the diagonal scale, a pair of compasses must be employed, to take off the lengths of the principal lines very accurately. But a scale with a thin edge divided, is much readier for laying down the perpendicular offsets to crooked hedges, and for marking the places of those offsets on the station-line; which is done at only one application of the edge of the scale to that line, and then pricking off all at once the distances along it. Angles are to be laid down, either with a good scale of chords, which is perhaps the most accurate way, or with a large protractor, which is much readier when many angles are to be laid down at one point, as they are pricked off all at once round the edge of the protractor.

In general, all lines and angles must be laid down on the plan in the same order in which they were measured in the field, and in which they are written in the field-book; laying down first the angles for the position of lines, next the lengths of the lines, with the places of the offsets, and then the lengths of the offsets themselves, all with dry or obscure lines; then a black line drawn through the extremities of all the offsets, will be the edge or bounding line of the field, &c. After the principal bounds and lines are laid down, and made to fit or close properly, proceed next to the smaller objects, till you have entered every thing that ought to appear in the plan, as houses, brooks, trees, hills, gates, stiles, roads, lanes, mills, bridges, woodlands, &c. &c.

The north side of a map or plan is commonly placed uppermost, and a meridian is somewhere drawn, with the compass or flower-de-luce pointing north. Also, in a vacant part, a scale of equal parts or chains is drawn, with the title of the map in conspicuous characters, and embellished with a compartment. Hills are shadowed, to distinguish them in the map. Colour the hedges with different colours; represent hilly grounds by broken hills and valleys; draw single dotted lines for foot-paths, and double ones for horse or carriage roads. Write the name of each field and remarkable place within it, and, if you choose, its contents in acres, roods, and perches.

In a very large estate, or a county, draw vertical and horizontal lines through the map, denoting the spaces between them by letters placed at the top, and bottom, and sides, for readily finding any field or other object mentioned in a table.

In mapping counties, and estates that have uneven grounds of hills and valleys, reduce all oblique lines, measured uphill and down-hill, to horizontal straight lines, if that was not done during the survey, before they were entered in the field-book, by making a proper allowance to shorten them. For which purpose there is commonly a small table engraven on some of the instruments for surveying.

THE NEW METHOD OF SURVEYING.

PROBLEM XV.

To survey and plan by the new Method.

In the former method of measuring a large estate, the accuracy of it depends both on the correctness of the instruments, and on the care in taking the angles. To avoid the errors incident to such a multitude of angles, other methods have of late years been used by some few skilful surveyors: the most practical, expeditious, and correct, seems to be the following, which is performed, without taking angles, by measuring with the chain only.

Choose two or more eminences, as grand stations, and measure a principal base line from one station to another; noting every hedge, brook, or other remarkable object, as you pass by it; measuring also such short perpendicular lines to the bends of hedges as may be near at hand. From the extremities of this base line, or from any convenient parts of the same, go off with other lines to some remarkable object situated towards the sides of the estate, without regarding the angles they make with the base line or with one another; still remembering to note every hedge, brook, or other object, that you pass by. These lines, when laid down by intersections, will, with the base line, form a grand triangle on the estate; several of which, if need be, being thus measured and laid down, you may proceed to form other smaller triangles and trapezoids on the sides of the former: and so on till you finish with the enclosures individually. By which means a kind of skeleton of the estate may first be obtained, and the chief lines serve as the bases of such triangles and trapezoids as are necessary to fill up all the interior parts.

The field-book is ruled into three columns, as usual. In the middle one are set down the distances on the chain-line, at which any mark, offset, or other observation, is made; and in the right and left hand columns are entered the offsets and observations made on the right and left hand respectively of the chain-line; sketching on the sides the shape or resemblance of the fences or boundaries.

It is of great advantage, both for brevity and perspicuity, to begin at the bottom of the leaf, and write upwards; denoting the crossing of fences, by lines drawn across the middle column, or only a part of such a line on the right and left opposite the figures, to avoid confusion; and the corners of fields, and other remarkable turns in the fences where offsets are taken to, by lines joining in the manner the fences do; as will be best seen by comparing the book with the plan annexed to the field-book following, p. 454.

The letter in the left-hand corner at the beginning of every line, is the mark or place measured *from*; and that at the right-hand corner at the end, is the mark measured *to*: but when it is not convenient to go exactly from a mark, the place measured from is described *such a distance* from *one mark* towards *another*; and where a former mark is not measured to, the exact place is ascertained by saying, turn to the right or left hand, *such a distance to such a mark*, it being always understood that those distances are taken in the chain-line.

The characters used are, ⌈ for *turn to the right hand*, ⌉ for *turn to the left hand*, and ⁀ placed over an offset, to show that it is not taken at right angles with the chain-line, but in the direction of some straight fence; being chiefly used when crossing their directions; which is a better way of obtaining their true places than by offsets at right angles.

When a line is measured whose position is determined, either by former work (as in the case of producing a given line, or measuring from one known place or mark to another) or by itself (as in the third side of the triangle), it is called a *fast line*, and a double line across the book is drawn at the conclusion of it; but if its position is not determined (as in the second side of the triangle), it is called a *loose line*, and a single line is drawn across the book. When a line becomes determined in position, and is afterwards continued farther, a double line half through the book is drawn.

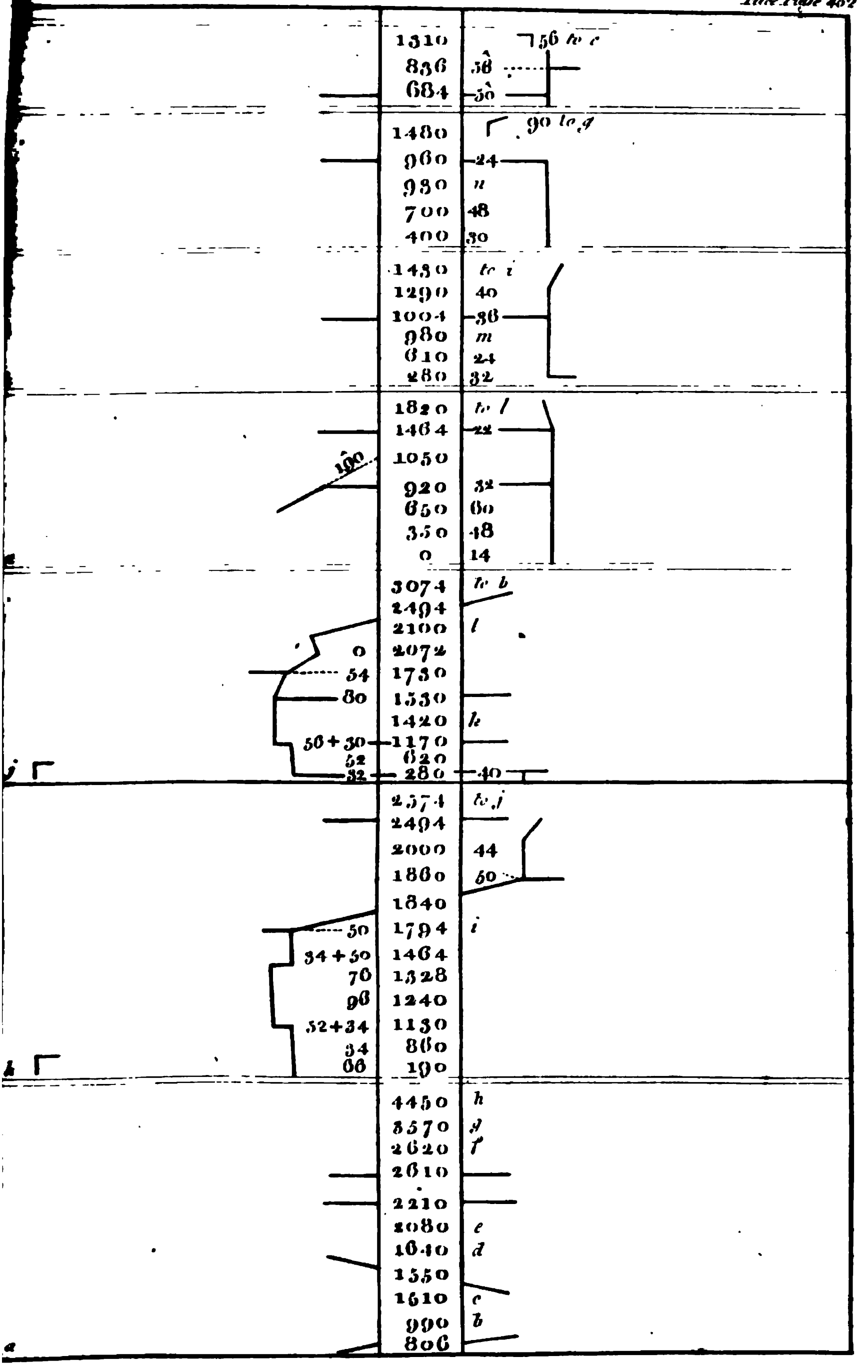
When a loose line is measured, it becomes absolutely necessary to measure some other line that will determine its position. Thus, the first line *ah* or *bh*, being the base of a triangle, is always determined; but the position of the second side *hj* does not become determined, till the third side *jb* is measured; then the position of both is determined, and the triangle may be constructed.

At the beginning of a line, to fix a loose line to the mark or place measured from, the sign of turning to the right or left hand must be added, as at *h* in the second, and *j* in the third line; otherwise a stranger, when laying down the work, may as easily construct the triangle *hjb* on the wrong side of the line *ah*, as on the right one: but this error cannot be fallen into, if the sign above named be carefully observed.

In choosing a line to fix a loose one, care must be taken that it does not make a very acute or obtuse angle; as in the triangle *p*B*r*, by the angle at B being very obtuse, a small deviation from truth, even the breadth of a point at *p* or *r*, would make the error at B, when constructed, very considerable; but by constructing the triangle *p*B*q*, such a deviation is of no consequence.

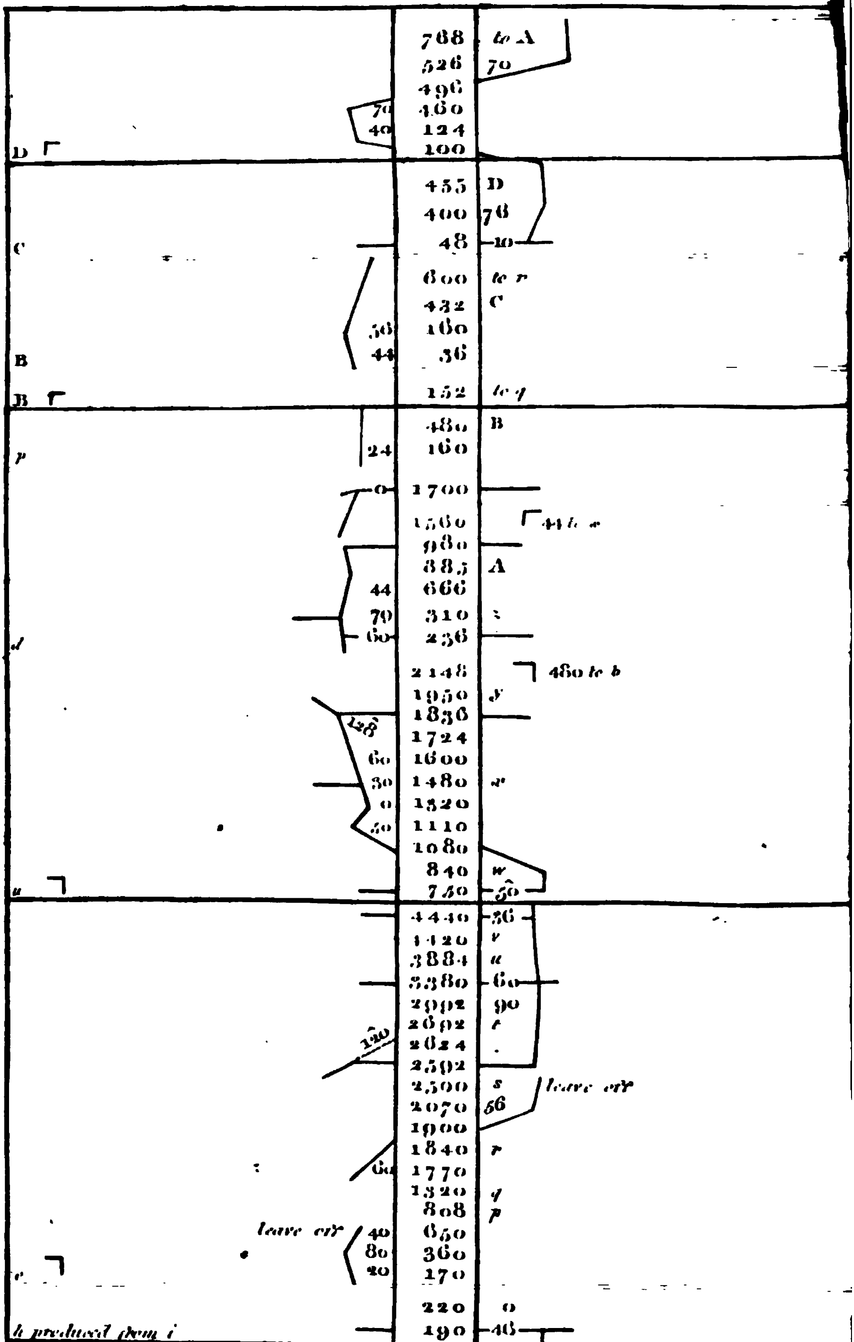
Where the words *leave off* are written in the field-book, it signifies that the taking of offsets is from thence discontinued; and of course something is wanting between that and the next offset, to be afterwards determined by measuring some other line.

The field-book for this method, and the plan drawn from it, are contained in the four following pages, engraven on copper plates; answerable to which the pupil is to draw a

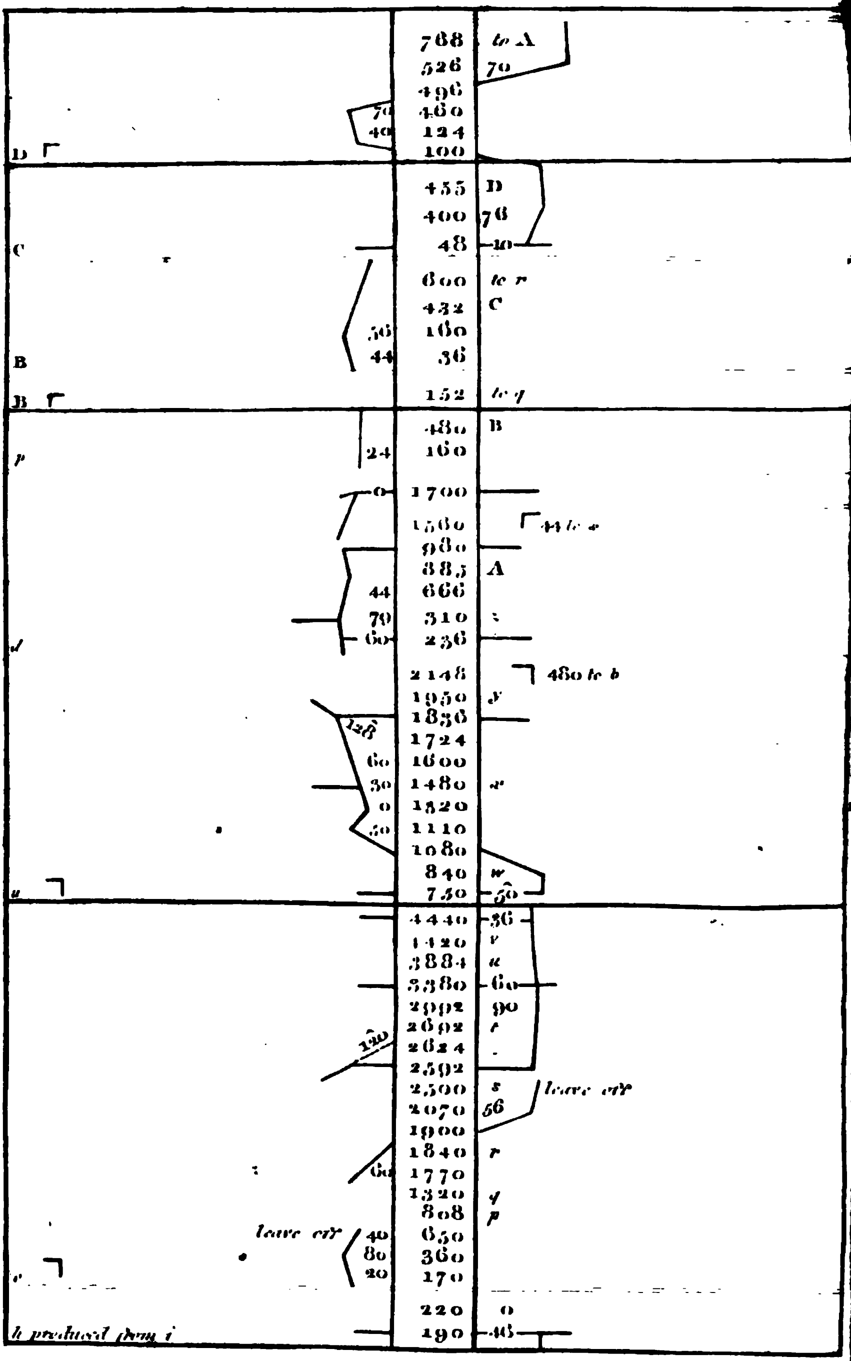


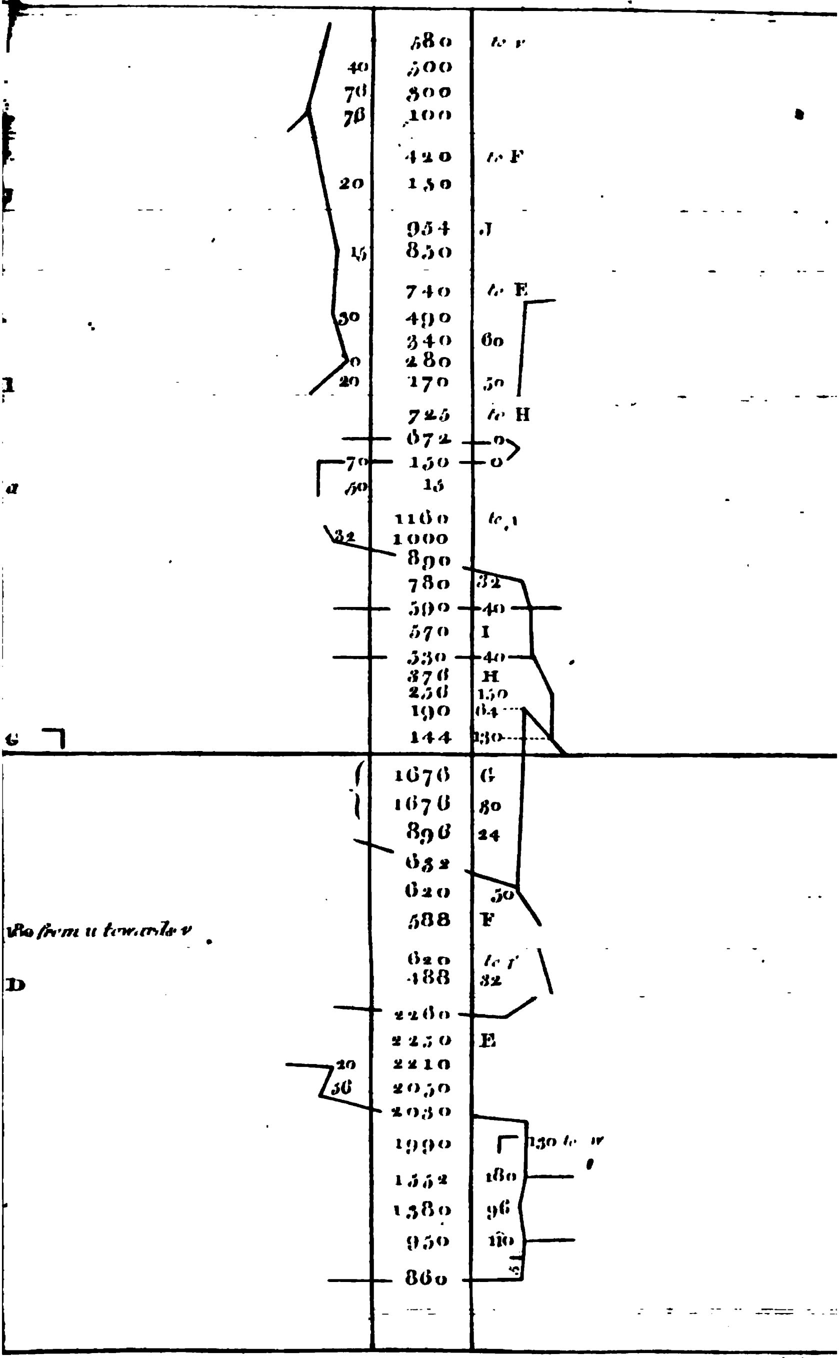
1310 56 to c
836 58
684 50
1480 90 to q
960 24
930 n
700 48
400 30
1430 to i
1290 40
1004 38
980 m
610 24
280 32
1820 to l
1464 22
100
1050
920 32
650 60
350 48
0 14
3074 to b
2494
2100 l
0 2072
54 1730
80 1530
1420 k
56+30 1170
52 620
32 280 40
j
2574 to j
2494
2000 44
1860 50
1840
50 1794 i
34+50 1464
70 1328
96 1240
52+34 1130
34 860
66 190
h
4450 h
3570 g
2620 f
2610
2210
2080 e
1640 d
1550
1610 c
990 b
806
a

1.

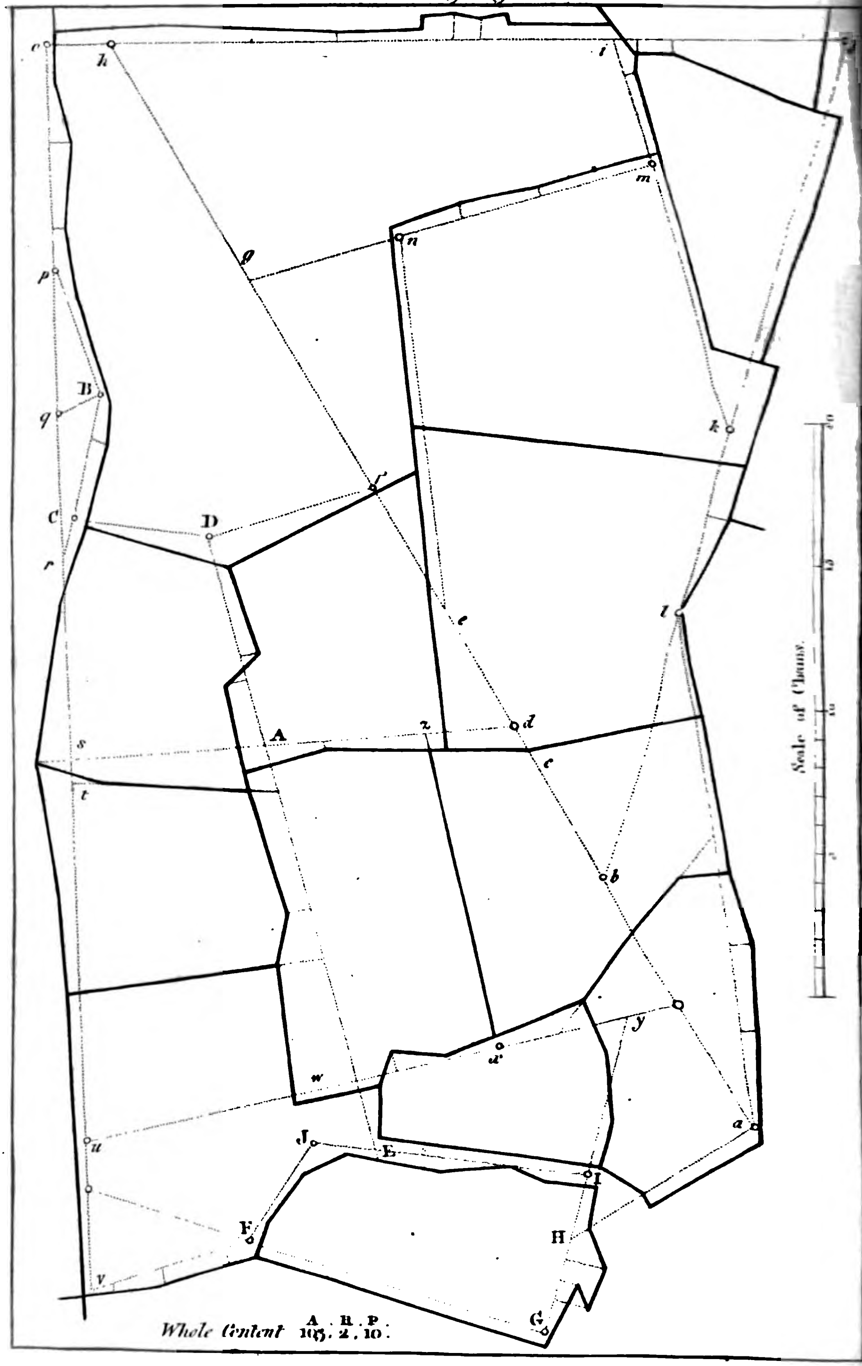

788 to A
526 70
496
70 460
40 124
D 100
455 D
400 78
C 48 10
600 to r
432 C
56 160
B 44 36
B 152 to q
480 B
p 24 160
0 1700
1560 44 to x
980
885 A
44 666
79 310
d 60 236
2148 480 to b
1950 y
128 1836
1724
60 1600
30 1480 x
0 1320
50 1110
1080
840 w
u 740 50
4440 36
4420 v
3884 u
3380 60
2992 90
2692 t
120 2624
2592
2500 s leave off
2070 56
1900
1840 r
60 1770
1320 q
808 p
leave off 40 650
80 360
c 20 170
220 o
h produced from i 190 46

	580	to v
40	500	
70	300	
70	100	
	420	to F
20	150	
	954	J
15	850	
	740	to E
30	490	
	340	60
0	280	
20	170	50
	725	to H
	672	0
70	150	0
50	15	
	1160	to v
32	1000	
	890	
	780	32
	590	40
	570	I
	530	40
	376	H
	256	150
	190	64
G	144	130
	1676	G
	1676	80
	896	24
	632	
	620	50
180 from u towards v	588	F
	620	to f
D	488	32
	2260	
	2250	E
20	2210	
36	2050	
	2030	
	1990	130 to w
	1552	180
	1380	96
	950	110
	860	


768 to A
526 70
496
70 460
40 124
D 100
455 D
400 76
C 48 10
600 to r
432 C
56 160
B 44 36
B 152 to q
480 B
p 24 160
0 1700
1560 44 to a
980
885 A
44 666
70 310
d 60 236
2148 480 to b
1950
1836
128 1724
60 1600
30 1480
0 1320
30 1110
1080
840 w
u 750 50
4440 36
4420 v
3884 u
3380 60
2992 90
2692 t
120 2624
2592
2500 s leave off
2070 56
1900
1840 r
60 1770
1320 q
808 p
leave off 40 650
80 360
o 20 170
220 o
h produced from i 190 46


580 to v
40 500
70 300
70 100
420 to F
20 150
954 J
15 850
740 to E
30 490
340 60
0 280
20 170 30
I
725 to H
672 0
70 150 0
a
50 15
1160 to t
32 1000
890
780 32
590 40
570 I
530 40
376 H
250 150
190 64
G
144 130
1676 G
1676 80
896 24
632
620 50
588 F
180 from u towards v
620 to f
D
488 32
2260
2250 E
20 2210
56 2050
2030
1990 130 to w
1552 180
1380 96
950 110
860

Plan from the foregoing Field Book.



plan from the measures in the field-book, of a larger size, viz. to a scale of a double size will be convenient, such a scale being also found on most instruments. In doing this, begin at the commencement of the field-book, or bottom of the first page, and draw the first line *ah* in any direction at pleasure, and then the next two sides of the first triangle *bhj* by sweeping intersected arcs; and so all the triangles in the same manner, after each other in their order; and afterwards setting the perpendicular and other offsets at their proper places, and through the ends of them drawing the bounding fences.

Note. That the field-book begins at the bottom of the first page, and reads up to the top; hence it goes to the bottom of the next page, and to the top; and thence it passes from the bottom of the third page to the top, which is the end of the field-book. The several marks measured to or from, are here denoted by the letters of the alphabet, first the small ones, *a*, *b*, *c*, *d*, &c, and after them the capitals *A*, *B*, *C*, *D*, &c. But instead of these letters, some surveyors use the numbers in order, 1, 2, 3, 4, &c.

OF THE OLD KIND OF FIELD-BOOK.

In surveying with the plain table, a field-book is not used, as every thing is drawn on the table immediately when it is measured. But in surveying with the theodolite, or any other instrument, some kind of a field book must be used to write down in it a register or account of all that is done and occurs relative to the survey in hand.

This book every one contrives and rules as he thinks fittest for himself. The following is a specimen of a form which has been formerly used. It is ruled in three columns, as in the next page.

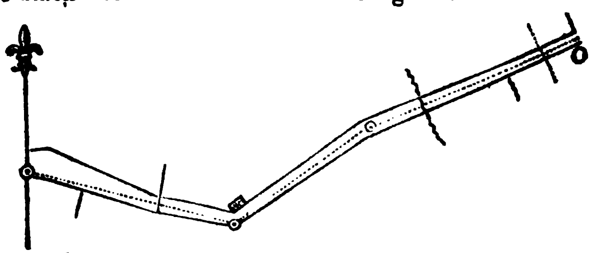
Here ⊙ 1 is the first station, where the angle or bearing is 105° 25′. On the left, at 73 links in the distance or principal line, is an offset of 92; and at 610 an offset of 24 to a cross hedge. On the right, at 0, or the beginning, an offset 25 to the corner of the field; at 248 Brown's boundary hedge commences; at 610 an offset 35; and at 954, the end of the first line, the 0 denotes its terminating in the hedge. And so on for the other stations.

A line is drawn under the work, at the end of every station line, to prevent confusion.

Form of this Field-Book.

Offsets and Remarks on the left.	Stations, Bearings, and Distances.	Offsets and Remarks on the right.
	⊙ 1	
	105° 25′	
80	00	25 corner
92	73	
	248	Brown's hedge
a cross hedge 24	610	35
	954	00
	⊙ 2	
	53° 10′	
house corner 51	25	21
	120	29 a tree
34	764	40 a stile
	⊙ 3	
	67° 20′	
	61	35
a brook 30	248	
	639	16 a spring
foot-path 16	810	
cross hedge 18	973	20 a pond

Then the plan, on a small scale drawn from the above field-book, will be as in the following figure. But the pupil may draw a plan of 3 or 4 times the size on his paper book. The dotted lines denote the 3 chain or measured lines, and the black lines the boundaries on the right and left.



But some skilful surveyors now make use of a different method for the field-book, namely, beginning at the bottom of the page and writing upwards; sketching also a neat boundary on either hand, resembling the parts near the measured lines as they pass along; an example of which was given in the new method of surveying, in the preceding pages.

In smaller surveys and measurements, a good way of setting down the work, is to draw by the eye, on a piece of paper, a figure resembling that which is to be measured; and so writing the dimensions, as they are found, against the corresponding parts of the figure. And this method may be practised to a considerable extent, even in the larger surveys.

SECTION III.

OF COMPUTING AND DIVIDING.

PROBLEM XVI.

To compute the Contents of Fields.

1. Compute the contents of the figures as divided into triangles, or trapeziums, by the proper rules for these figures laid down in measuring; multiplying the perpendiculars by the diagonals or bases, both in links, and divide by 2; the quotient is acres, after having cut off five figures on the right for decimals. Then bring these decimals to roods and perches, by multiplying first by 4, and then by 40. An example of which is given in the description of the chain, pag. 430.

2. In small and separate pieces, it is usual to compute their contents from the measures of the lines taken in surveying them, without making a correct plan of them.

3. In pieces bounded by very crooked and winding hedges, measured by offsets, all the parts between the offsets are most accurately measured separately as small trapezoids.

4. Sometimes such pieces as that last mentioned are computed by finding a mean breadth, by adding all the offsets together, and dividing the sum by the number of them, accounting that for one of them where the boundary meets the station-line (which increases the number of them by 1, for the divisor, though it does not increase the sum or quantity to be divided); then multiply the length by that mean breadth.

5. But in larger pieces and whole estates, consisting of

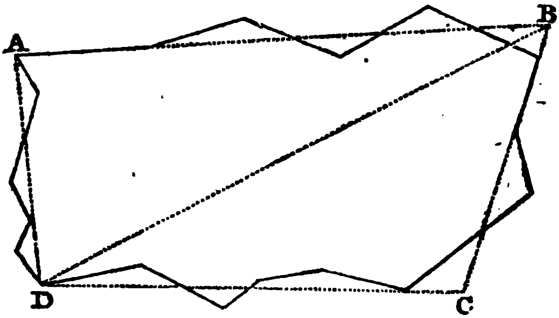
many fields, it is the common practice to make a rough plan of the whole, and from it compute the contents, quite independent of the measures of the lines and angles that were taken in surveying. For then new lines are drawn in the fields on the plans, so as to divide them into trapeziums and triangles, the bases and perpendiculars of which are measured on the plan by means of the scale from which it was drawn, and so multiplied together for the contents. In this way, the work is very expeditiously done, and sufficiently correct; for such dimensions are taken as afford the most easy method of calculation; and among a number of parts, thus taken and applied to a scale, though it be likely that some of the parts will be taken a small matter too little, and others too great, yet they will, on the whole, in all probability, very nearly balance one another, and give a sufficiently accurate result. After all the fields and particular parts are thus computed separately, and added all together into one sum; calculate the whole estate independent of the fields, by dividing it into large and arbitrary triangles and trapeziums, and add these also together. Then if this sum be equal to the former, or nearly so, the work is right; but if the sums have any considerable difference, it is wrong, and they must be examined, and re-computed, till they nearly agree.

6. But the chief art in computing, consists in finding the contents of pieces bounded by curved or very irregular lines, or in reducing such crooked sides of fields or boundaries to straight lines, that shall enclose the same or equal area with those crooked sides, and so obtain the area of the curved figure by means of the right-lined one, which will commonly be a trapezium. Now this reducing the crooked sides to straight ones, is very easily and accurately performed in this manner:—Apply the straight edge of a thin, clear piece of lantern-horn to the crooked line, which is to be reduced, in such a manner, that the small parts cut off from the crooked figure by it, may be equal to those which are taken in: which equality of the parts included and excluded you will presently be able to judge of very nicely by a little practice: then with a pencil, or point of a tracer, draw a line by the straight edge of the horn. Do the same by the other sides of the field or figure. So shall you have a straight-sided figure equal to the curved one; the content of which, being computed as before directed, will be the content of the crooked figure proposed.

Or, instead of the straight edge of the horn, a horse-hair, or fine thread, may be applied across the crooked sides in the same manner; and the easiest way of using the thread, is to string a small slender bow with it, either of wire, or cane, or whale-bone, or such-like slender elastic matter; for the bow keeping it always stretched, it can be easily and neatly applied with one hand, while the other is at liberty to make two marks by the side of it, to draw the straight line by.

EXAMPLE.

Thus, let it be required to find the contents of the same figure as in Prob. IX, page 443, to a scale of 4 chains to an inch.



Draw the 4 dotted straight lines AB, BC, CD, DA, cutting off equal quantities on both sides of them, which they do as near as the eye can judge: so is the crooked figure reduced to an equivalent right-lined one of 4 sides, ABCD. Then draw the diagonal BD, which, by applying a proper scale to it, measures suppose 1256. Also the perpendicular, or nearest distance from A to this diagonal, measures 456; and the distance of C from it, is 428.

Then, half the sum of 456 and 428, multiplied by the diagonal 1256, gives 555152 square links, or 5 acres, 2 roods, 8 perches, the content of the trapezium, or of the irregular crooked piece.

As a general example of this practice, let the contents be computed of all the fields separately in the foregoing plan facing page 453, and, by adding the contents altogether, the whole sum or content of the estate will be found nearly equal to 103¼ acres. Then, to prove the work, divide the whole plan into two parts, by a pencil line drawn across it any way near the middle, as from the corner *l* on the right, to the corner near *s* on the left; then, by computing these two large parts separately, their sum must be nearly equal to the former sum, when the work is all right.

PROBLEM XVII.

To Transfer a Plan to Another Paper, &c.

After the rough plan is completed, and a fair one is wanted; this may be done by any of the following methods.

First Method.—Lay the rough plan on the clean paper, keeping them always pressed flat and close together, by weights laid on them. Then with the point of a fine pin or pricker, prick through all the corners of the plan to be copied. Take them asunder, and connect the pricked points on the clean paper, with lines; and it is done. This method is only to be practised in plans of such figures as are small and tolerably regular, or bounded by right lines.

Second Method.—Rub the back of the rough plan over with black-lead powder; and lay this blacked part on the clean paper on which the plan is to be copied, and in the proper position. Then, with the blunt point of some hard substance, as brass, or such-like, trace over the lines of the whole plan; pressing the tracer so much, as that the black lead under the lines may be transferred to the clean paper: after which, take off the rough plan, and trace over the leaden marks with common ink, or with Indian ink.—Or, instead of blacking the rough plan, we may keep constantly a blacked paper to lay between the plans.

Third Method.—Another method of copying plans, is by means of squares. This is performed by dividing both ends and sides of the plan which is to be copied into any convenient number of equal parts, and connecting the corresponding points of division with lines; which will divide the plan into a number of small squares. Then divide the paper, on which the plan is to be copied, into the same number of squares, each equal to the former when the plan is to be copied of the same size, but greater or less than the others, in the proportion in which the plan is to be increased or diminished, when of a different size. Lastly, copy into the clean squares the parts contained in the corresponding squares of the old plan; and you will have the copy, either of the same size, or greater or less in any proportion.

Fourth Method.—A fourth method is by the instrument called a pentagraph, which also copies the plan in any size required: for this purpose, also, Professor Wallace's eidograph may be advantageously employed.

Fifth method.—A very neat method, at least in copying from a fair plan, is this. Procure a copying frame or glass, made in this manner; namely, a large square of the best window glass, set in a broad frame of wood, which can be raised up to any angle, when the lower side of it rests on a table. Set this frame up to any angle before you, facing a strong light; fix the old plan and clean paper together, with several pins quite around, to keep them together, the clean paper being laid uppermost, and over the face of the plan to be copied. Lay them, with the back of the old plan, on the glass; namely, that part which you intend to begin at to copy first; and by means of the light shining through the papers, you will very distinctly perceive every line of the plan through the clean paper. In this state then trace all the lines on the paper with a pencil. Having drawn that part which covers the glass, slide another part over the glass, and copy it in the same manner. Then another part: and so on, till the whole is copied. Then take them asunder, and trace all the pencil lines over with a fine pen and Indian ink, or with common ink. And thus you may copy the finest plan, without injuring it in the least.

OF ARTIFICERS' WORKS,

AND

TIMBER MEASURING.

1. OF THE CARPENTER'S OR SLIDING RULE.

The Carpenter's or Sliding Rule, is an instrument much used in measuring of timber and artificers' works, both for taking the dimensions, and computing the contents.

The instrument consists of two equal pieces, each a foot in length, which are connected together by a folding joint.

One side or face of the rule is divided into inches, and eighths, or half-quarters. On the same face also are several plane scales divided into twelfth parts by diagonal lines; which are used in planning dimensions that are taken in feet and inches. The edge of the rule is commonly divided decimally, or into tenths; namely, each foot into ten equal parts, and each of these into ten parts again; so that by means of this last scale, dimensions are taken in feet, tenths, and hundredths, and multiplied as common decimal numbers, which is the best way.

On the one part of the other face are four lines, marked A, B, C, D; the two middle ones B and C being on a slider, which runs in a groove made in the stock. The same numbers serve for both these two middle lines, the one being above the numbers, and the other below.

These four lines are logarithmic ones, and the three A, B, C, which are all equal to one another, are double lines, as they proceed twice over from 1 to 10. The other or lowest line, D, is a single one, proceeding from 4 to 40. It is also called the girt line, from its use in computing the contents of trees and timber; and on it are marked WG at 17·15, and AG at 18·95, the wine and ale gage points, to make this instrument serve the purpose of a gaging rule.

On the other part of this face, there is a table of the value of a load, or 50 cubic feet of timber, at all prices, from 6 pence to 2 shillings a foot.

When 1 at the beginning of any line is accounted 1, then the 1 in the middle will be 10, and the 10 at the end 100; but when 1 at the beginning is counted 10, then the 1 in the middle is 100, and the 10 at the end 1000; and so on. And all the smaller divisions are altered proportionally.

II. ARTIFICERS' WORK.

ARTIFICERS compute the contents of their works by several different measures. As,

Glazing and masonry, by the foot; Painting, plastering, paving, &c. by the yard, of 9 square feet: Flooring, partitioning, roofing, tiling, &c. by the square of 100 square feet:

And brickwork, either by the yard of 9 square feet, or by the perch, or square rod or pole, containing 272¼ square feet, or 30¼ square yards, being the square of the rod or pole of 16½ feet or 5½ yards long.

As this number 272¼ is troublesome to divide by, the ¼ is often omitted in practice, and the content in feet divided only by the 272.

All works, whether superficial or solid, are computed by the rules proper to the figure of them, whether it be a triangle, or rectangle, a parallelopiped, or any other figure.

III. BRICKLAYERS' WORK.

BRICKWORK is estimated at the rate of a brick and a half thick. So that if a wall be more or less than this standard thickness, it must be reduced to it, as follows:

Multiply the superficial content of the wall by the number of half bricks in the thickness, and divide the product by 3.

The dimensions of a building may be taken by measuring half round on the outside and half round on the inside; the sum of these two gives the compass of the wall, to be multiplied by the height, for the content of the materials.

Chimneys are commonly measured as if they were solid, deducting only the vacuity from the hearth to the mantle, on account of the trouble of them. All windows, doors, &c. are to be deducted out of the contents of the walls in which they are placed.

The dimensions of a common bare brick are, $8\frac{1}{4}$ inches long, 4 inches broad, and $2\frac{1}{2}$ thick; but including the half inch joint of mortar, when laid in brickwork, every dimension is to be counted half an inch more, making its length 9 inches, its breadth $4\frac{1}{2}$, and thickness 3 inches. So that every 4 courses of proper brickwork measures just 1 foot or 12 inches in height.

EXAMPLES.

EXAM. 1. How many yards and rods of standard brickwork are in a wall whose length or compass is 57 feet 3 inches, and height 24 feet 6 inches; the wall being $2\frac{1}{2}$ bricks or 5 half bricks thick? Ans. 8 rods, $17\frac{7}{9}$ yards.

EXAM. 2. Required the content of a wall 62 feet 6 inches long, and 14 feet 8 inches high, and $2\frac{1}{2}$ bricks thick? Ans. 169·753 yards.

EXAM. 3. A triangular gable is raised $17\frac{1}{2}$ feet high, on an end wall whose length is 24 feet 9 inches, the thickness being 2 bricks: required the reduced content? Ans. $32{\cdot}08\frac{1}{4}$ yards.

EXAM. 4. The end wall of a house is 28 feet 10 inches long, and 55 feet 8 inches high, to the eaves; 20 feet high is $2\frac{1}{2}$ bricks thick, other 20 feet high is 2 bricks thick, and the remaining 15 feet 8 inches is $1\frac{1}{2}$ brick thick; above which is a triangular gable, of 1 brick thick, which rises 42 courses of bricks, of which every 4 courses make a foot. What is the whole content in standard measure? Ans. 253·626 yards.

IV. MASONS' WORK.

To Masonry belong all sorts of stone work ; and the measure made use of is a foot, either superficial or solid.

Walls, columns, blocks of stone or marble, &c. are measured by the cubic foot ; and pavements, slabs, chimney-pieces, &c. by the superficial or square foot.

Cubic or solid measure is used for the materials, and square measure for the workmanship.

In the solid measure, the true length, breadth, and thickness are taken and multiplied continually together. In the superficial, there must be taken the length and breadth of every part of the projection which is seen without the general upright face of the building.

EXAMPLES.

Exam. 1. Required the solid content of a wall, 53 feet 6 inches long, 12 feet 3 inches high, and 2 feet thick ?

Ans. 1310¾ feet.

Exam. 2. What is the solid content of a wall, the length being 24 feet 3 inches, height 10 feet 9 inches, and 2 feet thick ?

Ans. 521·375 feet.

Exam. 3. Required the value of a marble slab, at 8*s*. per foot ; the length being 5 feet 7 inches, and breadth 1 foot 10 inches ?

Ans. 4*l*. 1*s*. 10½*d*.

Exam. 4. In a chimney-piece, suppose the

	feet	inches
length of the mantle and slab, each	4	6
breadth of both together -	3	2
length of each jamb - -	4	4
breadth of both together. -	1	9

Required the superficial content ? Ans. 21 feet 10 inches.

V. CARPENTERS' AND JOINERS' WORK.

To this branch belongs all the wood-work of a house, such as flooring, partitioning, roofing, &c.

Large and plain articles are usually measured by the square foot or yard, &c. ; but enriched mouldings, and some other articles, are often estimated by running or lineal measure ; and some things are rated by the piece.

In measuring of Joists, take the dimensions of one joist, and multiply its content by the number of them; considering that each end is let into the wall about ⅔ of the thickness, as it ought to be.

Partitions are measured from wall to wall for one dimension, and from floor to floor, as far as they extend, for the other.

The measure of Centering for Cellars is found by making a string pass over the surface of the arch for the breadth, and taking the length of the cellar for the length: but in groin centering, it is usual to allow double measure, on account of their extraordinary trouble.

In Roofing, the dimensions, as to length, breadth, and depth, are taken as in flooring joists, and the contents computed the same way.

In Floor-boarding, take the length of the room for one dimension, and the breadth for the other, to multiply together for the content.

For Stair-cases, take the breadth of all the steps, by making a line ply close over them, from the top to the bottom, and multiply the length of this line by the length of a step, for the whole area.—By the length of a step is meant the length of the front and the returns at the two ends; and by the breadth is to be understood the girts of its two outer surfaces, or the tread and riser.

For the Balustrade, take the whole length of the upper part of the hand-rail, and girt over its end till it meet the top of the newel-post, for the one dimension; and twice the length of the baluster on the landing, with the girt of the hand-rail, for the other dimension.

For Wainscoting, take the compass of the room for the one dimension; and the height from the floor to the ceiling, making the string ply close into all the mouldings, for the other.

For Doors, take the height and the breadth, to multiply them together for the area.—If the door be panneled on both sides, take double its measure for the workmanship; but if one side only be panneled, take the area and its half for the workmanship. *For the Surrounding Architrave*, girt it about the uppermost part for its length; and measure over it, as far as it can be seen when the door is open, for the breadth.

Window-shutters, *Bases*, &c. are measured in like manner.

In measuring of Joiners' work, the string is made to ply close into all mouldings, and to every part of the work over which it passes.

EXAMPLES.

Exam. 1. Required the content of a floor, 48 feet 6 inches long, and 24 feet 3 inches broad? Ans. 11 sq. 76⅛ feet.

Exam. 2. A floor being 36 feet 3 inches long, and 16 feet 6 inches broad, how many squares are in it?

Ans. 5 sq. 98⅛ feet.

Exam. 3. How many squares are there in 173 feet 10 inches in length, and 10 feet 7 inches height, of partitioning?

Ans. 18·3973 squares.

Exam. 4. What cost the roofing of a house at 10*s.* 6*d.* a square; the length within the walls being 52 feet 8 inches, and the breadth 30 feet 6 inches; reckoning the roof $\frac{3}{2}$ of the flat? Ans. 12*l.* 12*s.* 11¾*d.*

Exam. 5. To how much, at 6*s.* per square yard, amounts the wainscoting of a room; the height, taking in the cornice and mouldings, being 12 feet 6 inches, and the whole compass 83 feet 8 inches; also the three window-shutters are each 7 feet by 8 inches by 3 feet 6 inches, and the door 7 feet by 3 feet 6 inches; the door and shutters, being worked on both sides, are reckoned work and half work?

Ans. 36*l.* 12*s.* 2¼*d.*

VI. SLATERS' AND TILERS' WORK.

In these articles, the content of a roof is found by multiplying the length of the ridge by the girt over from eaves to eaves; making allowance in this girt for the double row of slates at the bottom, or for how much one row of slates or tiles is laid over another.

When the roof is of a true pitch, that is, forming a right angle at top; then the breadth of the building, with its half added, is the girt added over both sides nearly.

In angles formed in a roof, running from the ridge to the eaves, when the angle bends inwards, it is called a valley; but when outwards, it is called a hip.

Deductions are made for chimney shafts or window holes.

EXAMPLES.

Exam. 1. Required the content of a slated roof, the length being 45 feet 9 inches, and the whole girt 34 feet 3 inches? Ans. $174\frac{5}{12}$ yards.

Exam. 2. To how much amounts the tiling of a house, at 25*s*. 6*d*. per square; the length being 43 feet 10 inches, and the breadth on the flat 27 feet 5 inches; also the eaves projecting 16 inches on each side, and the roof of a true pitch? Ans. 24*l*. 9*s*. $8\frac{1}{4}$*d*.

VII. PLASTERERS' WORK.

Plasterers' work is of two kinds; namely, ceiling, which is plastering on laths; and rendering, which is plastering on walls: which are measured separately.

The contents are estimated either by the foot or the yard, or the square, of 100 feet. Enriched mouldings, &c. are rated by running or lineal measure.

Deductions are made for chimneys, doors, windows, &c.

EXAMPLES.

Exam. 1. How many yards contains the ceiling which is 43 feet 3 inches long, and 25 feet 6 inches broad? Ans. $122\frac{1}{2}$.

Exam. 2. To how much amounts the ceiling of a room, at 10*d*. per yard: the length being 21 feet 8 inches, and the breadth 14 feet 10 inches? Ans. 1*l*. 9*s*. $8\frac{3}{4}$*d*.

Exam. 3. The length of a room is 18 feet 6 inches, the breadth 12 feet 3 inches, and height 10 feet 6 inches; to how much amounts the ceiling and rendering, the former at 8*d*. and the latter at 3*d*. per yard: allowing for the door of 7 feet by 3 feet 8, and a fire-place of 5 feet square? Ans. 1*l*. 13*s*. $3\frac{1}{2}$*d*.

Exam. 4. Required the quantity of plastering in a room, the length being 14 feet 5 inches, breadth 13 feet 2 inches, and height 9 feet 3 inches to the under side of the cornice, which girts $8\frac{1}{2}$ inches, and projects 5 inches from the wall on the upper part next the ceiling; deducting only for a door 7 feet by 4?

Ans.	53 yards	5 feet	$3\frac{1}{2}$ inches	of rendering
	18	5	6	of ceiling
		39	$0\frac{1}{12}$	of cornice.

VIII. PAINTERS' WORK.

PAINTERS' work is computed in square yards. Every part is measured where the colour lies; and the measuring line is forced into all the mouldings and corners.

Windows are done at so much a piece. And it is usual to allow double measure for carved mouldings, &c.

EXAMPLES.

EXAM. 1. How many yards of painting contains the room which is 65 feet 6 inches in compass, and 12 feet 4 inches high? Ans. $89\frac{41}{54}$ yards.

EXAM. 2. The length of a room being 20 feet, its breadth 14 feet 6 inches, and height 10 feet 4 inches; how many yards of painting are in it, deducting a fire-place of 4 feet by 4 feet 4 inches, and two windows each 6 feet by 3 feet 2 inches? Ans. $73\frac{2}{27}$ yards.

EXAM. 3. What cost the painting of a room, at 6d. per yard; its length being 24 feet 6 inches, its breadth 16 feet 3 inches, and height 12 feet 9 inches; also the door is 7 feet by 3 feet 6, and the window-shutters to two windows each 7 feet 9 by 3 feet 6; but the breaks of the windows themselves are 8 feet 6 inches high, and 1 foot 3 inches deep; including also the window cills or seats, and the soffits above, the dimensions of which are known from the other dimensions: but deducting the fire-place of 5 feet by 5 feet 6? Ans. 3*l*. 3*s*. $10\frac{3}{4}$*d*.

IX. GLAZIERS' WORK.

GLAZIERS take their dimensions, either in feet, inches, and parts, or feet, tenths, and hundredths. And they compute their work in square feet.

In taking the length and breadth of a window, the cross bars between the squares are included. Also windows of round or oval forms are measured as square, measuring them to their greatest length and breadth, on account of the waste in cutting the glass.

EXAMPLES.

EXAM. 1. How many square feet contains the window which is 4·25 feet long, and 2·75 feet broad? Ans. $11\frac{3}{4}$.

Exam. 2. What will the glazing a triangular sky-light come to, at 10*d.* per foot; the base being 12 feet 6 inches, and the perpendicular height 6 feet 9 inches?

Ans. 1*l.* 15*s.* 1¾*d.*

Exam. 3. There is a house with three tiers of windows, three windows in each tier, their common breadth 3 feet 11 inches:

now the height of the first tier, is	7 feet	10 inches
of the second	6	8
of the third	5	4

Required the expense of glazing at 14*d* per foot?

Ans. 13*l.* 11*s.* 10½*d.*

Exam. 4. Required the expense of glazing the windows of a house at 13*d.* a foot; there being three stories, and three windows in each story:

the height of the lower tier is	7 feet	9 inches
of the middle	6	6
of the upper	5	3¼
and of an oval window over the door	1	10¾

the common breadth of all the windows being 3 feet 9 inches? Ans. 12*l.* 5*s.* 6*d.*

X. PAVERS' WORK.

Pavers' work is done by the square yard. And the content is found by multiplying the length by the breadth.

EXAMPLES.

Exam. 1. What cost the paving a foot-path, at 3*s.* 4*d.* a yard; the length being 35 feet 4 inches, and breadth 8 feet 3 inches? Ans. 5*l.* 7*s.* 11½*d.*

Exam. 2. What cost the paving a court, at 3*s.* 2*d.* per yard; the length being 27 feet 10 inches, and the breadth 14 feet 9 inches? Ans. 7*l.* 4*s.* 5¼*d.*

Exam. 3. What will be the expense of paving a rectangular court-yard, whose length is 63 feet, and breadth 45 feet; in which there is laid a foot-path of 5 feet 3 inches broad, running the whole length, with broad stones, at 3*s.* a yard; the rest being paved with pebbles at 2*s.* 6*d.* a yard;

Ans. 40*l.* 5*s.* 10½*d.*

XI. PLUMBERS' WORK.

PLUMBERS' work is rated at so much a pound, or else by the hundred weight of 112 pounds.

Sheet lead, used in roofing, guttering, &c. is from 6 to 10lb. to the square foot. And a pipe of an inch bore is commonly 13 or 14lb. to the yard in length.

EXAMPLES.

EXAM. 1. How much weighs the lead which is 39 feet 6 inches long, and 3 feet 3 inches broad, at $8\frac{1}{4}$lb. to the square foot? Ans. $1091\frac{3}{16}$lb.

EXAM. 2. What cost the covering and guttering a roof with lead, at 18*s*. the cwt.; the length of the roof being 43 feet, and breadth or girt over it 32 feet; the guttering 57 feet long, and 2 feet wide; the former 9·831lb. and the latter 7·373lb. to the square foot? Ans. 115*l*. 9*s*. $1\frac{1}{4}$*d*.

XII. TIMBER MEASURING.

PROBLEM I.

To find the Area, or Superficial Content of a Board or Plank.

MULTIPLY the length by the mean breadth.

Note. When the board is tapering, add the breadths at the two ends together, and take half the sum for the mean breadth. Or else take the mean breadth in the middle.

By the Sliding Rule.

Set 12 on B to the breadth in inches on A; then against the length in feet on B, is the content on A, in feet and fractional parts.

EXAMPLES.

EXAM. 1. What is the value of a plank, at $1\frac{1}{2}$*d*. per foot, whose length is 12 feet 6 inches, and mean breadth 11 inches? Ans. 1*s*. 5*d*.

Exam. 2. Requred the content of a board, whose length is 11 feet 2 inches, and breadth 1 foot 10 inches?

Ans. 20 feet 5 inches 8″.

Exam. 3. What is the value of a plank, which is 12 feet 9 inches long, and 1 foot 3 inches broad, at $2\frac{1}{2}d.$ a foot?

Ans. $3s.\ 3\frac{3}{4}d.$

Exam. 4. Required the value of 5 oaken planks at $3d.$ per foot, each of them being $17\frac{1}{2}$ feet long; and their several breadths as follows, namely, two of $13\frac{1}{2}$ inches in the middle, one of $14\frac{1}{2}$ inches in the middle, and the two remaining ones, each 18 inches at the broader end, and $11\frac{1}{4}$ at the narrower?

Ans. $1l.\ 5s.\ 9\frac{1}{4}d.$

PROBLEM II.

To find the Solid Content of Squared or Four-sided Timber.

Multiply the mean breadth by the mean thickness, and the product again by the length, for the content nearly.

By the Sliding Rule.

C		D		D		C
As length	:	12 or 10	: :	quarter girt	:	solidity.

That is, as the length in feet on C, is to 12 on D, when the quarter girt is in inches, or to 10 on D, when it is in tenths of feet; so is the quarter girt on D, to the content on C.

Note 1. If the tree taper regularly from the one end to the other; either take the mean breadth and thickness in the middle, or take the dimensions at the two ends, and half their sum will be the mean dimensions: which multiplied as above, will give the content nearly.

2. If the piece do not taper regularly, but be unequally thick in some parts and small in others; take several different dimensions, add them all together, and divide their sum by the number of them, for the mean dimensions.

EXAMPLES.

Exam. 1. The length of a piece of timber is 18 feet 6 inches, the breadths at the greater and less end 1 foot 6 inches and 1 foot 3 inches, and the thickness at the greater and less end 1 foot 3 inches and 1 foot; required the solid content?

Ans. 28 feet 7 inches.

Exam. 2. What is the content of the piece of timber, whose length is $24\frac{1}{2}$ feet, and the mean breadth and thickness each 1·04 feet? Ans. $26\frac{1}{2}$ feet.

Exam. 3. Required the content of a piece of timber, whose length is 20·38 feet, and its ends unequal squares, the side of the greater being $19\frac{1}{2}$ inches, and the side of the less $9\frac{3}{4}$ inches? Ans. 29·7562 feet.

Exam. 4. Required the content of the piece of timber, whose length is 27·36 feet; at the greater end the breadth is 1·78, and thickness 1·23; and at the less end the breadth is 1·04, and thickness 0·91 feet? Ans. 41·278 feet.

PROBLEM III.

To find the Solidity of Round or Unsquared Timber.

Multiply the square of the quarter girt, or of $\frac{1}{4}$ of the mean circumference, by the length, for the content.

By the Sliding Rule.

As the length upon C : 12 or 10 upon D ::
quarter girt, in 12ths, or 10ths, on D : content on C.

Note 1. When the tree is tapering take the mean dimensions as in the former problems, either by girting it in the middle, for the mean girt, or at the two ends, and taking half the sum of the two; or by girting it in several places, then adding all the girts together, and dividing the sum by the number of them, for the mean girt. But when the tree is very irregular, divide it into several lengths, and find the content of each part separately.

2. This rule, which is commonly used, gives the answer about $\frac{1}{4}$ less than the true quantity in the tree, or nearly what the quantity would be, after the tree is hewed square in the usual way: so that it seems intended to make an allowance for the squaring of the tree.

On this subject, however, Hutton's Mensuration, part v. sect. 4, may be advantageously consulted.

EXAMPLES.

Exam. 1. A piece of round timber being 9 feet 6 inches long, and its mean quarter girt 42 inches; what is the content? Ans. $116\frac{3}{8}$ feet.

Exam. 2. The length of a tree is 24 feet, its girt at the thicker end 14 feet, and at the smaller end 2 feet; required the content? Ans. 96 feet.

EXAM. 3. What is the content of a tree whose mean girt is 3·15 feet, and length 14 feet 6 inches?

Ans. 8·9922 feet.

EXAM. 4. Required the content of a tree, whose length is $17\frac{1}{4}$ feet, which girts in five different places as follows, namely, in the first place 9·43 feet, in the second 7·92, in the third 6·15, in the fourth 4·74, and in the fifth 3·16?

Ans. 42·519525.

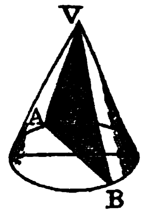
CONIC SECTIONS.

DEFINITIONS.

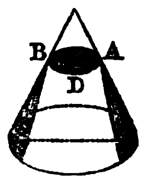
1. Conic Sections are the figures made by a plane cutting a cone.

2. According to the different positions of the cutting plane there arise five different figures or sections, namely, a triangle, a circle, an ellipsis, an hyperbola, and a parabola: the three last of which only are peculiarly called Conic Sections.

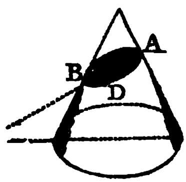
3. If the cutting plane pass through the vertex of the cone, and any part of the base, the section will evidently be a triangle; as VAB.



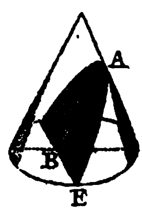
4. If the plane cut the cone parallel to the base, or make no angle with it, the section will be a circle; as ABD.



5. The section DAB is an ellipse when the cone is cut obliquely through both sides, or when the plane is inclined to the base in a less angle than the side of the cone is.

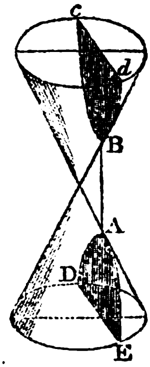


6. The section is a parabola, when the cone is cut by a plane parallel to the side, or when the cutting plane and the side of the cone make equal angles with the base.



7. The section is an hyperbola, when the cutting plane makes a greater angle with the base than the side of the cone makes.

8. And if all the sides of the cone be continued through the vertex, forming an opposite equal cone, and the plane be also continued to cut the opposite cone, this latter section will be the opposite hyperbola to the former; as *d*BE.



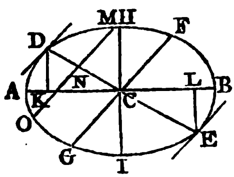
9. The Vertices of any section, are the points where the cutting plane meets the sides of that vertical triangular section which is perpendicular to it; as A and B.

Hence the ellipse and the opposite hyperbolas, have each two vertices; but the parabola only one; unless we consider the other as at an infinite distance.

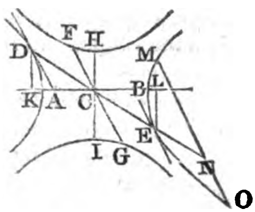
10. The Axis, or Transverse Diameter, of a conic section, is the line or distance AB between the vertices.

Hence the axis of a parabola is infinite in length, A*b* being only a part of it.

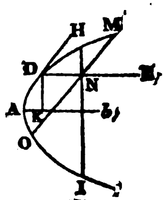
Ellipse.



Hyperbolas.



Parabola.



11. The centre C is the middle of the axis.

Hence the centre of a parabola is infinitely distant from the vertex. And of an ellipse, the axis and centre lie within the curve; but of an hyperbola, without.

12. A Diameter is any right line, as AB or DE, drawn through the centre, and terminated on each side by the curve; and the extremities of the diameter, or its intersections with the curve, are its vertices.

Hence all the diameters of a parabola are parallel to the axis, and infinite in length. Hence also every diameter of the ellipse and hyperbola has two vertices; but of the parabola, only one; unless we consider the other as at an infinite distance.

13. The Conjugate to any diameter, is the line drawn through the centre, and parallel to the tangent of the curve at the vertex of the diameter. So, FG, parallel to the tangent at D, is the conjugate to DE; and HI, parallel to the tangent at A, is the conjugate to AB.

Hence the conjugate HI, of the axis AB, is perpendicular to it.

14. An Ordinate to any diameter, is a line parallel to its conjugate, or to the tangent at its vertex, and terminated by the diameter and curve. So DK, EL, are ordinates to the axis AB; and MN, NO, ordinates to the diameter DE.

Hence the ordinates of the axis are perpendicular to it.

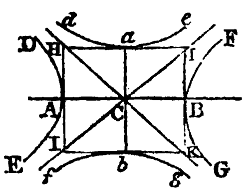
15. An Absciss is a part of any diameter contained between either of its vertices and an ordinate to it; as AK or BK, or DN or EN.

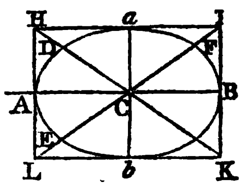
Hence, in the ellipse and hyperbola, every ordinate has two determinate abscisses; but in the parabola only one; the other vertex of the diameter being infinitely distant.

16. The Parameter of any diameter, is a third proportional to that diameter and its conjugate, in the ellipse and hyperbola, and to one absciss and its ordinate in the parabola.

17. The Focus is the point in the axis where the ordinate is equal to half the parameter. As K and L, where DK or EL is equal to the semi-parameter. The name focus being given to this point from the peculiar property of it mentioned in the corol. to theor. 9 in the Ellipse and Hyperbola following, and to theor. 6 in the Parabola.

Hence, the ellipse and hyperbola have each two foci; but the parabola only one.





18. If DAE, FBG, be two opposite hyperbolas, having AB for their first or transverse axis, and *ab* for their second or conjugate axis. And if *dae*, *fbg*, be two other opposite hyperbolas having the same axes, but in the contrary order, namely, *ab* their first axis, and AB their second; then these

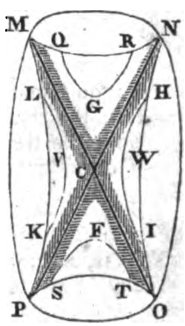
two latter curves *dae*, *fbg*, are called the conjugate hyperbolas to the two former DAE, FBG; and each pair of opposite curves mutually conjugate to the other; being all for convenience of investigation referred to one plane, though they are only posited two and two in one plane; as will appear more evidently from the demonstration of th. 2. Hyperbola.

19. And if tangents be drawn to the four vertices of the curves, or extremities of the axes, forming the inscribed rectangle HIKL; the diagonals HCK, ICL, of this rectangle, are called the asymptotes of the curves. And if these asymptotes intersect at right angles, or the inscribed rectangle be a square, or the two axes AB and *ab* be equal, then the hyberbolas are said to be right-angled, or equilateral.

SCHOLIUM.

The rectangle inscribed between the four conjugate hyperbolas, is similar to a rectangle circumscribed about an ellipse, by drawing tangents in like manner, to the four extremities of the two axes; and the asymptotes or diagonals in the hyperbola, are analogous to those in the ellipse, cutting this curve in similar points, and making that pair of conjugate diameters which are equal to each other. Also, the whole figure formed by the four hyperbolas, is as it were, an ellipse turned inside out, cut open at the extremities, D, E, F, G, of the said equal conjugate diameters, and these four points drawn out to an infinite distance; the curvature being turned the contrary way, but the axes, and the rectangle passing through their extremities, continuing fixed.

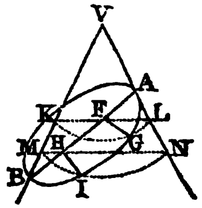
And further, if there be four cones CMN, COP, CMP, CNO, having all the same vertex C, and all their axes in the same plane, and their sides touching or coinciding in the common intersecting lines MCO, NCP; then if these four cones be all cut by one plane, parallel to the common plane of their axes, there will be formed the four hyperbolas, GQR, FST, VKL, WHI, of which each two opposites are equal; and each pair resembles the conjugates to the other two, as here in the annexed figure; but they are not accurately the conjugates, except only when the four cones are all equal, and then the four hyperbolic sections are all equal also.



OF THE ELLIPSE.

THEOREM I.

The Squares of the Ordinates of the Axis are to each other as the Rectangles of their Abscisses.



Let AVB be a plane passing through the axis of the cone; AGIH another section of the cone perpendicular to the plane of the former; AB the axis of this elliptic section; and FG, HI, ordinates perpendicular to it. Then it will be, as $FG^2 : HI^2 :: AF \cdot FB : AH \cdot HB$.

For, through the ordinates FG, HI, draw the circular sections KGL, MIN, parallel to the base of the cone, having KL, MN, for their diameters, to which FG, HI, are ordinates, as well as to the axis of the ellipse.

Now, by the similar trangles AFL, AHN, and BFK, BHM,

it is $AF : AH :: FL : HN$,

and $FB : HB :: KF : MH$;

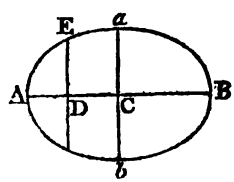
hence, taking the rectangles of the corresponding terms, it is, the rect. $AF \cdot FB : AH \cdot HB :: KF \cdot FL : MH \cdot HN$.

But, by the circle, $KF \cdot FL = FG^2$, and $MH \cdot HN = HI^2$;
Therefore the rect. $AF \cdot FB : AH \cdot HB :: FG^2 : HI^2$. Q. E. D.

THEOREM II.

As the Square of the Transverse Axis :
Is to the Square of the Conjugate ::
So is the Rectangle of the Abscisses :
To the Square of their Ordinate.

That is, $AB^2 : ab^2$ or $AC^2 : ac^2 :: AD \cdot DB : DE^2$.



For, by theor. 1, $AC \cdot CB : AD \cdot DB :: Ca^2 : DE^2$;
But, if C be the centre, then $AC \cdot CB = AC^2$, and Ca is the semi-conjugate.

Therefore $AC^2 : AD \cdot DB :: ac^2 : DE^2$;
or, by permutation, $AC^2 : ac^2 :: AD \cdot DB : DE^2$;
or, by doubling, $AB^2 : ab^2 :: AD \cdot DB : DE^2$. Q. E. D.

Corol. Or, by div. $AB : \frac{ab^2}{AB} :: AD \cdot DB$ or $CA^2 - CD^2 : DE^2$,

that is, $AB : p :: AD \cdot DB$ or $CA^2 - CD^2 : DE^2$;

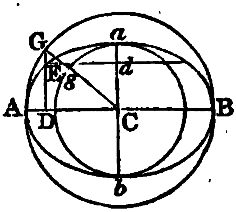
where p is the parameter $\frac{ab^2}{AB}$, by the definition of it.

That is, As the transverse,
Is to its parameter,
So is the rectangle of the abscisses,
To the square of their ordinate.

THEOREM III.

As the Square of the Conjugate Axis
Is to the Square of the Transverse Axis,
So is the Rectangle of the Abscisses of the Conjugate, or the difference of the Squares of the Semi-conjugate and Distance of the centre from any Ordinate of that Axis,
To the Square of the Ordinate.

That is,
$ca^2 : CB^2 :: ad \cdot db$ or $ca^2 - cd^2 : dE^2$.



For, draw the ordinate ED to the transverse AB.
Then, by theor. 1, $ca^2 : CA^2 :: DE^2 : AD \cdot DB$ or $CA^2 - CD^2$,
or - - - - - - $ca^2 : CA^2 :: cd^2 : CA^2 - dE^2$,
But - - - - - - $ca^2 : CA^2 :: ca^2 : CA^2$,
theref. by subtr. $ca^2 : CA^2 :: ca^2 - cd^2$ or $ad \cdot db : dE^2$.

Q. E. D.

Corol. 1. If two circles be described on the two axes as diameters, the one inscribed within the ellipse, and the other circumscribed about it; then an ordinate in the circle will be to the corresponding ordinate in the ellipse, as the axis of this ordinate, is to the other axis.

That is, $CA : ca :: DG : DE$,
and $ca : CA :: dg : dE$.

For, by the nature of the circle, $AD \cdot DB = DG^2$; theref. by the nature of the ellipse, $CA^2 : ca^2 :: AD \cdot DB$ or $DG^2 : DE^2$,
or $CA : ca :: DG : DE$

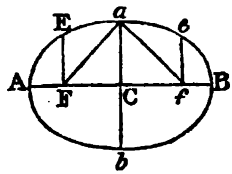
In like manner . $ca : CA :: dg : dE$.
Also, by equality . $DG : DE$ or $cd :: dE$ or $DC : dg$.
Therefore cgG is a continued straight line.

Corol. 2. Hence also, as the ellipse and circle are made up of the same number of corresponding ordinates, which are all in the same proportion of the two axes, it follows that the areas of the whole circle and ellipse, as also of any like parts of them, are in the same proportion of the two axes, or as the square of the diameter to the rectangle of the two axes; that is, the areas of the two circles, and of the ellipse, are as the square of each axis and the rectangle of the two; and therefore the ellipse is a mean proportional between the two circles.

THEOREM IV.

The Square of the Distance of the Focus from the Centre, is equal to the Difference of the Squares of the Semi-axes.
Or, the square of the Distance between the Foci, is equal to the Difference of the Squares of the two Axes.

That is, $CF^2 = CA^2 - ca^2$
or $Ff^2 = AB^2 - ab^2$



For, to the focus F draw the ordinate FE; which, by the definition, will be the semi-parameter. Then, by the nature of the curve . . $CA^2 : ca^2 :: CA^2 - CF^2 : FE^2$;
and by the def. of the para. $CA^2 : ca^2 :: ca^2 : FE^2$;
therefore . . $ca^2 = CA^2 - CF^2$;
and by addit. and subtr. $CF^2 = CA^2 - ca^2$;
or, by doubling, . $Ff^2 = AB^2 - ab^2$. Q. E. D.

Corol. 1. The two semi-axes, and the focal distance from the centre, are the sides of a right-angled triangle CFa; and the distance Fa from the focus to the extremity of the conjugate axis, is $=$ AC the semi-transverse.

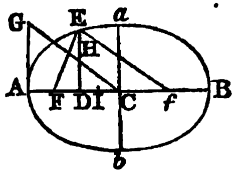
Corol. 2. The conjugate semi-axis ca is a mean proportional between AF, FB, or between Af, fB, the distances of either focus from the two vertices.

For $ca^2 = CA^2 - CF^2 = (CA + CF) \cdot (CA - CF) = AF \cdot FB$.

THEOREM V.

The Sum of two lines drawn from the two Foci to meet at any Point in the Curve, is equal to the Transverse Axis.

That is,
$FE + fe = AB$



For, draw AG parallel and equal to ca the semi-conjugate; and join CG meeting the ordinate DE in H; also take CI a 4th proportional to CA, CF, CD.

Then by theor. 2, $CA^2 : AG^2 :: CA^2 - CD^2 : DE^2$;
and, by sim. tri. $CA^2 : AG^2 :: CA^2 - CD^2 : AG^2 - DH^2$;
consequently $DE^2 = AG^2 - DH^2 = ca^2 - DH^2$.

Also, $FD = CF \sim CD$, and $FD^2 = CF^2 - 2CF \cdot CD + CD^2$;
And, by right-angled triangles, $FE^2 = FD^2 + DE^2$;
therefore $FE^2 = CF^2 + ca^2 - 2CF \cdot CD + CD^2 - DH^2$;

But by theor. 4, $CF^2 + ca^2 = CA^2$,
and by supposition, $2CF \cdot CD = 2CA \cdot CI$;
theref. $FE^2 = CA^2 - 2CA \cdot CI + CD^2 - DH^2$.

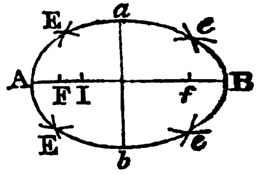
Again, by supp. $CA^2 : CD^2 :: CF^2$ or $CA^2 - AG^2 : CI^2$;
and, by sim. tri. $CA^2 : CD^2 :: CA^2 - AG^2 : CD^2 - DH^2$;
therefore $CI^2 = CD^2 - DH^2$;
consequently $FE^2 = CA^2 - 2CA \cdot CI + CI^2$.

And the root or side of this square is $FE = CA - CI = AI$.
In the same manner it is found that $fE = CA + CI = BI$.
Conseq. by addit. $FE + fE = AI + BI = AB$. Q. E. D.

Corol. 1. Hence CI or $CA - FE$ is a 4th proportional to CA, CF, CD.

Corol. 2. And $fE - FE = 2CI$; that is, the difference between two lines drawn from the foci, to any point in the curve, is double the 4th proportional to CA, CF, CD.

Corol. 3. Hence is derived the common method of describing this curve mechanically by points, or with a thread, thus:



In the transverse take the foci F, f, and any point I. Then with the radii AI, BI, and centres F, f, describe arcs intersecting in E, which will be a point in the curve. In like manner, assuming other points I, as many other points will be found in the curve. Then with a steady hand, the curve line may be drawn through all the points of intersection E.

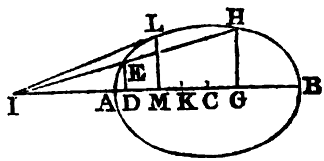
Or, take a thread of the length AB of the transverse axis, and fix its two ends in the foci F, f, by two pins. Then carry a pen or pencil round by the thread, keeping it always stretched, and its point will trace out the curve line.

THEOREM VI.

If from any Point I in the Axis produced, a Line IL be drawn touching the Curve in one Point L; and the Ordinate LM be drawn; and if C be the Centre or Middle of AB: Then shall CM be to CI as the Square of AM to the Square of AI.

That is,

$CM : CI :: AM^2 : AI^2$.



For, from the point I draw any other line IEH to cut the curve in two points E and H; from which let fall the perpendiculars ED and HG; and bisect DG in K.

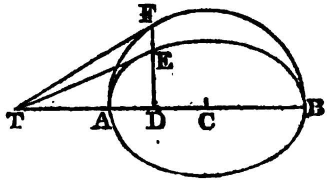
Then, by theor. 1, $AD \cdot DB : AG \cdot GB :: DE^2 : GH^2$,
and by sim. triangles, $ID^2 : IG^2 :: DE^2 : GH^2$;
theref. by equality, $AD \cdot DB : AG \cdot GB :: ID^2 : IG^2$.
But $DB = CB + CD = AC + CD = AG + DC - CG = 2CK + AG$,
and $GB = CB - CG = AC - CG = AD + DC - CG = 2CK + AD$;
theref. $AD \cdot 2CK + AD \cdot AG : AG \cdot 2CK + AD \cdot AG :: ID^2 : IG^2$,
and, by div. $DG \cdot 2CK : IG^2 - ID^2$ or $DG \cdot 2IK :: AD \cdot 2CK + AD \cdot AG : ID^2$,
or $2CK : 2IK :: AD \cdot 2CK + AD \cdot AG : ID^2$,
or $AD \cdot 2CK : AD \cdot 2IK :: AD \cdot 2CK + AD \cdot AG : ID^2$;
theref. by div. $CK : IK :: AD \cdot AG : ID^2 - AD \cdot 2IK$,
and, by comp. $CK : IC :: AD \cdot AG : ID^2 - AD \cdot \overline{ID + IA}$,
or $CK : CI :: AD \cdot AG : AI^2$.

But, when the line IH, by revolving about the point I, comes into the position of the tangent IL, then the points E and H meet in the point L, and the points D, K, G, coincide with the point M; and then the last proportion becomes CM : CI : : AM² : AI². Q. E. D.

THEOREM VII.

If a Tangent and Ordinate be drawn from any Point in the Curve, meeting the Transverse Axis; the Semi-transverse will be a Mean Proportional between the Distances of the said two Intersections from the Centre.

That is,
CA is a mean proportional between CD and CT;
or CD, CA, CT, are continued proportionals.



For, by theor. 6, $CD : CT :: AD^2 : AT^2$
that is, $CD : CT :: (CA - CD)^2 : (CT - CA)^2$,
or - $CD : CT :: CD^2 + CA^2 : CA^2 + CT^2$,
and - $CD : DT :: CD^2 + CA^2 : CT^2 - CD^2$,
or - $CD : DT :: CD^2 + CA^2 : (CT + CD)DT$,
or - $CD^2 : CD \cdot DT :: CD^2 + CA^2 : (CD \cdot DT) + (CT \cdot DT)$,
hence $CD^2 : CA^2 :: CD \cdot DT : CT \cdot DT$,
and - $CD^2 : CA^2 :: CD : CT$.
therefore (th. 78, Geom.) $CD : CA :: CA : CT$. Q. E. D.

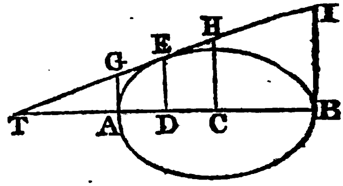
Corol. 1. Since CT is always a third proportional to CD, CA; if the points D, A, remain constant, then will the point T be constant also; and therefore all the tangents will meet in this point T, which are drawn from the point E, of every ellipse described on the same axis AB, where they are cut by the common ordinate DEF drawn from the point D.

Corol. 2. When the outer ellipse, by enlarging, becomes a circle, as at the upper figure at E, then by drawing ET perp. to CE, and joining T to the lower E, the tangent to the point E at the ellipse is obtained.

THEOREM VIII.

If there be any Tangent meeting four Perpendiculars to the Axis drawn from these four Points, namely, the Centre, the two Extremities of the Axis, and the Point of Contact; those four Perpendiculars will be Proportions.

That is,
AG : DE : : CH : BI.



For, by theor. 7, TC : AC : : AC : DC,
theref. by div. TA : AD : : TC : AC or CB,
and by comp. TA : TD : : TC : TB,
and by sim. tri. AG : DE : : CH : BI. Q. E. D.

Corol. 1. Hence TA, TD, TC, TB and TG, TE, TH, TI are also proportionals.
For these are as AG, DE, CH, BI, by similar triangles.

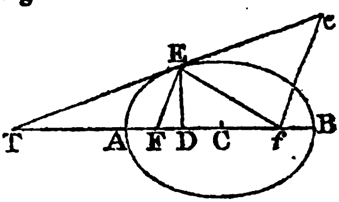
Corol. 2. Draw AI to bisect DE in P; then since TA : TE : : TC : TI, the triangles TAE, TCI are similar, as well as the triangles AED, CBI, and ADP, ABI.

Hence . AD : DE : : CB : BI
and . AD : DP : : AB : BI
∴ DE : DP : : AB : CB : : 2 : 1; which suggests another simple practical method of drawing a tangent to an ellipse.

THEOREM IX.

If there be any Tangent, and two Lines drawn from the Foci to the Point of Contact; these two lines will make equal Angles with the Tangent.

That is,
the ∠ FET = ∠ *f*Ee.



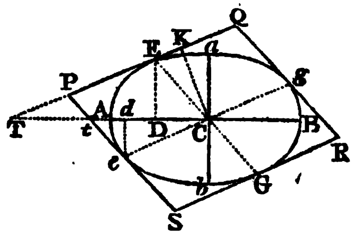
For, draw the ordinate DE and *fe* parallel to FE.
By cor. 1, theor. 5, CA : CD : : CF : CA — FE,
and by theor. 7, CA : CD : : CT : CA;
therefore CT : CF : : CA : CA — FE;
and by add. and sub. TF : T*f* : : FE : 2CA — FE or *f*E by th. 5.
But by sim. tri. TF : T*f* : : FE : *fe*;
therefore *f*E = *fe*, and conseq. ∠*e* = ∠*f*E*e*.
But because FE is parallel to *fe*, the ∠*e* = ∠FET;
therefore ∠FET = ∠*f*E*e*. Q. E. D.

Corol. As opticians find that the angle of incidence is equal to the angle of reflection, it appears from this theorem, that rays of light issuing from the one focus, and meeting the curve in every point, will be reflected into lines drawn from those points to the other focus. So the ray fE is reflected into FE. And this is the reason why the points F, f, are called the *foci*, or burning points.

THEOREM X.

All the Parallelograms circumscribed about an Ellipse are equal to one another, and each equal to the Rectangle of the two Axes.

That is,
the parallelogram PQRS =
the rectangle AB . ab.



Let EG, eg, be two conjugate diameters parallel to the sides of the parallelogram, and dividing it into four less and equal parallelograms. Also, draw the ordinates DE, de, and CK perpendicular to PQ; and let the axis CA produced meet the sides of the parallelogram, produced if necessary, in T and t.

Then, by theor. 7, $CT : CA :: CA : CD$,
and - - $Ct : CA :: CA : Cd$;
theref. by equality, $CT : Ct :: Cd : CD$;
but, by sim. triangles, $CT : Ct :: TD : td$,
theref. by equality, $TD : td :: Cd : CD$,
and the rectangle $TD . DC$ = is the square cd^2.
Again, by theor. 7, $CD : CA :: CA : CT$,
or, by division, $CD : CA :: DA : AT$,
and by composition, $CD : DB :: AD : DT$;
conseq. the rectangle $CD . DT = cd^2 = AD . DB$*.
But, by theor. 1, $CA^2 : ca^2 ::$ ($AD . DB$ or) $cd^2 : DE^2$,
therefore - $CA : ca :: cd : DE$;
or - - - $ca . DE :: CA : Cd$;
By th. 7, - $Ct : CA :: CA : Cd$,

---

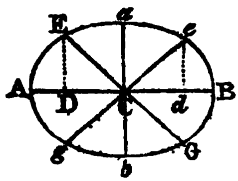
* *Corol.* Because $cd^2 = AD . DB = CA^2 - CD^2$,
therefore $CA^2 = CD^2 + cd^2$.
In like manner, $ca^2 = DE^2 + de^2$.

by equality .	ct : CA : : ca : DE,
by sim. tri, .	ct : CT : : de : DE,
theref. by equality,	CT : CA : : ca : de.
But, by sim. tri.	CT : CK : : ce : de;
theref. by equality,	CK : CA : : ca : ce,
and the rectangle	CK . ce = CA . ca.
But the rect.	CK . ce = the parallelogram GEre,
theref. the rect.	CA . ca = the parallelogram CEre,
conseq. the rect.	AB . ab = the parallelogram PQRS. Q.E.D.

THEOREM XI.

The Sum of the Squares of every Pair of Conjugate Diameters, is equal to the same constant Quantity, namely, the Sum of the Squares of the two Axes.

That is,
$AB^2 + ab^2 = EG^2 + eg^2$;
where EG, eg, are any pair of conjugate diameters.

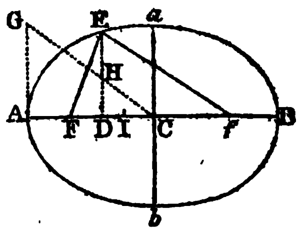


For, draw the ordinates ED, ed,

Then, by cor. to Theor. 10,	$CA^2 = CD^2 + cd^2$,
and . . .	$ca^2 = DE^2 + de^2$;
therefore the sum	$CA^2 + ca^2 = CD^2 + DE^2 + cd^2 + de^2$.
But, by right-angled $\triangle$s,	$CE^2 = CD^2 + DE^2$,
and . . .	$ce^2 = cd^2 + de^2$;
therefore the sum	$CE^2 + ce^2 = CD^2 + DE^2 + cd^2 + de^2$.
consequently .	$CA^2 + ca^2 = CE^2 + ce^2$;
or, by doubling,	$AB^2 + ab^2 = EG^2 + eg^2$. Q. E. D.

THEOREM XII.

The difference between the semi-transverse and a line drawn from the focus to any point in the curve, is equal to a fourth proportional to the semi-transverse, the distance from the centre to the focus, and the distance from the centre to the ordinate belonging to that point of the curve.



That is,

$AC - FE = CI$, or $FE = AI$;
and $fE - AC = CI$, or $fE = BI$.
Where $CA : CF :: CD : CI$ the 4th proportional to CA, CF, CD.

For, draw AG parallel and equal to Ca the semi-conjugate; and join CG meeting the ordinate DE in H.
Then, by theor. 2, $CA^2 : AG^2 :: CA^2 - CD^2 : DE^2$:
and, by sim. tri. $CA^2 : AG^2 :: CA^2 - CD^2 : AG^2 - DH^2$;
consequently $DE^2 = AG^2 - DH^2 = Ca^2 - DH^2$.
Also - $FD = CF \backsim CD$, and $FD^2 = CF^2 - 2CF \cdot CD + CD^2$;
but by right-angled triangles, $FD^2 + DE^2 = FE^2$;
therefore - $FE^2 = CF^2 + Ca^2 - 2CF \cdot CD + CD^2 - DH^2$.
But by theor. 4, $Ca^2 + CF^2 = CA^2$;
and, by supposition, $2CF \cdot CD = 2CA \cdot CI$;
therefore - - $FE^2 = CA^2 - 2CA \cdot CI + CD^2 - DH^2$.
But by supposition, $CA^2 : CD^2 :: CF^2$ or $CA^2 - AG^2 : CI^2$.
and, by sim. tri. $CA^2 : CD^2 :: CA^2 - AG^2 : CD^2 - DH^2$;
therefore - - $CI^2 = CD^2 - DH^2$;
consequently - $FE^2 = CA^2 - 2CA \cdot CI + CI^2$.
And the root or side of this square is $FE = CA - CI = AI$.
In the same manner is found $fE = CA + CI = BI$. Q. E. D.

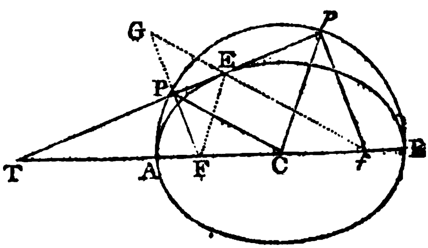
Corol. 1. Hence CI or $CA - FE$ is a 4th proportional to CA, CF, CD.

Corol. 2. And $fE - FE = 2CI$; that is, the difference between two lines draw from the foci, to any point in the curve, is double the 4th proportional to CA, CF, CD.

THEOREM XIII.

If a line be drawn from either focus, perpendicular to a tangent to any point of the curve; the distance of their intersections from the centre will be equal to the semi-transverse axis.

That is, if FP, fp, be perpendicular to the tangent TPp, then shall CP and Cp be each equal to CA or CB.



For through the point of contact E draw FE, and *f*E meeting FP produced in G. Then the ∠ GEP = ∠ FEP, being each equal to the ∠ *f*E*p*, and the angles at P being right, and the side PE being common, the two triangles GEP, FEP are equal in all respects, and so GE = FE, and GP = FP. Therefore, since FP = ½FG, and FC = ½F*f*, and the angle at F common, the side CP will be = ½*f*G or ½AB, that is CP = CA or CB. And in the same manner C*p* = CA or CB. Q. E. D.

Corol. 1. A circle described on the transverse axis, as a diameter, will pass through the points P, *p*; because all the lines CA, CP, C*p*, CB, being equal, will be radii of the circle.

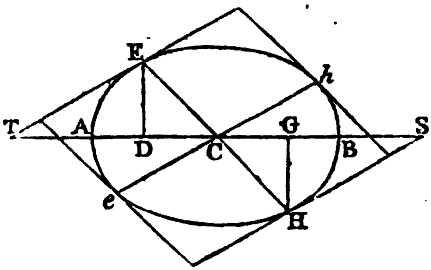
Corol. 2. CP is parallel to *f*E, and C*p* parallel to FE.

Corol. 3. If at the intersections of any tangent, with the circumscribed circle, perpendiculars to the tangent be drawn, they will meet the transverse axis in the two foci. That is, the perpendiculars PF, *pf* give the foci F, *f*.

THEOREM XIV.

The equal ordinates, or the ordinates at equal distances from the centre, on the opposite sides and ends of an ellipse, have their extremities connected by one right line passing through the centre, and that line is bisected by the centre.

That is, if CD = CG, or the ordinate DE = GH;
then shall CE = CH, and ECH will be a right line.



For when CD = CG, then also is DE = GH by cor. 2, th. 1. But the ∠ D = ∠ G, being both right angles; therefore the third side CE = CH, and the ∠ DCE = ∠ GCH, and consequently ECH is a right line.

Corol. 1. And, conversely, if ECH be a right line passing through the centre; then shall it be bisected by the centre, or have CE = CH; also DE will be = GH, and CD = CG.

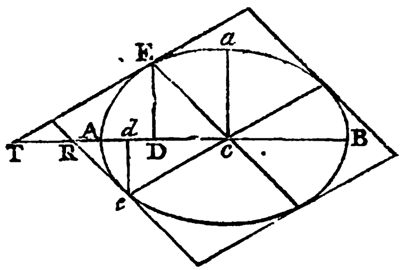
Corol. 2. Hence also, if two tangents be drawn to the two ends E, H of any diameter EH; they will be parallel to each other, and will cut the axis at equal angles, and at equal distances from the centre. For, the two CD, CA being equal to the two CG, CB, the third proportionals CT, CS will be equal also; then the two sides CE, CT being equal to the two CH, CS, and the included angle ECT equal to the included angle HCS, all the other corresponding parts are equal: and so the ∠ T = ∠ S, and TE parallel to HS.

Corol. 3. And hence the four tangents, at the four extremities of any two conjugate diameters, form a parallelogram circumscribing the ellipse, and the pairs of opposite sides are each equal to the corresponding parallel conjugate diameters. For, if the diameter *eh* be drawn parallel to the tangent TE or HS, it will be the conjugate to EH by the definition; and the tangents to *e*, *h* will be parallel to each other, and to the diameter EH for the same reason.

THEOREM XV.

If two ordinates ED, *ed* be drawn from the extremities E, *e*, of two conjugate diameters, and tangents be drawn to the same extremities, and meeting the axis produced in T and R;

Then shall CD be a mean proportional between C*d*, *d*R, and C*d* a mean proportional between CD, DT.



For, by theor. 7,	CD : CA : : CA : CT,
and by the same,	C*d* : CA : : CA : CR ;
theref. by equality,	CD : C*d* : : CR : CT,
But, by sim. tri.	DT : C*d* : : CT : CR ;
theref. by equality,	CD : C*d* : : C*d* : DT.
In like manner,	C*d* : CD : : CD : *d*R. Q. E. D.

Corol. 1. Hence CD : C*d* : : CR : CT.

Corol. 2. Hence also CD : cd : : de : DE.
And the rectangle CD . DE = cd . de, or △ CDE = △ cde.

Corol. 3. Also Dd^2 = CD . DT,
and $CD^2 = cd . d$E,

Or cd a mean proportional between CD, DT;
and CD a mean proportional between cd, dE.

THEOREM XVI.

The same figure being constructed as in the last theorem, each ordinate will divide the axis, and the semi-axis added to the external part, in the same ratio.

[See the last fig.]

That is, DA : DT : : DC : DE,
and dA : dE : : dC : dB.

For, by Theor. 7, CD : CA : : CA : CT,
and by div. CD : CA : : AD : AT,
and by comp. CD : DB : : AD : DT,
or, - - - DA : DT : : DC : DB.
In like manner, dA : dE : : dC : dB. Q. E. D.

Corol. 1. Hence, and from cor. 3 to the last, it is,
cd^2 = CD . DT = AD . DB = $CA^2 - CD^2$,
CD^2 = cd . dE = Ad . dB = $CA^2 - cd^2$.

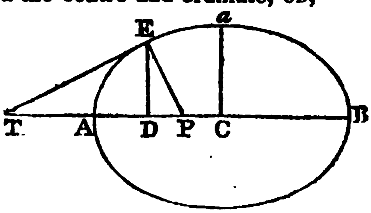
Corol. 2. Hence also, $CA^2 = CD^2 + cd^2$,
and $ca^2 = DE^2 + de^2$.

Corol. 3. Further, because $CA^2 : ca^2$: : AD . DB or $cd^2 : DE^2$,
therefore CA : ca : : cd : DE,
likewise CA : ca : : CD : de.

THEOREM XVII.

If from any point in the curve there be drawn an ordinate, and a perpendicular to the curve, or to the tangent at that point: then, the

Dist. on the trans. between the centre and ordinate, CD,
Will be to the dist. PD,
As sq. of the trans. axis
To sq. of the conjugate.



That is,
$CA^2 : ca^2$: : DC : DP.

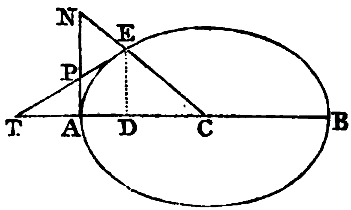
For, by theor. 2, $CA^2 : ca^2$: : AD . DB : DE^2,

But, by rt. angled Δs, the rect. TD . DP $=$ DE^2;
and, by cor. 1, theor. 16, CD . DT $=$ AD . DB;
therefore - - - - CA^2 : ca^2 : : TD . DC : TD . DP,
or - - - - - - AC^2 : ca^2 : : DC : DP. Q. E. D.

THEOREM. XVIII.

If there be two tangents drawn, the one to the extremity of the transverse, and the other to the extremity of any other diameter, each meeting the other's diameter produced; the two tangential triangles so formed, will be equal.

That is, the triangle CET $=$ the triangle CAN.



For, draw the ordinate DE. Then
By sim. triangles, CD : CA : : CE : CN;
but, by theor. 7, CD : CA : : CA : CT;
theref. by equal. CA : CT : : CE : CN.

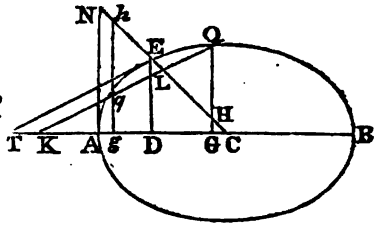
The two triangles CET, CAN, have then the angle C common, and the sides about that angle reciprocally proportional; those triangles are therefore equal, namely, the ΔCET $=$ ΔCAN.

Corol. 1. From each of the equal tri. CET, CAN, take the common space CAPE, and there remains the external ΔPAT $=$ ΔPNE.

Corol. 2. Also from the equal triangles CET, CAN, take the common triangle CED, and there remains the ΔTED $=$ trapez. ANED.

THEOREM XIX.

The same being supposed as in the last proposition; then any lines KQ, QG, drawn parallel to the two tangents, shall also cut off equal spaces. That is,



Δ KQG = trapez. ANHG,
and Δ Kqg = trapez. ANhg

For, draw the ordinate DE. Then
the three sim. triangles CAN, CDE, CGH,
are to each other as CA^2, CD^2, CG^2;
th. by div. the trap. ANED : trap. ANHG :: $CA^2-CD^2 : CA^2-CG^2$.
But, by theor. 1, $DE^2 : GQ^2 :: CA^2-CD^2 : CA^2-CG^2$.
theref. by equ. trap. ANED : trap. ANHG :: $DE^2 : GQ^2$.
But, by sim. Δs, tri. TED : tri. KQG :: $DE^2 : GQ^2$.
theref. by equality, ANED : TED :: ANHG : KQG.
But, by cor. 2, theor. 18, the trap. ANED = Δ TED;
and therefore the trap. ANHG = Δ KQG.
In like manner the trap. ANhg = Δ Kqg. Q. E. D.

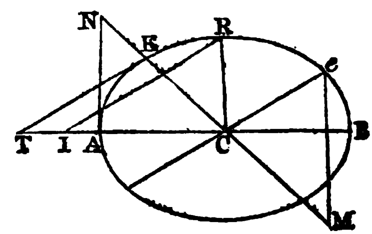
Corol. 1. The three spaces ANHG, TEHG, KQG, are all equal.

Corol. 2. From the equals ANHG, KQG,
take the equals ANhg, Kqg,
and there remains ghHG = gqQG.

Corol. 3. And from the equals ghHG, gqQG,
take the common space gqLHG,
and there remains the Δ LQH = Δ Lqh.

Corol. 4. Again, from the equals KQG, TEHG,
take the common space KLHG,
and there remains TELK = Δ LQH.

Corol. 5. And when by the lines KQ, GH, moving with a parallel motion, KQ comes into the position IR, where CR is the conjugate to CA; then



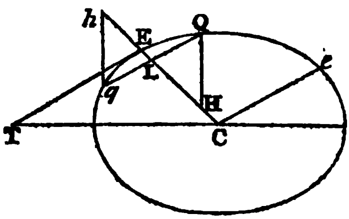
the triangle KQG becomes the triangle IRC,
and the space ANHG becomes the triangle ANC;
and therefore the Δ IRC = Δ ANC = TEC.

Corol. 6. Also when the lines KQ and HQ, by moving with a parallel motion, come into the position Ce, Me,

the triangle LQH becomes the triangle CEM.
and the space TELK becomes the triangle TEC;
and theref. the ΔCEM = ΔTEC = ΔANC = ΔIRC.

THEOREM XX.

Any diameter bisects all its double ordinates, or the lines drawn parallel to the tangent at its vertex, or to its conjugate diameter.



That is, if Q*q* be parallel to the tangent TE, or to C*e*, then shall LQ = L*q*.

For, draw QH, *qh* perpendicular to the transverse.
Then by cor. 3, theor. 19, the ΔLQH = L*qh*;
but these triangles are also equiangular;
consequently their like sides are equal, or LQ = L*q*.

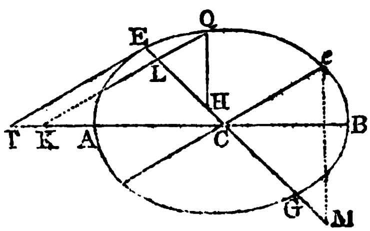
Corol. Any diameter divides the ellipse into two equal parts.

For, the ordinates on each side being equal to each other, and equal in number; all the ordinates, or the area, on one side of the diameter, is equal to all the ordinates, or the area, on the other side of it.

THEOREM XXI.

As the square of any diameter
Is to the square of its conjugate,
So is the rectangle of any two abscisses
To the square of their ordinate.

That is, $CE^2 : Ce^2 :: EL \cdot LG$ or $CE^2 - CL^2 : LQ^2$.



For, draw the tangent TE, and produce the ordinate QL to the transverse at K. Also draw QH, *e*M perpendicular to the transverse, and meeting EG in H and M.

Then, similar triangles

being as the squares of their like sides, it is,

by sim. triangles, $\triangle$CET : $\triangle$CLK : : $CE^2 : CL^2$;

or, by division, $\triangle$CET : trap. TELK : : $CE^2 : CE^2 - CL^2$.

Again, by sim. tri. $\triangle$ceM : $\triangle$LQH : : $ce^2 : LQ^2$.

But, by cor. 5 theor. 19, the $\triangle$ceM = $\triangle$CET,

and, by cor. 4 theor. 19, the $\triangle$IQH = trap. TELK ;

theref. by equality, $CE : ce^2 : : CE^2 - CL^2 : LQ^2$,

or - - $CE^2 : ce^2 : :$ EL . LG : LQ^2. Q. E. D.

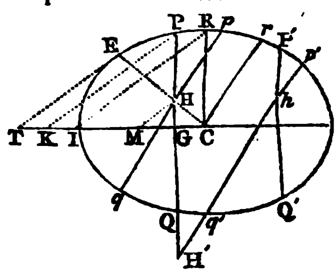
Corol. 1. The squares of the ordinates to any diameter, are to one another as the rectangles of their respective abscisses, or as the difference of the squares of the semi-diameter and of the distance between the ordinate and centre. For they are all in the same ratio of CE^2 to ce^2.

Corol. 2. The above being a similar property to that belonging to the two axes, all the other properties before laid down, for the axes, may be understood of any two conjugate diameters whatever, using only the oblique ordinates of these diameters, instead of the perpendicular ordinates of the axes; namely, all the properties in theorens 6, 7, 8, 14, 15, 16, 18, and 19.

THEOREM XXII.

If any two lines, that any where intersect each other, meet the curve each in two points; then the rectangle of the segments of the one is to the rectangle of the segments of the other, as the square of the diam. parallel to the former to the square of the diam. parallel to the latter.

That is, if CR and Cr be parallel to any two lines PHQ, pHq : then shall $CR^2 : Cr^2 : :$ PH . HQ : pH . Hq.



For, draw the diameter CHE, and the tangent TE, and its parallels PK, RI, MH, meeting the conjugate of the diameter CR in the points T, K, I, M. Then, because similar triangles are as the squares of their like sides, it is,

by sim. triangles $CR^2 : GP^2 : :$ $\triangle$CRI : $\triangle$GPK,

and - - - - $CR^2 : GH^2 : :$ $\triangle$CRI : $\triangle$GHM ;

theref. by division, $CR^2 : GP^2 - GH^2 : :$ CRI : KPHM.

Again, by sim. tri. $CE^2 : CH^2 :: \triangle CTE : \triangle CMH$;
and by division, $CE^2 : CE^2 - CH^2 :: \triangle CTE : TEHM$.
But, by cor. 5 theor. 19, the $\triangle CTE = \triangle CIR$,
and by cor. 1 theor. 19, $TKHG = KPHG$, or $TEHM = KPHM$;
theref. by equ. $CE^2 : CE^2 - CH^2 :: CR^2 : GP^2 - GH^2$ or $PH . HQ$.
In like manner $CE^2 : CE^2 - CH^2 :: cr^2 : pH . Hq$.
Theref. by equ. $CR^2 : cr^2 :: PH . HQ : pH . Hq$. Q. E. D.

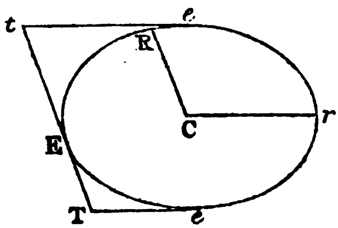
Corol. 1. In like manner, if any other line $p'H'q'$, parallel to cr or to pq, meet PHQ; since the rectangles $PH'Q$, $p'H'q'$ are also in the same ratio of CR^2 to cr^2; therefore rect. $PHQ : pHq :: PH'Q : p'H'q'$.

Also, if another line $P\,hQ'$ be drawn parallel to PQ or CR; because the rectangles, $P\,hQ'$, $p\,hq$ are still in the same ratio, therefore, in general, the rect. $PHQ : pHq :: P\,hQ' : p\,hq'$.

That is, the rectangles of the parts of two parallel lines, are to one another, as the rectangles of the parts of two other parallel lines, any where intersecting the former.

Corol. 2. And when any of the lines only touch the curve, instead of cutting it, the rectangles of such become squares, and the general property still attends them.

That is,
$CR^2 : cr^2 :: TE^2 : Te^2$,
or $CR : cr :: TE : Te$.
and $CR : cr :: tE : te$.

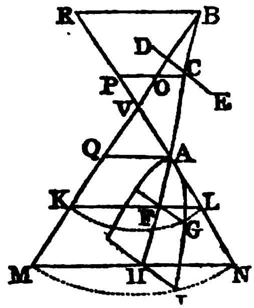


Corol. 3. And hence $TE : Te :: tE : te$.

OF THE HYPERBOLA.

THEOREM I.

The Squares of the Ordinates of the Axis are to each other as the Rectangles of their Abscisses.



Let AVB be a plane passing through the vertex and axis of the opposite cones; AGIH another section of them perpendicular to the plane of the former; AB the axis of the hyperbolic sections; and FG, HI. ordinates perpendicular to it. Then it will be, as $FG^2 : HI^2 :: AF . FB : AH . HB$.

For, through the ordinates FG, HI, draw the circular sections KGL, MIN, parallel to the base of the cone, having KL, MN, for their diameters, to which FG, HI, are ordinates, as well as to the axis of the hyperbola.

Now, by the similar triangles AFL, AHN, and BFK, BHM,

it is $AF : AH :: FL : HN$,

and $FB : HB :: KF : MH$;

hence, taking the rectangles of the corresponding terms,

it is, the rect. $AF . FB : AH . HB :: KF . FL : MH . HN$.

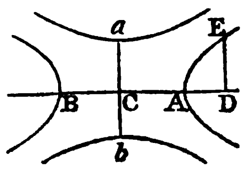
But, by the circle, $KF . FL = FG^2$, and $MH . HN = HI^2$;

Therefore the rect. $AF . FB : AH . HB :: FG^2 : HI^2$.

Q. E. D.

THEOREM II.

As the Square of the Transverse Axis :
Is to the Square of the Conjugate : :
So is the Rectangle of the Abscisses :
To the Square of their Ordinate.



That is, $AB^2 : ab^2$ or $AC^2 : ac^2 :: AD . DB : DE^2$.

For, by theor. 1, $CA \cdot CB : AD \cdot DB :: Ca^2 : DE^2$;
But, if C be the centre, then $AC \cdot CB = AC^2$, and Ca is the semi-conj.

Therefore $\quad AC^2 : AD \cdot DB :: aC^2 : DE^2$;
or, by permutation, $AC^2 : aC^2 :: AD \cdot DB : DE^2$;
or, by doubling, $AB^2 : ab^2 :: AD \cdot DB : DE^2$. Q. E. D.

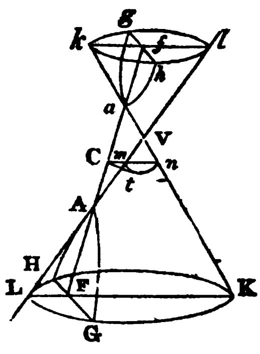
Corol. Or, by div. $AB : \frac{ab^2}{AB} :: AD \cdot DB$ or $CD^2 - CA^2 : DE^2$,

that is, $AB : p :: AD \cdot DB$ or $CD^2 - CA^2 : DE^2$;

where p is the parameter $\frac{ab^2}{AB}$ by the definition of it.

That is, As the transverse,
Is to its parameter,
So is the rectangle of the abscisses,
To the square of their ordinate.

Otherwise, thus:



Let a continued plane, cut from the two opposite cones, the two mutually connected opposite hyperbolas HAG, *hag*, whose vertices are A, *a*, and bases HG, *hg*, parallel to each other, falling in the planes of the two parallel circles LGK, *lgk*. Through C, the middle point of A*a*, let a plane be drawn parallel to that of LGK, it will cut in the cone LVK a circular section whose diameter is *mn*; to which circular section, let C*t* be a tangent at *t*.

Then, by sim. tri. ACm, AFL } $AC : Cm :: AF : FL$;
and, by sim. tri. aCn, aFK } $aC : Cn :: aF : FK$.

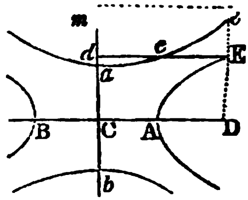
$\therefore AC \cdot Ca : Cm \cdot Cn :: AF \cdot Fa : LF \cdot FK$,
or, $AC^2 : Ct^2 :: AF \cdot Fa : FG^2$.

In like manner, for the opposite hyperbola
$AC^2 : Ct^2 :: Af \cdot fa : fg^2$.

Here C*t* is what is usually denominated the semi-conjugate to the opposite hyperbolas HAK, *hak*: but it is evidently not *in the same plane* with them.

THEOREM III.

As the Square of the Conjugate Axis
Is to the Square of the Transverse Axis,
So is the Sum of the Squares of the Semi-conjugate, and distance of the Centre from any Ordinate of the Axis,
To the Square of the Ordinate.



That is,
$ca^2 : CA^2 :: ca^2 + cd^2 : dE^2$.

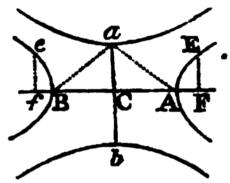
For, draw the ordinate ED to the transverse AB.

Then, by theor. 1, $ca^2 : CA^2 :: DE^2 : AD . DB$ or $CD^2 - CA^2$,
or - - $ca^2 : CA^2 :: cd^2 : dE^2 - CA^2$.
But - - $ca^2 : CA^2 :: ca^2 : CA^2$.
theref. by compos. $ca^2 : CA^2 :: CA^2 + cd^2 : dE^2$.
In like manner, $CA^2 : ca^2 :: CA^2 + CD^2 : De^2$. Q. E. D.

Corol. By the last theor. $CA^2 : ca^2 :: CD^2 - CA^2 : DE^2$,
and by this theor. $CA^2 : ca^2 :: CD^2 + CA^2 : De^2$,
therefore - $DE^2 : De^2 :: CD^2 - CA^2 : CD^2 + CA^2$.
In like manner, $De^2 : dE^2 :: cd^2 - ca^2 : CD^2 + ca^2$.

THEOREM IV.

The Square of the Distance of the Focus from the Centre, is equal to the Sum of the Squares of the Semi-axes.
Or, the Square of the Distance between the Foci, is equal to the Sum of the Squares of the two Axes.



That is,
$CF^2 = CA^2 + ca^2$, or
$Ff^2 = AB^2 + ab^2$.

For, to the focus F draw the ordinate FE; which, by the definition, will be the semi-parameter. Then, by the nature
of the curve - $CA^2 : ca^2 :: CF^2 - CA^2 : FE^2$;
and by the def. of the para. $CA^2 : ca^2 :: ca^2 : FE^2$;
therefore - - $ca^2 = CF^2 - CA^2$;
and by addition - $CF^2 = CA^2 + ca^2$;
or, by doubling, - $Ff^2 = AB^2 + ab^2$. Q. E. D.

Corol. 1. The two semi-axes, and the focal-distance from the centre, are the sides of a right-angled triangle CAa; and the distance Aa is $= CF$ the focal distance.

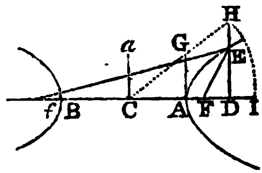
Corol. 2. The conjugate semi-axis Ca is a mean proportional between AF, FB, or between Af, fB, the distances of either focus from the two vertices.

For $Ca^2 = CF^2 - CA^2 = (CF + CA) . (CF - CA) = AF . FB$.

THEOREM V.

The Difference of two Lines drawn from the two Foci to meet at any Point in the Curve, is equal to the Transverse Axis.

That is,
$fE - FE = AB$.



For, draw AG parallel and equal to Ca the semi-conjugate; and join CG, meeting the ordinate DE produced in H; also take CI a 4th proportional to CA, CF, CD.

Then, by th. 2, $CA^2 : AG^2 :: CD^2 - CA^2 : DE^2$;
and, by sim. $\triangle$s, $CA^2 : AG^2 :: CD^2 - CA^2 : DH^2 - AG^2$;
consequently $DE^2 = DH^2 - AG^2 = DH^2 - Ca^2$.
Also, $FD = CF \sim CD$, and $FD^2 = CF^2 - 2CF . CD + CD^2$;
and, by right-angled triangles, $FE^2 = FD^2 + DE^2$.
therefore $FE^2 = CF^2 - Ca^2 - 2CF . CD + CD^2 + DH^2$.
But, by theor. 4, $CF^2 - Ca^2 = CA^2$,
and, by supposition, $2CF . CD = 2CA . CI$;
theref. $FE^2 = CA^2 - 2CA . CI + CD^2 + DH^2$;
Again, by suppos. $CA^2 : CD^2 :: CF^2$ or $CA^2 + AG^2 : CI^2$;
and, by sim. tri. $CA^2 : CD^2 :: CA^2 + AG^2 : CD^2 + DH^2$;
therefore $CI^2 = CD^2 + DH^2 = CH^2$;
consequently $FE^2 = CA^2 - 2CA . CI + CI^2$.
And the root or side of this square is $FE = CI - CA = AI$.
In the same manner, it is found that $fE = CI + CA = BI$.
Conseq. by subtract. $fE - FE = BI - AI = AB$. Q. E. D.

Corol. 1. Hence $CH = CI$ is a 4th proportional to CA, CF, CD.

Corol. 2. And $fE + FE = 2CH$ or $2CI$; or FE, CH, fE, are in continued arithmetical progression, the common difference being CA the semi-transverse.

Corol. 3. Hence is derived the common method of describing this curve mechanically by points, thus:

In the transverse AB, produced, take the foci F, *f*, and any point I. Then with the radii AI, BI, and centres F, *f*, describe arcs intersecting in E, which will be a point in the curve. In like manner, assuming other points I, as many other points will be found in the curve.

Then, with a steady hand, the curve line may be drawn through all the points of intersection E.

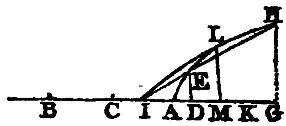
In the same manner are constructed the other two or conjugate hyperbolas, using the axis *ab* instead of AB.

THEOREM VI.

If from any Point I in the Axis, a Line IL be drawn touching the Curve in one Point L; and the Ordinate LM be drawn; and if C be the Centre or the Middle of AB: Then shall CM be to CI as the Square of AM to the Square of AI.

That is,

$$CM : CI :: AM^2 : AI^2.$$



For, from the point I draw any line IEH to cut the curve in two points E and H: from which let fall the perps. ED, HG; and bisect DG in K.

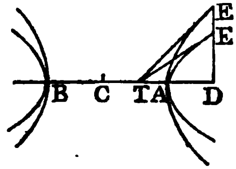
Then, by theor. 1, $AD . DB : AG . GB :: DE^2 : GH^2$,
and by sim. triangles, $ID^2 : IG^2 :: DE^2 : GH^2$;
theref. by equality, $AD . DB : AG . GB :: ID^2 : IG^2$.
But $DB = CB + CD = CA + CD = CG + CD - AG = 2CK - AG$,
and $GB = CB + CG = CA + CG = CG + CD - AD = 2CK - AD$;
theref. $AD . 2CK - AD . AG : AG . 2CK - AD . AG :: ID^2 : IG^2$,
and, by div. $DG . 2CK : IG^2 - ID^2$ or $DG . 2IK :: AD . 2CK - AD . AG : ID^2$;
or - $2CK : 2IK :: AD . 2CK - AD . AG : ID^2$;
or $AD . 2CK : AD . 2IK :: AD . 2CK - AD . AG : ID^2$;
theref. by div. $CK : IK :: AD . AG : AD . 2IK - ID^2$,
and, by div. $CK : CI :: AD . AG : ID^2 - AD . (ID + IA)$,
or - $CK : CI :: AD . AG : AI^2$.

But, when the line IH, by revolving about the point I, comes into the position of the tangent IL, then the points E and H meet in the point L, and the points D, K, G, coincide with the point M; and then the last proportion becomes $CM : CI :: AM^2 : AI^2$.

Q. E. D.

THEOREM VII.

If a Tangent and Ordinate be drawn from any Point in the Curve, meeting the Transverse Axis; the Semi-transverse will be a Mean Proportional between the Distances of the said Two Intersections from the Centre.



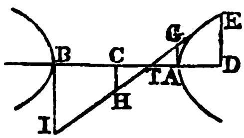
That is,
CA is a mean proportional between CD and CT; or CD, CA, CT, are continued proportionals.

For, by th. 6, $CD : CT :: AD^2 : AT^2$,
that is, - $CD : CT :: (CD - CA)^2 : (CA - CT)^2$,
or - - $CD : CT :: CD^2 + CA^2 : CA^2 + CT^2$,
and - - $CD : DT :: CD^2 + CA^2 : CD^2 - CT^2$,
or - - $CD : DT :: CD^2 + CA^2 : (CD + CT)\ DT$,
or $CD^2 : CD . DT :: CD^2 + CA^2 : CD . DT + CT . TD$;
hence $CD^2 : CA^2 :: CD . DT : CT . TD$,
and $CD^2 : CA^2 :: CD : CT$,
theref. (th. 78, Geom.) $CD : CA :: CA : CT$. Q. E. D.

Corol. Since CT is always a third proportional to CD, CA; if the points D, A, remain constant, then will the point T be constant also; and therefore all the tangents will meet in this point T, which are drawn from the point E, of every hyperbola described on the same axis AB, where they are cut by the common ordinate DEE drawn from the point D.

THEOREM VIII.

If there be any Tangent meeting four Perpendiculars to the Axis drawn from these four Points, namely, the Centre, the two Extremities of the Axis, and the Point of Contact; those four Perpendiculars will be Proportionals.



That is,
AG : DE :: CH : BI.

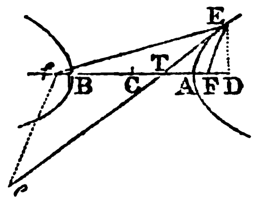
For, by theor. 7, $TC : AC :: AC : DC$,
theref. by div. $TA : AD :: TC : AC$ or CB,
and by comp. $TA : TD :: TC : TB$,
and by sim. tri. $AG : DE :: CH : BI$. Q. E. D.

Corol. Hence TA, TD, TC, TB and TG, TE, TH, TI } are also proportionals.
For these are as AG, DE, CH, BI, by similar triangles.

THEOREM IX.

If there be any Tangent, and two Lines drawn from the Foci to the Point of Contact; these two Lines will make equal Angles with the Tangent.

That is,
the ∠ FET = ∠ *f*Ee.



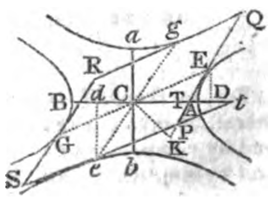
For, draw the ordinate DE, and *fe* parallel to FE.
By cor. 1, theor. 5, CA : CD : : CF : CA + FE,
and by th. 7, CA : CD : : CT : CA;
therefore - CT : CF : : CA : CA + FE;
and by add. and sub. TF : T*f* : : FE : 2CA + FE or *f*E by th. 5.
But by sim. tri. TF : T*f* : : FE : *f*E;
therefore - *f*E = *fe*, and conseq. ∠*e* = ∠*f*E*e*.
But, because FE is parallel to *fe*, the ∠*e* = ∠FET;
therefore the ∠ FET = ∠*f*E*e*. Q. E. D.

Corol. As opticians find that the angle of incidence is equal to the angle of reflection, it appears, from this proposition, that rays of light issuing from the one focus, and meeting the curve in every point, will be reflected into lines drawn from the other focus. So the ray *f*E is reflected into FE. And this is the reason why the points F, *f*, are called *foci*, or burning points.

THEOREM X.

All the Parallelograms inscribed between the four Conjugate Hyperbolas are equal to one another, and each equal to the Rectangle of the two Axes.

That is,
the parallelogram PQRS = the rectangle AB . *ab*.



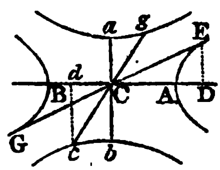
Let EG, *eg*, be two conjugate diameters parallel to the sides of the parallelogram, and dividing it into four less and equal parallelograms. Also, draw the ordinates DE, *de*, and CK perpendicular to PQ; and let the axis produced meet the sides of the parallelograms, produced, if necessary, in T and *t*.

Then, by theor. 7,	CT : CA : : CA : CD,
and - -	*ct* : CA : : CA : *cd*;
theref. by equality,	CT : *ct* : : *cd* : CD;
but, by sim. triangles,	CT : *ct* : : TD : *cd*,
theref. by equality,	TD : *cd* : : *cd* : CD,
and the rectangle	TD . DC is = the square cd^2.
Again, by theor. 7,	CD : CA : : CA : CT,
or, by division,	CD : CA : : DA : AT,
and, by composition,	CD : DB : : DA : DT;
conseq. the rectangle	CD . DT : : cd^2 = AD . DB*.
But, by theor. 1,	$CA^2 : ca^2$: : (AD . DB or) $cd^2 : DE^2$,
therefore -	CA : *ca* : : *cd* : DE,
or - -	*ca* : DE : : CA : *cd*.
By theor. 7, -	CA : *ct* : : *cd* : CA.
By equality -	*ct* : CA : : *ca* . DE.
By sim. tri. -	*ct* : CT : : *de* : DE;
theref. by equality,	CT : CA : : *ca* : *de*.
But, by sim. tri.	CT : CK : : *ce* : *de*;
theref. by equality,	CK : CA : : *ca* : *ce*.
and the rectangle	CK . *ce* = CA . *ca*.
But the rect.	CK . *ce* = the parallelogram CEPe,
theref. the rect.	CA . *ce* = the parallelogram CEPe,
conseq. the rect.	AB . *ab* = the paral. PQRS. Q. E. D.

THEOREM XI.

The Difference of the Squares of every Pair of Conjugate Diameters, is equal to the same constant Quantity, namely, the Difference of the Squares of the two Axes.

That is.
$AB^2 - ab^2 = EG^2 - eg^2$;
where EG, *eg* are any conjugate diameters.



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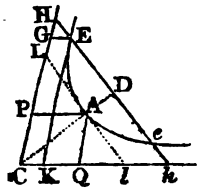
* *Corol.* Because $cd^2 = AD . DB = CD^2 - CA^2$,
therefore $CA^2 = CD^2 - cd^2$.
In like manner $CA^2 = de^2 - DE^2$.

For, draw the ordinates ED, ed.

Then, by cor. to theor. 10, $CA^2 = CD^2 - cd^2$,
and . . . . $ca^2 = de^2 - DE^2$;
theref. the difference $CA^2 - ca^2 = CD^2 + DE^2 - cd^2 - de^2$.
But, by right-angled $\triangle$s, $CE^2 = CD^2 + DE^2$,
and . . . . $ce^2 = cd^2 + de^2$;
theref. the difference $CE^2 - ce^2 = CD^2 + DE^2 - cd^2 - de^2$,
consequently . $CA^2 - ca^2 = CE^2 - ce^2$;
or, by doubling, $AB^2 - ab^2 = EG^2 - eg^2$. Q. E. D.

THEOREM XII.

All the Parallelograms are equal which are formed between the Asymptotes and Curve, by Lines drawn Parallel to the Asymptotes.



That is, the lines GE, EK, AP, AQ, being parallel to the asymptotes CH, Cl; then the paral. CGEK = paral. CPAQ.

For, let A be the vertex of the curve, or extremity of the semi-transverse axis AC, perp. to which draw AL or Al, which will be equal to the semi-conjugate, by definition 19. Also, draw HEDeh parallel to Ll,

Then, by theor. 2, $CA^2 : AL^2 :: CD^2 - CA^2 : DE^2$,
and, by parallels, $CA^2 : AL^2 :: CD^2 : DH^2$;
theref. by subtract. $CA^2 : AL^2 :: CA^2 : DH^2 - DE^2$ or rect. $HE \cdot Eh$;
conseq. the square $AL^2 =$ the rect. $HE \cdot Eh$.

But, by sim. tri. $PA : AL :: GE : EH$,
and, by the same, $QA : Al :: EK : Eh$;
theref. by comp. $PA \cdot AQ \cdot AL^2 :: GE \cdot EK : HE \cdot Eh$;
and because $AL^2 = HE \cdot Eh$, theref. $PA \cdot AQ = GE \cdot EK$.

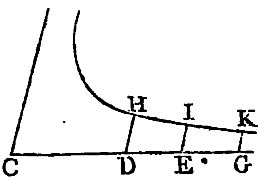
But the parallelograms CGEK, CPAQ, being equiangular, are as the rectangles $GE \cdot EK$ and $PA \cdot AQ$.

Therefore the parallelogram GK = the paral. PQ.

That is, all the inscribed parallelograms are equal to one another. Q. E. D.

Corol. 1. Because the rectangle GEK or CGE is constant, therefore GE is reciprocally as CG, or CG : CP :: PA : GE. And hence the asymptote continually approaches towards the curve, but never meets it: for GE decreases continually as CG increases; and it is always of *some* magnitude, except when CG is supposed to be infinitely great, for then GK is infinitely small, or nothing. So that the asymptote CG may be considered as a tangent to the curve at a point infinitely distant from C.

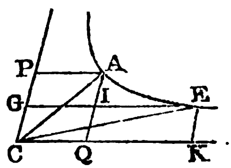
Corol. 2. If the abscisses CD, CE, CG, &c. taken on the one asymptote, be in geometrical progression increasing; then shall the ordinates DH, EI, GK, &c. parallel to the other asymptote, be a decreasing geometrical progression, having the same ratio. For, all the rectangles CDH, CEI, CGK, &c. being equal, the ordinates DH, EI, GK, &c, are reciprocally as the abscisses CD, CE, CG, &c. which are geometricals. And the reciprocals of geometricals are also geometricals, and in the same ratio, but decreasing, or in converse order.



THEOREM XIII.

The three following Spaces between the Asymptotes and the Curve, are equal; namely, the Sector or Trilinear Space contained by an Arc of the Curve and two Radii, or Lines drawn from its Extremities to the Centre; and each of the two Quadrilaterals, contained by the said Arc, and two Lines drawn from its Extremities parallel to one Asymptote, and the intercepted Part of the other Asymptote.

That is,
The sector CAE = PAEG = QAEK,
all standing on the same arc AE.



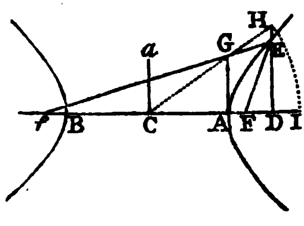
For, by theor. 12, CPAQ = CGEK;
subtract the common space CGIQ,
there remains the paral. PI = the par. IK;
To each add the trilineal IAE, then
the sum is the quadr. PAEG = QAEK.
Again, from the quadrilateral CAEK
take the equal triangles, CAQ, CEK,
and there remains the sector CAE = QAEK.
Therefore CAE = QAEK = PAEG. Q. E. D.

SCHOLIUM.

In the figure to theorem 12, cor. 2, if CD = 1, and CE, CG, &c. be any numbers, the hyperbolic spaces HDEI, IEGK, &c. are analogous to the *logarithms* of those numbers. For, whilst the numbers CD, CE, CG, &c. proceed in geometrical progression, the correspondent spaces proceed in arithmetical progression; and therefore, from the nature of logarithms are respectively proportional to the logarithms of those numbers. If the angle C were a right angle, and CD = DH = 1; then if CE were = 10, the space DEIH would be 2·30258509, &c.; if CG were = 100, then the space DGKH would be 4·60517018: these being the Napierean logarithms to 10 and 100 respectively. Intermediate arears corresponding to intermediate abscissæ would be the appropriate logarithms. These are usually called *Hyperbolic* logarithms; but the term is improper: for by drawing other hyperbolic curves between HIK and its asymptotes, other systems of logarithms would be obtained. Or, by changing the angle between the asymptotes, the same thing may be effected. Thus, when the angle C is a right angle, or has its sine = 1, the hyperbolic spaces indicate the Napierean logarithms; but when the angle is 25° 44′ 27½″, whose sine is = ·43429448, &c. the modulus to the common, or Briggs's, logarithms, the spaces DEIH, &c. measure those logarithms. In both cases, if spaces to the right of DH are regarded as *positive*, those to the left will be *negative*; whence it follows that the logarithms of numbers less than 1 are negative also.

THEOREM XIV.

The sum or difference of the semi-transverse and a line drawn from the focus to any point in the curve is equal to a fourth proportional to the semi-transverse, the distance from the centre to the focus, and the distance from the centre to the ordinate belonging to that point of the curve.



That is,
FE + AC = CI, or FE = AI;
and *f*E − AC = CI, or *f*E = BI.
Where CA : CF : : CD : CI the 4th propor. to CA, CF, CD.

For, draw AG parallel and equal to ca the semi-conjugate; and join CG meeting the ordinate DE produced in H.

Then, by theor. 2, $CA^2 : AG^2 :: CD^2 - CA^2 : DE^2$;
and, by sim. $\triangle$s, $CA^2 : AG^2 :: CD^2 - CA^2 : DH^2 - AG^2$;
consequently $DE^2 = DH^2 - AG^2 = DH^2 - ca^2$.

Also $FD = CF \backsim CD$, and $FD^2 = CF^2 - 2CF \cdot CD + CD^2$;
but, by right-angled triangles, $FD^2 + DE^2 = FE^2$;
therefore $FE^2 = CF^2 - ca^2 - 2CF \cdot CD + CD^2 + DH^2$.

But by theor. 4, $CF^2 - ca^2 = CA^2$,
and, by supposition, $2CF \cdot CD = 2CA \cdot CI$;
theref. $FE^2 = CA^2 - 2CA \cdot CI + CD^2 + DH^2$.

But, by supposition, $CA^2 : CD^2 :: CF^2$ or $CA^2 + AG^2 : CI^2$;
and, by sim. $\triangle$s, $CA^2 : CD^2 :: CA^2 + AG^2 : CD^2 + DH^2$;
therefore . $CI^2 = CD^2 + DH^2 = CH^2$;
consequently . $FE^2 = CA^2 - 2CA \cdot CI + CI^2$.

And the root or side of this square is $FE = CI - CA = AI$.
In the same manner is found $fE = CI + CA = BI$. Q. E. D.

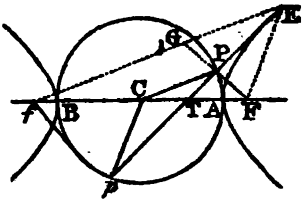
Corol. 1. Hence $CH = CI$ is a 4th propor. to CA, CF, CD.

Corol. 2. And $fE + FE = 2CH$ or $2CI$; or FE, CH, fE are in continued arithmetical progression, the common difference being CA the semi transverse.

Corol. 3. From the demonstration it appears, that $DE^2 = DH^2 - AG^2 = DH^2 - ca^2$. Consequently DH is every where greater than DE; and so the asymptote CGH never meets the curve, though they be ever so far produced: but DH and DE approach nearer and nearer to a ratio of equality as they recede farther from the vertex, till at an infinite distance they become equal, and the asymptote is a tangent to the curve at an infinite distance from the vertex.

THEOREM XV.

If a line be drawn from either focus, perpendicular to a tangent to any point of the curve; the distance of their intersection from the centre will be equal to the semi-transverse axis.



That is, if FP, fp be perpendicular to the tangent TPp, then shall CP and cp be each equal to CA or CB.

For, through the point of contact E draw FE, and fE, meeting FP produced in G. Then, the ∠GEP = ∠FEP, being each equal to the ∠fEp, and the angles at P being right, and the side PE being common, the two triangles, GEP, FEP are equal in all respects, and so GE = FE, and GP = FP. Therefore, since FP = ½FG, and FC = ½Ff, and the angle at F common, the side CP will be = ½fG or ½AB, that is CP = CA or CB.

And in the same manner cp = CA or CB. Q. E. D.

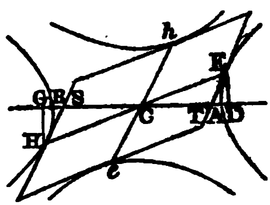
Corol. 1. A circle described on the transverse axis, as a diameter, will pass through the points P, p; because all the lines CA, Cp, CB, being equal, will be radii of the circle.

Corol. 2. CP is parallel to fE, and Cp parallel to FE.

Corol. 3. If at the intersections of any tangent with the circumscribed circle perpendiculars to the tangent be drawn, they will meet the transverse axis in the two foci. That is, the perpendiculars PF, pf give the foci F, f.

THEOREM XVI.

The equal ordinates, or the ordinates at equal distances from the centre, on the opposite sides and ends of an hyperbola, have their extremities connected by one right line passing through the centre, and that line is bisected by the centre.



That is, if CD = CG, or the ordinate DE = GH; then shall CE = CH, and ECH will be a right line.

For, when CD = CG, then also is DE = GH by cor. 2 theor. 1. But the ∠ D = ∠ G, being both right angles; therefore the third side CE = CH, and the ∠ DCE = ∠ GCH, and consequently ECH is a right line.

Corol. 1. And, conversely, if ECH be a right line passing through the centre; then shall it be bisected by the centre, or have CE = CH, also DE will be = GH, and CD = CG.

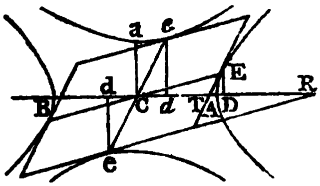
Corol. 2. Hence also, if two tangents be drawn to the two ends E, H of any diameter EH; they will be parallel to each other, and will cut the axis at equal angles, and at equal distances from the centre. For, the two CD, CA being equal to the two CG, CB, the third proportionals CT, CS will be equal also; then the two sides CE, CT being equal to the two CH, CS, and the included angle ECT equal to the included angle HCS, all the other corresponding parts are equal: and so the ∠ T = ∠ S, and TE parallel to HS.

Corol. 3. And hence the four tangents, at the four extremities of any two conjugate diameters, form a parallelogram inscribed between the hyperbolas, and the pairs of opposite sides are each equal to the corresponding parallel conjugate diameters.—For, if the diameter *eh* be drawn parallel to the tangent TE or HS, it will be the conjugate to EH by the definition; and the tangents to *e*, *h* will be parallel to each other, and to the diameter EH, for the same reason.

THEOREM XVII.

If two ordinates ED, *ed* be drawn from the extremities E, *e*, of two conjugate diameters, and tangents be drawn to the same extremities, and meeting the axis produced in T and R;

Then shall CD be a mean proportional between C*d*, *d*R, and C*d* a mean proportional between CD, DT.



For, by theor. 7,	CD : CA : : CA : CT,
and by the same,	C*d* : CA : : CA : CR;
theref. by equality,	CD : C*d* : : CR : CT.
But by sim. tri.	DT : C*d* : : CT : CR;
theref. by equality,	CD : C*d* : : C*d* : DT.
In like manner,	C*d* : CD : : CD : *d*R. Q. E. D.

Corol. 1. Hence CD : C*d* : : CR : CT.

Corol. 2. Hence also CD : C*d* : : *de* : DE.

And the rect. CD . DE = C*d* . *de*, or △ CDE = △ C*de*.

Corol. 3. Also $cd^2 = CD \cdot DT$, and $CD^2 = cd \cdot dR$.
Or cd a mean proportional between CD, DT;
and CD a mean proportional between cd, dR.

THEOREM XVIII.

The same figure being constructed as in the last proposition, each ordinate will divide the axis, and the semi-axis added to the external part, in the same ratio.

[See the last fig.]

That is, $DA : DT :: DC : DB$,
and $dA : dR :: dc : dB$.

For, by theor. 7, $CD : CA :: CA : CT$,
and by div. $CD : CA :: AD : AT$,
and by comp. $CD : DB :: AD : DT$,
or $DA : DT :: DC : DB$.
In like manner, $dA : dR :: dc : dB$. Q. E. D.

Corol. 1. Hence, and from cor. 3 to the last prop. it is
$cd^2 = CD \cdot DT = AD \cdot DB = CD^2 - CA^2$,
and $CD^2 = cd \cdot dR = Ad \cdot dB = CA^2 = cd^2$.

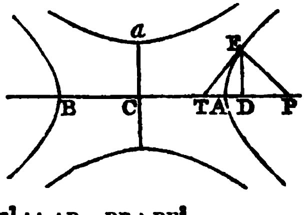
Corol. 2. Hence also $CA^2 = CD^2 - cd^2$, and $ca^2 = de^2 - DE^2$.

Corol. 3. Farther, because $CA^2 : ca^2 :: AD \cdot DB$ or $cd^2 : DE^2$.
therefore $CA : ca :: cd : DE$.
likewise $CA : ca :: CD : de$.

THEOREM XIX.

If from any point in the curve there be drawn an ordinate, and a perpendicular to the curve, or to the tangent at that point: then the

**Dist. on the trans. between the centre and ordinate, CD,
Will be to the dist. PD,
As square of trans. axis
To square of the conjugate.**



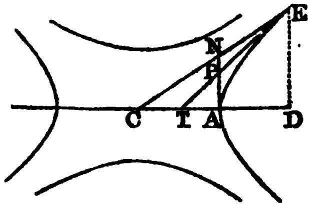
That is,
$CA^2 : ca^2 :: DC : DP$.

For, by theor. 2, $CA^2 : ca^2 :: AD \cdot DB : DE^2$,
But, by rt. angled Δs, the rect. $TD \cdot DP = DE^2$,
and, by cor. 1 theor. 16, $CD \cdot DT = AD \cdot DB$;
therefore $CA^2 : ca^2 :: TD \cdot DC : TD \cdot DP$,
or $CA^2 : ca^2 :: DC : DP$. Q. E. D.

THEOREM XX.

If there be two tangents drawn, the one to the extremity of the transverse, and the other to the extremity of any other diameter, each meeting the other's diameter produced: the two tangential triangles so formed, will be equal.

That is,
the triangle CET =
the triangle CAN.



For, draw the ordinate DE. Then
By sim. triangles, CD : CA : : CE : CN;
but, by theor. 7, CD : CA : : CA : CT;
theref. by equal. CA : CE : : CE : CN.

The two triangles CET, CAN have then the angle C common, and the sides about that angle reciprocally proportional; those triangles are therefore equal, viz. the $\triangle$CET = $\triangle$CAN.

Q. E. D.

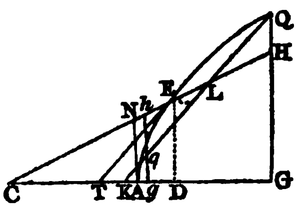
Corol. 1. Take each of the equal triangles CET, CAN, from the common space CAPE,
and there remains the external $\triangle$PAT = $\triangle$PNE.

Corol. 2. Also take the equal triangles CET, CAN, from the common triangle CED,
and there remains the $\triangle$TED = trapez. ANED.

THEOREM XXI.

The same being supposed as in the last proposition; then any lines KQ, GQ, drawn parallel to the two tangents, shall also cut off equal spaces.

That is,
the $\triangle$KQG = trapez. ANHG.
and $\triangle$K*qg* = trapez. AN*hg*.



For, draw the ordinate DE. Then
The three sim. triangles CAN, CDE, CGH,
are to each other as CA^2, CD^2, CG^2;
th. by div. the trap. ANED : trap. ANHG :: $CD^2 - CA^2 : CG^2 - CA^2$.
But, by theor. 1, $DE^2 : GQ^2 :: CD^2 - CA^2 : CG^2 - CA^2$;
theref. by equ. trap. ANED : trap. ANHG :: $DE^2 : GQ^2$.
But, by sim. Δs, tri. TED : tri. KQG :: $DE^2 : GQ^2$;
theref. by equal. ANED : TED :: ANHG : KQG.
But, by cor. 2 theor. 20, the trap. ANED = Δ TED;
and therefore the trap. ANHG = Δ KQG.
In like manner the trap. AN*hg* = Δ K*qg*. Q. E. D.

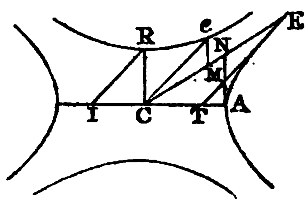
Corol. 1. The three spaces ANHG, TEHG, KQG are all equal.

Corol. 2. From the equals ANHG, KQG,
take the equals AN*hg*, K*qg*,
and there remains *gh*HG = *gq*QG.

Corol. 3. And from the equals *gh*HG, *gq*QG,
take the common space *gq*LHG,
and there remains the Δ LQH = Δ L*qh*.

Corol. 4. Again, from the equals KQG, TEHG,
take the common space KLHG,
and there remains TELK = Δ LQH.

Corol. 5. And when by the lines KQ, GH, moving with a parallel motion, KQ comes into the position IR, where CR is the conjugate to CA; then



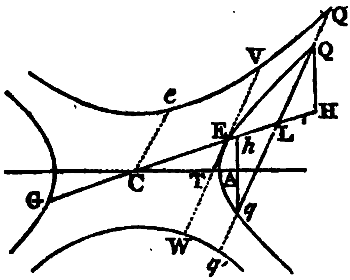
the triangle KQG becomes the triangle IRC,
and the space ANHG becomes the triangle ANC;
and therefore the Δ IRC = Δ ANC = Δ T. C.

Corol. 6. Also when the lines KQ and HQ, by moving with a parallel motion, come into the position C*e*, M*e*,
the triangle LQH becomes the triangle C*e*M,
and the space TELK becomes the triangle T*e*C;
and theref. the Δ C*e*M = Δ TEC = Δ ANC = Δ IRC.

THEOREM XXII.

Any diameter bisects all its double ordinates, or the lines drawn parallel to the tangent at its vertex, or to its conjugate diameter.

That is, if Qq be parallel to the tangent TE, or to Ce, then shall LQ = Lq.



For, draw QH, qh perpendicular to the transverse.
Then by cor. 3 theor. 21, the $\triangle$LQH = $\triangle$Lqh;
but these triangles are also equiangular;
conseq. their like sides are equal, or LQ = Lq.

Corol. 1. Any diameter divides the hyperbola into two equal parts.

For, the ordinates on each side being equal to each other, and equal in number; all the ordinates, or the area, on one side of the diameter, is equal to all the ordinates, or the area, on the other side of it.

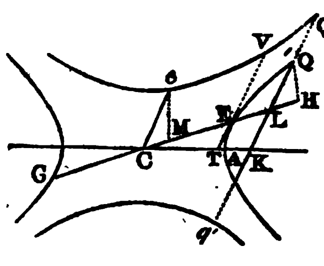
Corol. 2. In like manner, if the ordinate be produced to the conjugate hyperbolas at Q′, q′, it may be proved that LQ′ = Lq′. Or if the tangent TE be produced, then EV = EW. Also the diameter GCEH bisects all lines drawn parallel to TE or Qq, and limited either by one hyperbola, or by its two conjugate hyperbolas.

THEOREM XXIII.

As the square of any diameter
Is to the square of its conjugate,
So is the rectangle of any two abscisses
To the square of their ordinate.

That is, $CE^2 : Ce^2 :: EL \cdot LG$ or $CL^2 - CE^2 : LQ^2$.

For, draw the tangent TE, and produce the ordinate QL to the transverse at K. Also draw QH, eM perpendicular to the transverse, and meeting EG in H and M. Then, similar triangles being as the squares of their like sides, it is,



by sim. triangles, $\triangle CET : \triangle CLK :: CE^2 : CL^2$;

or, by division, $\triangle$ CET : trap. TELK :: $CE^2 : CL^2 - CE^2$.
Again, by sim. tri. $\triangle$ cem : $\triangle$ LQH :: ce^2 LQ^2.
But, by cor. 5 theor. 21, the $\triangle$ cem = $\triangle$ CET,
and, by cor. 4 theor. 21, the $\triangle$ LQH = trap. TELK;
theref. by equality, $CE^2 : ce^2 :: CL^2 - CE^2 : LQ^2$,
or - - - $CE^2 : ce^2 :: EL \cdot LG : LQ^2$. Q. E. D.

Corol. 1. The squares of the ordinates to any diameter, are to one another as the rectangles of their respective abscisses, or as the difference of the squares of the semi-diameter and of the distance between the ordinate and centre. For they are all in the same ratio of CE^2 to ce^2.

Corol. 2. The above being a similar property to that belonging to the two axes, all the other properties before laid down, for the axes, may be understood of any two conjugate diameters whatever, using only the oblique ordinates of these diameters instead of the perpendicular ordinates of the axes; namely, all the properties in theorems 6, 7, 8, 16, 17, 20, 21.

Corol. 3. Likewise, when the ordinates are continued to the conjugate hyperbolas at Q′, q', the same properties still obtain, substituting only the sum for the difference of the squares of CE and CL,

That is, $CE^2 : ce^2 :: CL^2 + CE^2 : LQ'^2$.
And so $LQ^2 : LQ'^2 :: CL^2 - CE^2 : CL^2 + CE^2$.

Corol. 4. When, by the motion of LQ′ parallel to itself, that line coincides with EV, the last corollary becomes

$CE^2 : ce^2 :: 2CE^2 : EV^2$,
or $ce^2 : EV^2 :: 1 : 2$,
or $ce : EV :: 1 : \sqrt{2}$,
or as the side of a square to its diagonal.

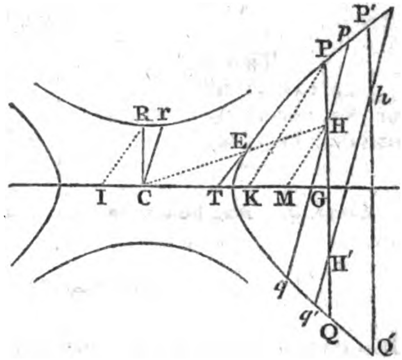
That is, in all conjugate hyperbolas, and all their diameters, any diameter is to its parallel tangent, in the constant ratio of the side of a square to its diagonal.

THEOREM XXIV.

If any two lines, that any where intersect each other, meet the curve each in two points; then

The rectangle of the segments of the one
Is to the rectangle of the segments of the other,
As the square of the diam. parallel to the former
To the square of the diam. parallel to the latter.

That is, if CR and cr be parallel to any two lines PHQ, phq; then shall $CR^2 : cr^2 ::$ PH . HQ : ph . Hq.



For, draw the diameter CHE, and the tangent TE, and its parallels PK, RI, MH, meeting the conjugate of the diameter CR in the points T, K, I, M. Then, because similar triangles are as the squares of their like sides, it is,

by sim. triangles, $CR^2 : GP^2 :: \triangle CRI : \triangle GPK$,
and - - $CR^2 : GH^2 :: \triangle CRI : \triangle GHM$;
theref. by division, $CR^2 : GP^2 - GH^2 :: CRI : KPHM$.
Again, by sim. tri. $CE^2 : CH^2 :: \triangle CTE : \triangle CMH$;
and by division, $CE^2 : CH^2 - CE^2 :: \triangle CTE : TEHM$.

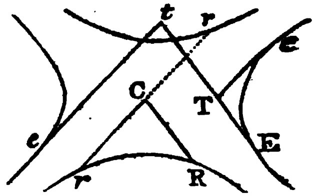
But, by cor. 5 theor. 21, the $\triangle CTE = \triangle CIR$,
and by cor. 1 theor. 21, $TEHG = KPHG$, or $TEHM = KPHM$;
theref. by equ. $CE^2 : CH^2 - CE^2 :: CR^2 : GP^2 - GH^2$ or PH . HQ.
In like manner $CE^2 : CH^2 - CE^2 :: cr^2 : ph$. Hq.
Theref. by equ. $CR^2 : cr^2 ::$ PH . HQ : ph . Hq. Q. E. D.

Corol. 1. In like manner, if any other line p'H$'q'$, parallel to cr or to pq, meet PHQ; since the rectangles PH Q. p H q' are also in the same ratio of CR^2 to cr^2; therefore the rect. PHQ : phq :: PH'Q : p H'q'.

Also, if another line P hQ' be drawn parallel to PQ or CR; because the rectangles P hQ', p hq are still in the same ratio, therefore, in general, the rectangle PHQ : phq :: P'hQ' : p hq'. That is, the rectangles of the parts of two parallel lines, are to one another, as the rectangles of the parts of two other parallel lines, any where intersecting the former.

Corol. 2. And when any of the lines only touch the curve, instead of cutting it, the rectangles of such become squares, and the general property still attends them.

That is,
$CR^2 : Cr^2 :: TE^2 : Te^2$,
or $CR : Cr :: TE : Te$,
and $CR : Cr :: tE : te$.

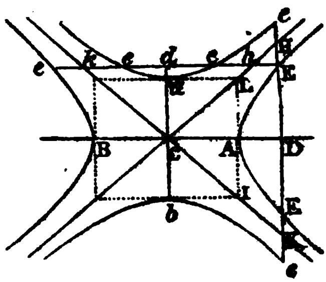


Corol. 3. And hence $TE : Te :: tE : te$.

THEOREM XXV.

If a line be drawn through any point of the curves, parallel to either of the axes, and terminated at the asymptotes; the rectangle of its segments, measured from that point, will be equal to the square of the semi-axis to which it is parallel.

That is,
the rect. HEK or $HeK = Ca^2$,
and rect. hEk or $hek = CA^2$.



For, draw AL parallel to Ca, and aL to CA. Then
by the parallels, $CA^2 : Ca^2$ or $AL^2 :: CD^2 : DH^2$;
and, by theor. 2, $CA^2 : Ca^2 :: CD^2 - CA^2 : DE^2$;
theref. by subtr. $CA^2 : Ca^2 :: CA^2 : DH^2 - DE^2$ or HEK.
But the antecedents CA^2, CA^2 are equal,
theref. the consequents Ca^2, HEK must also be equal.

In like manner it is again,
by the parallels, $CA^2 : Ca^2$ or $AL^2 :: CD^2 : DH^2$;
and by theor. 3, $CA^2 : Ca^2 :: CD^2 + CA^2 : De^2$;
theref. by subtr. $CA^2 : Ca^2 :: CA^2 : De^2 - DH^2$ or HeK.
But the antecedents CA^2, CA^2 are the same,
theref the conseq. Ca^2, HeK must be equal.
In like manner, by changing the axes, is hEk or $hek = CA^2$.

Corol. 1. Because the rect. HEK = the rect. HeK.
therefore $EH : eH :: eK : EK$.
And consequently HE is always greater than He.

Corol. 2. the rectangle hek = the rect. HEK,
For, by sim. tri. eh : EH : : ek : EK.

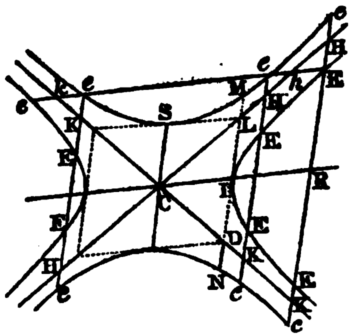
SCHOLIUM.

It is evident that this proposition is general for any line oblique to the axis also, namely, that the rectangle of the segments of any line, cut by the curve, and terminated by the asymptotes, is equal to the square of the semi-diameter to which the line is parallel. Since the demonstration is drawn from properties that are common to all diameters.

THEOREM XXVI.

All the rectangles are equal which are made of the segments of any parallel lines cut by the curve, and limited by the asymptotes.

That is,
the rect. HEK = HeK,
and rect. hek = hek.



For, each of the rectangles HEK or HeK is equal to the square of the parallel semi-diameter CS; and each of the rectangles hek or hek is equal to the square of the parallel semi-diameter CE. and therefore the rectangles of the segments of all parallel lines are equal to one another. Q. E. D.

Corol. 1. The rectangle HEK being constantly the same, whether the point E is taken on the one side or the other of the point of contact I of the tangent parallel to HK, it follows that the parts HE, KE, of any such line HK, are equal.

And because the rectangle HeK is constant, whether the point e is taken in the one or the other of the opposite hyperbolas, it follows, that the parts He, Ke, are also equal.

Corol. 2. And when HK comes into the position of the tangent DL, the last corollary becomes IL = ID, and IM = IN, and LM = DN.

Hence also the diameter CIR bisects all the parallels to DL which are terminated by the asymptote, namely RH = RK.

Corol. 3. From the proposition, and the last corollary, it follows that the constant rectangle HEK or KHE is = IL^2. And the equal constant rect. HEK or eHE = MLN or $IM^2 - IL^2$.

Corol. 4. And hence IL = the parallel semi-diameter CS.

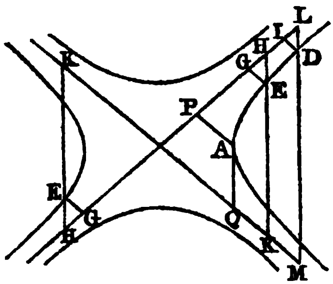
For, the rect. EHE = IL^2,
and the equal rect. eHE = $IM^2 - IL^2$,
theref. $IL^2 = IM^2 - IL^2$, or $IM^2 = 2IL^2$;
but, by cor. 4 theor. 23, $IM^2 = 2CS^2$,
and therefore . . IL = CS.

And so the asymptotes pass through the opposite angles of all the inscribed parallelograms.

THEOREM XXVII.

The rectangle of any two lines drawn from any point in the curve, parallel to two given lines, and limited by the asymptotes, is a constant quantity.

That is, if AP, EG, DI be parallels,
as also AQ, EK, DM parallels,
then shall the rect. PAQ = rect. GEK = rect. IDM.

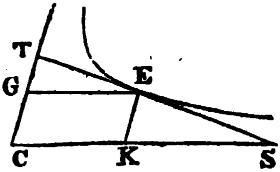


For, produce KE, MD to the other asymptote at H, L.
Then, by the parallels, HE : GE : : LD : ID;
but . . . EK : EK : : DM : DM;
theref. the rectangle HEK : GEK : : LDM : IDM.
But, by the last theor. the rect. HEK = LDM;
and therefore the rect. GEK = IDM = PAQ. Q. E. D.

THEOREM XXVIII.

Every inscribed triangle, formed by any tangent and the two intercepted parts of the asymptotes is equal to a constant quantity; namely, double the inscribed parallelogram.

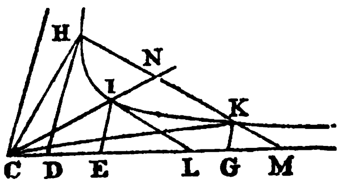
That is, the triangle CTS = 2 paral. GK.



For, since the tangent TS is bisected by the point of contact E, (th. 26, cor. 2), and KE is parallel to TC, and GE to CK; therefore CK, KS, GE, are all equal, as are also CG, GT, KE. Consequently the triangle GTE = the triangle KES, and each equal to half the constant inscribed parallelogram GK. And therefore the whole triangle CTS, which is composed of the two smaller triangles and the parallelogram, is equal to double the constant inscribed parallelogram GK. Q. E. D.

THEOREM XXIX.

If from the point of contact of any tangent, and the two intersections of the curve with a line parallel to the tangent, three parallel lines be drawn in any direction, and terminated by either asymptote; those three lines shall be in continued proportion.

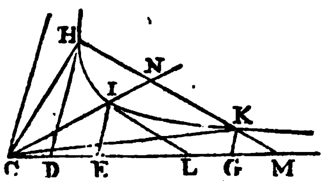


That is, if HKM and the tangent IL be parallel, then are the parallels DH, EI, GK in continued proportion.

For, by the parallels, EI : IL :: DH : HM;
and, by the same, EI : IL :: GK : KM;
theref. by compos. $EI^2 : IL^2 :: DH . GK : HMK$;
but, by theor. 26, the rect. $HMK = IL^2$;
and theref. the rect. $DH . GK = EI^2$,
or - - - DH : EI :: EI : GK. Q. E. D.

THEOREM XXX.

Draw the semi-diameters CH, CIN, CK;
Then shall the sector CHI = the sector CIK.



For, because HK and all its parallels are bisected by CIN, therefore the triangle CNH = tri. CNK,

and the segment INH = seg. IXK;
consequently the sector CIH = sec. CIK.

Corol. If the geometrical proportionals DH, EI, GK be parallel to the other asymptote, the spaces DHIE, EIKG will be equal; for they are equal to the equal sectors CHI, CIK.

So that by taking any geometrical proportionals CD, CE, CG, &c. and drawing DH, EI, GK, &c. parallel to the other asymptote, as also the radii CH, CI, CK;

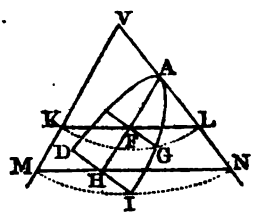
then the sectors CHI, CIK, &c.
or the spaces DHIE, EIKG, &c.
will be all equal among themselves.
Or the sectors CHI, CHK, &c.
or the spaces DHIE, DHKG, &c.
will be in arithmetical progression.

And therefore these sectors, or spaces, will be analogous to the logarithms of the lines or bases CD, CE, CG, &c.; namely, CHI or DHIE the log. of the ratio of
CD to CE, or of CE to CG, &c.; or of EI to DH, or of GK to EI, &c.;
and CHK or DHKG the log. of the ratio of CD to CG, &c.
or of GK to DH, &c.

OF THE PARABOLA.

THEOREM I.

The Abscisses are proportional to the Squares of their Ordinates.



Let AVM be a section through the axis of the cone, and AGIH a parabolic section by a plane perpendicular to the former, and parallel to the side VM of the cone; also let AFH be the common intersection of the two planes, or the axis of the parabola, and FG, HI ordinates perpendicular to it.

Then it will be, as AF : AH : : FG^2 : HI^2.

For, through the ordinates FG, HI, draw the circular sections, KGL, MIN, parallel to the base of the cone, having KL, MN for their diameters, to which FG, HI are ordinates, as well as to the axis of the parabola.

Then, by similar triangles, AF : AH : : FL : HN ;
but, because of the parallels, KF = MH ;
therefore - - - AF : AH : : KF . FL : MH . HN.
But, by the circle, KF . FL = FG^2, and MH . HN = HI^2 ;
Therefore - - - AF : AH : : FG^2 : HI^2. Q. E. D.

Corol. Hence the third proportional $\frac{FG^2}{AF}$ or $\frac{HI^2}{AH}$ is a constant quantity, and is equal to the parameter of the axis, by defin. 16.

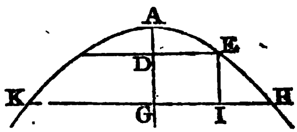
Or AF : FG : : FG : P the parameter.

Or the rectangle P . AF = FG^2.

THEOREM II.

As the Parameter of the Axis :
Is to the Sum of any Two Ordinates : :
So is the Difference of those Ordinates :
To the Difference of their Abscisses.

That is,
P : GH + DE : : GH − DE : DG,
Or, P : KI : : IH : IE.



For, by cor. theor. 1, P . AG = GH^2,
and - - - P . AD = DE^2 ;
theref. by subtraction, P . DG = $GH^2 - DE^2$.
Or, - - - P . DG = KI . IH,
therefore - - P : KI : : IH : DG or EI. Q. E. D.

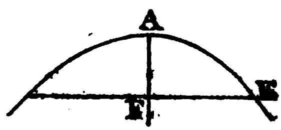
Corol. Hence, because P . EI = KI . IH,
and, by cor. theor. 1, P . AG = GH^2,
therefore - - AG : EI : : GH^2 : KI . IH.

So that any diameter EI is as the rectangle of the segments KI, IH of the double ordinate KH.

THEOREM III.

The Distance from the Vertex to the Focus is equal to ¼ of the Parameter, or to Half the Ordinate at the Focus.

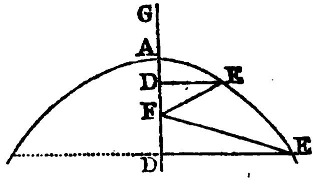
That is,
AF = ½FE = ¼P,
where F is the focus.



For, the general property is AF : FE :: FE : P.
But, by definition 17, - $FE = \frac{1}{2}P$;
therefore also - - $AF = \frac{1}{2}FE = \frac{1}{4}P$. Q. E. D.

THEOREM IV.

A Line drawn from the Focus to any Point in the Curve, is equal to the Sum of the Focal Distance and the Abscisa of the Ordinate to that Point.



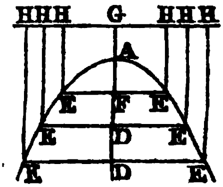
That is,
$FE = FA + AD = GD$,
taking $AG = AF$.

For, since $FD = AD \backsim AF$,

theref. by squaring,	$FD^2 = AF^2 - 2AF \cdot AD + AD^2$,
But, by cor. theor. 1,	$DE^2 = P \cdot AD = 4AF \cdot AD$;
theref. by addition,	$FD^2 + DE^2 = AF^2 + 2AF \cdot AD + AD^2$.
But, by right-ang. tri.	$FD^2 + DE^2 = FE^2$;
therefore - -	$FE^2 = AF^2 + 2AF \cdot AD + AD^2$,
and the root or side is	$FE = AF + AD$,
or . . - -	$FE = GD$, by taking $AG = AF$.

Q. E. D.

Corol. 1. If, through the point G, the line GH be drawn perpendicular to the axis, it is called the directrix of the parabola.* The property of which, from this theorem, it appears, is this: That drawing any lines HE parallel to the axis, HE is always equal to FE the distance of the focus from the point E.



Corol. 2. Hence also the curve is easily described by points. Namely, in the axis produced take $AG = AF$ the focal distance, and draw a number of lines EE perpendicular to the axis AD; then with the distances GD, GD, GD, &c. as radii, and the centre F, draw arcs crossing the parallel ordinates in E, E, E, &c. Then draw the curve through all the points E, E, E.

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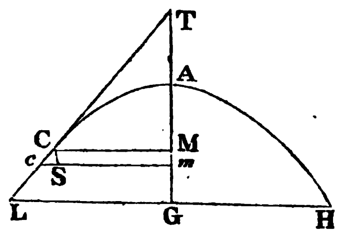
* Each of the other conic sections has a directrix; but the consideration of it does not occur in the mode here employed of investigating the general properties of the curves.

THEOREM V.

If a Tangent be drawn to any Point of the Parabola, meeting the Axis produced; and if an Ordinate to the Axis be drawn from the point of Contact; then the Abscis of that Ordinate will be equal to the external Part of the Axis, measured from the Vertex.

That is, if TC touch the curve at the point C, then is $AT = AM$.



Let cC, an indefinitely small portion of a parabolic curve, be produced to meet the prolongation of the axis in T; and let cm be drawn parallel to CM, and Cs parallel to AG the axis. Let, also, p = parameter of the parabola.

Then, by sim. tri. $cs : sC :: CM : MA + AT = MT$,

$$\therefore Cs = \frac{MT \cdot cs}{CM}.$$

Also, th. 1. cor. $p \cdot Am = mc^2 = ms^2 + 2ms \cdot sc + sc^2,$
$$= MC^2 + 2MC \cdot sc + sc^2,$$
and $p \cdot AM = MC^2$.

Consequently, omitting sc^2 as indefinitely small, and subtracting the latter equa. from the former, we have

$$p \cdot (Am - AM) = p \cdot Cs = 2cs \cdot MC:$$

or, substituting for Cs its value above,

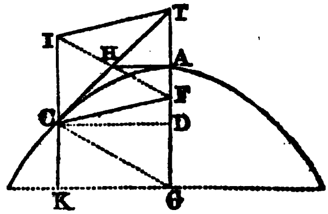
$$p \cdot \frac{MT \cdot cs}{CM} = 2cs \cdot MC;$$

or $p \cdot MT = 2MC^2 = 2p \cdot AM$ (th. 1.)
Consequently, $MT = 2AM$, and $MA = AT$. Q. E. D.

THEOREM VI.

If a Tangent to the Curve meet the Axis produced; then the Line drawn from the Focus to the Point of Contact, will be equal to the Distance of the Focus from the Intersection of the Tangent and Axis.

That is,
FG = FT.



For, draw the ordinate DC to the point of contact C.
Then, by theor. 5, AT = AD ;
therefore . FT = AF + AD.
But, by theor. 4. FC = AF + AD ;
theref. by equality, FC = FT. Q. E. D.

Corol. 1. If CG be drawn perpendicular to the curve, or to the tangent, at C ; then shall FG = FC = FT.

For, draw FH perpendicular to TC, which will also bisect TC, because FT = FC ; and therefore, by the nature of the parallels, FH also bisects TG in F. And consequently FG = FT = FC.

So that F is the centre of a circle passing through T, C, G.

Corol. 2. The subnormal DG is a constant quantity, and equal to half the parameter, or to 2AF, double the focal distance. For, since TCG is a right angle,
therefore TD or 2AD : DC : : DC : DG ;
but by the def AD : DC : : DC : parameter ;
therefore DG = half the parameter = 2AF.

Corol. 3. The tangent at the vertex AH, is a mean proportional between AF and AD.

For, because FHT is a right angle,
therefore . AH is a mean between AF, AT,
or between . AF, AD, because AD = AT.
Likewise, . FH is a mean between FA, FT,
or between FA, FC.

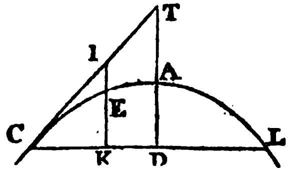
Corol. 4. The tangent TC makes equal angles with FC and the axis FT ; as well as with FC and CI.

For, because FT = FC,
Therefore the ∠ FCT = ∠ FTC.
Also, the angle GCF = the angle GCK,
drawing ICK parallel to the axis AG.

Corol. 5. And because the angle of incidence GCK is = the angle of reflection GCF ; therefore a ray of light falling on the curve in the direction KC, will be reflected to the focus F. That is, all rays parallel to the axis, are reflected to the focus, or burning point.

THEOREM VII.

If there be any Tangent, and a Double Ordinate drawn from the Point of Contact, and also any Line parallel to the Axis, limited by the Tangent and Double Ordinate: Then shall the Curve divide that Line in the same Ratio as the Line divides the Double Ordinate.

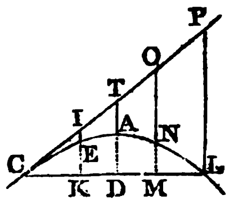


That is,
IE : EK : : CK : KL.

For, by sim. triangles, CK : KI : : CD : DT or 2DA ;
but, by the def. the param. P : CL : : CD : 2DA ;
therefore, by equality, P : CK : : CL : KI.
But, by theor. 2, P : CK : : KL : KE ;
therefore, by equality, CL : KL : : KI : KE ;
and, by division, - CK : KL : : IE : EK. Q. E. D.

THEOREM VIII.

The same being supposed as in theor. 7; then shall the External Part of the Line between the Curve and Tangent, be proportional to the Square of the intercepted Part of the Tangent, or to the Square of the intercepted Part of the Double Ordinate.



That is, IE is as CI^2 or as CK^2,
and IE, TA, ON, PL, &c.
are as CI^2, CT^2, CO^2, CP^2, &c.
or as CK^2, CD^2, CM^2, CL^2, &c.

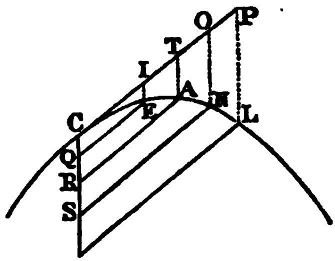
For, by theor. 7, IE : EK : : CK : KL,
or, by equality, IE : EK : : CK^2 : CK . KL.
But, by cor. th. 2, EK is as the rect. CK . KL,
therefore - IE is as CK^2, or as CI^2. Q. E. D.

Corol. As this property is common to every position of the tangent, if the lines IE, TA, ON, &c. be appended on the points I, T, O, &c. and moveable about them, and of such lengths as that their extremities E, A, N, &c. be in the curve of a parabola in some one position of the tangent; then making the tangent revolve about the point C, it appears that the extremities E, A, N, &c. will always form the curve of some parabola, in every position of the tangent.

THEOREM IX.

The Abscisses of any Diameter, are as the Squares of their Ordinates.

That is, CQ, CR, CS, &c. are as QE^2, RA^2, SN^2, &c. Or $CQ : CR :: QE^2 : RA^2$, &c.



For, draw the tangent CT, and the externals, EI, AT, NO, &c. parallel to the axis, or to the diameter, CS.

Then, because the ordinates, QE, RA, SN, &c. are parallel to the tangent CT, by the definition of them, therefore all the figures IQ, TR, OS, &c. are parallelograms, whose opposite sides are equal;

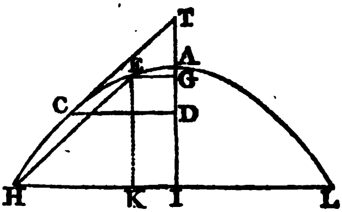
namely, - -	IE, TA, ON, &c.
are equal to -	CQ, CR, CS, &c.
Therefore, by theor. 8,	CQ, CR, CS, &c.
are as - -	CI^2, CT^2, CO^2, &c.
or as their equals -	QE^2, RA^2, SN^2, &c. Q. E. D.

Corol. Here, like as in theor. 2, the difference of the abscisses is as the difference of the squares of their ordinates, or as the rectangles under the sum and difference of the ordinates, the rectangle of the sum and difference of the ordinates being equal to the rectangle under the difference of the abscisses and the parameter of that diameter, or a third proportional to any absciss and its ordinate.

THEOREM X.

If a Line be drawn parallel to any Tangent, and cut the Curve in two Points; then, if two Ordinates be drawn to the Intersections, and a third to the Point of Contact, these three Ordinates will be in Arithmetical Progression, or the Sum of the Extremes will be equal to Double the Mean.

That is,
EG + HI = 2CD.



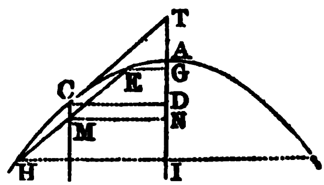
For, draw EK parallel to the axis, and produce HI to L.
Then, by sim. triangles, EK : HK :: TD or 2AD : CD ;
but, by theor. 2, EK : HK :: KL : P the param.
theref. by equality, 2AD : KL :: CD : P.
But, by the defin. 2AD : 2CD :: CD : P ;
theref. the 2d terms are equal, KL = 2CD,
that is, EG + HI = 2CD. Q. E. D.

Corol. When the point E is on the other side of AI ; then
HI — GE = 2CD.

THEOREM XI.

Any diameter bisects all its Double Ordinates, or Lines parallel to the Tangent at its Vertex.

That is,
ME = MH.



For, to the axis AI draw the ordinates EG, CD, HI, and MN parallel to them, which is equal to CD.

Then, by theor. 10, 2MN or 2CD = EG + HI,
therefore M is the middle of EH.

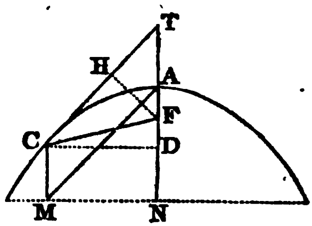
And, for the same reason, all its parallels are bisected.
Q. E. D.

SCHOL. Hence, as the abscisses of any diameter and their ordinates have the same relations as those of the axis, namely, that the ordinates are bisected by the diameter, and their squares proportional to the abscisses ; so all the other properties of the axis and its ordinates and abscisses, before demonstrated, will likewise hold good for any diameter and its ordinates and abscisses. And also those of the parameters, understanding the parameter of any diameter, as a third proportional to any abscis and its ordinate. Some of the most material of which are demonstrated in the four following theorems.

THEOREM XII.

The Parameter of any Diameter is equal to four Times the Line drawn from the Focus to the Vertex of that Diameter.

That is, 4FC = p, the param. of the diam. CM.



For, draw the ordinate MA parallel to the tangent CT: also CD, MN perpendicular to the axis AN, and FH perpendicular to the tangent CT.

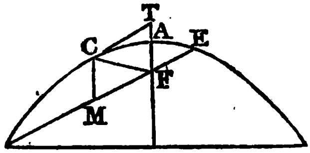
Then the abscisses AD, CM or AT, being equal, by theor. 5, the parameters will be as the squares of the ordinates CD, MA or CT, by the definition;

that is, - - $P : p :: CD^2 : CT^2$,
But by sim. tri. - $FH : FT :: CD : CT$;
therefore - - $P : p :: FH^2 : FT^2$.
But, by cor. 3, th. 6, $FH^2 = FA . FT$;
therefore - - $P : p :: FA . FT : FT^2$;
or, by equality, - $P : p :: FA : FT$ or FC.
But, by theor. 3, $P = 4FA$,
and therefore - $p = 4FT$ or 4FC. Q. E. D.

Corol. Hence the parameter p of the diameter CM is equal to 4FA + 4AD, or to P + 4AD, that is, the parameter of the axis added to 4AD.

THEOREM XIII.

If an Ordinate to any Diameter pass through the Focus, it will be equal to Half its Parameter; and its Abscise equal to One Fourth of the same Parameter.



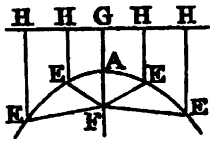
That is, CM = $\frac{1}{4}p$,
and ME = $\frac{1}{2}p$.

For, join FC, and draw the tangent CT.

By the parallels, CM = FT;
and, by theor. 6, FC = FT;
also, by theor. 12, FC = $\frac{1}{4}p$;
therefore . . CM = $\frac{1}{4}p$.

Again, by the defin. CM or $\frac{1}{4}p$: ME :: ME : p,
and consequently ME = $\frac{1}{2}p$ = 2CM. Q. E. D.

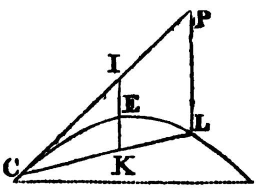
Corol. 1. Hence, of any diameter, the double ordinate which passes through the focus, is equal to the parameter, or to quadruple its absciss.



Corol. 2. Hence, and from cor 1, to theor. 4, and theor. 6 and 12, it appears, that if the directrix GH be drawn, and any lines HE, HE, parallel to the axis; then every parallel HE will be equal to EF, or $\frac{1}{4}$ of the parameter of the diameter to the point E.

THEOREM XIV.

If there be a Tangent, and any Line drawn from the Point of Contact and meeting the Curve in some other Point, as also another Line parallel to the Axis, and limited by the First Line and the Tangent: then shall the Curve divide this Second Line in the same Ratio as the Second Line divides the First Line.



That is,
IE : EK :: CK : KL.

For, draw LP parallel to IK, or to the axis.

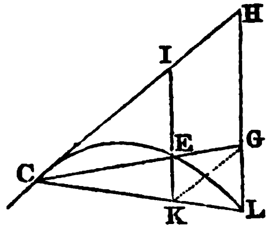
Then by theor. 8, IE : PL :: CI^2 : CP^2,
or, by sim. tri. . IE : PL :: CK^2 : CL^2.
Also, by sim. tri. IK : PL :: CK : CL,
or . . . IK : PL :: CK^2 : CK . CL;

therefore by equality, IE : IK :: CK . CL : CL^2;
or - - - IE : IK :: CK : CL;
and, by division, IE : EK :: CK : KL. Q. E. D.

Corol. When CK = KL, then IE = EK = ½IK.

THEOREM XV.

If from any Point of the Curve there be drawn a Tangent, and also Two Right Lines to cut the Curve; and Diameters be drawn through the Points of Intersection E and L, meeting those Two Right Lines in two other Points G and K: then will the Line KG joining these last Two Points be parallel to the Tangent.

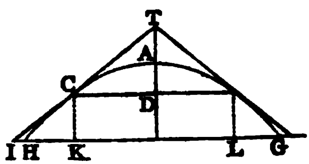


For, by theor. 14, CK : KL :: EI : EK;
and by composition, CK : CL :: KI : KI;
and by the parallels CK : CL :: GH : LH.
But, by sim. tri. - CK : CL :: KI : LH;
theref. by equal. - KI : LH :: GH : LH:
consequently - KI = GH,
and therefore - KG is parallel and equal to IH. Q. E. D.

THEOREM XVI.

If an ordinate be drawn to the point of contact of any tangent, and another ordinate produced to cut the tangent; it will be, as the difference of the ordinates
Is to the difference added to the external part,
So is double the first ordinate
To the sum of the ordinates.

That is, KH : KI :: KL : KG.



For, by cor. 1, theor. 1, P : DC :: DC : DA,

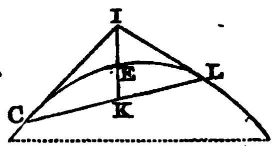
and . . . P : 2DC : : DC : DT or 2DA.
But, by sim. triangles, KI : KC : : DC : DT ;
therefore, by equality, P : 2DC : : KI : KC,
or, . . . P : KI : : KL : KC.
Again, by theor. 2, P : KH : : KG : KC ;
therefore by equality, KH : KI : : KL : KG. Q. E. D.

Corol. 1. Hence, by composition and division,
it is, KH : KI : : GK : GI,
and HI : HK : : HK : KL,
also IH : IK : : IK : IG ;
that is, IK is a mean proportional between IG and IH.

Corol. 2. And from this last property a tangent can easily be drawn to the curve from any given point I. Namely, draw IHG perpendicular to the axis, and take IK a mean proportional between IH, IG ; then draw KC parallel to the axis, and C will be the point of contact, through which and the given point I the tangent IC is to be drawn.

THEOREM XVII.

If a tangent cut any diameter produced, and if an ordinate to that diameter be drawn from the point of contact ; then the distance in the diameter produced, between the vertex and the intersection of the tangent, will be equal to the absciss of that ordinate.



That is, IE = EK.
For, by the last th. IE : EK : : CK : KL.
But, by theor. 11, CK = KL,
and therefore IE = EK.

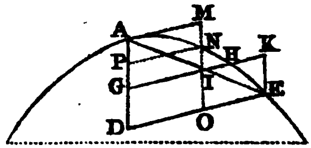
Corol. 1. The two tangents CI, LI, at the extremities of any double ordinate CL, meet in the same point of the diameter of that double ordinate produced. And the diameter drawn through the intersection of two tangents, bisects the line connecting the points of contact.

Corol. 2. Hence we have another method of drawing a tangent from any given point I without the curve. Namely, from I draw the diameter IK, in which take EK = EI, and through K draw CL parallel to the tangent at E ; then C and L are the points to which the tangents must be drawn from I.

THEOREM XVIII.

If a line be drawn from the vertex of any diameter, to cut the curve in some other point, and an ordinate of that diameter be drawn to that point, as also another ordinate any where cutting the line, both produced if necessary: The three will be continual proportionals, namely, the two ordinates and the part of the latter limited by the said line drawn from the vertex.

That is, DE, GH, GI are continual proportionals, or DE : GH : : GH : GI.



For, by theor. 9,	- - -	$DE^2 : GH^2 :: AD : AG$;
and, by sim. tri.	- - -	$DE : GI :: AD : AG$;
theref. by equality,	- -	$DE : GI :: DE^2 : GH^2$,
that is, of the three DE, GH, GI,		$1st : 3d :: 1st^2 : 2d^2$,
therefore	- - - - - -	$1st : 2d :: 2d : 3d$,
that is,	- - - - - - - -	$DE : GH :: GH : GI$. Q. E. D.

Corol. 1. Or their equals GK, GH, GI, are proportionals; where EK is parallel to the diameter AD.

Corol. 2. Hence it is DE : AG : : p : GI, where p is the parameter, or AG : GI : : DE : p.
For, by the defin. AG : GH : : GH : p.

Corol. 3. Hence also the three MN, MI, MO, are proportionals, where MO is parallel to the diameter, and AM parallel to the ordinates.

For, by theor. 9, - MN, MI, MO,
or their equals - AP, AG, AD,
are as the squares of PN, GH, DE,
or of their equals GI, GH, GK,
which are proportionals by cor. 1.

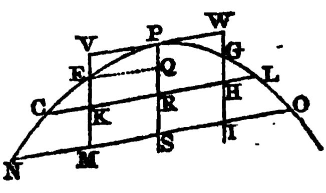
THEOREM XIX.

If a diameter cut any parallel lines terminated by the curve; the segments of the diameter will be as the rectangle of the segments of those lines.

That is, EK : EM : : CK . KL : NM . MO.
Or, EK is as the rectangle CK . KL.

For, draw the diameter PS to which the parallels CL, NO are ordinates, and the ordinate EQ parallel to them.

Then CK is the difference, and KL the sum of the ordinates EQ, CR; also



NM the difference, and MO the sum of the ordinates EQ, NS. And the differences, of the abscisses, are QR, QS, or EK, EM.

Then by cor. theor. 9, QR : QS :: CK . KL : NM . MO,
that is - - EK : EM :: CK . KL : NM . MO.

Corol. 1. The rect. CL . KL = rect, EK and the param. of PS.
For the rect. CK . KL = rect. QR and the param of PS.

Corol. 2. If any line CL be cut by two diameters, EK, GH; the rectangles of the parts of the line, are as the segments of the diameters.

For EK is as the rectangle CK . KL,
and GH is as the rectangle CH . HL;
therefore EK : GH :: CK . KL : CH . HL.

Corol. 3. If two parallels, CL, NO, be cut by two diameters, EM, GI; the rectangles of the parts of the parallels will be as the segments of the respective diameters.

For - - - EK : EM :: CK . KL : NM . MO,
and - - - EK : GH :: CK . KL : CH . HL,
theref. by equal. EM : GH :: NM . MO : CH . HL.

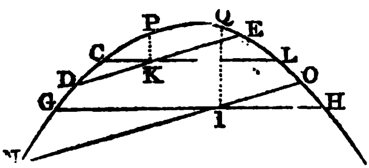
Corol. 4. When the parallels come into the position of the tangent at P, their two extremities, or points in the curve, unite in the point of contact P; and the rectangle of the parts becomes the square of the tangent, and the same properties still follow them.

So that, EV : PV :: PV : p the param.
GW : PW :: PW : p,
EV : GW :: PV^2 : PW^2,
EV : GH :: PV^2 : CH . HL.

THEOREM XX.

If two parallels intersect any other two parallels; the rectangles of the segments will be respectively proportional.

That is, CK . KL : DK . KE :: GI . IH : NI . IO.



For, by cor. 3 theor. 23, PK : QI :: CK . KL : GI . IH;
And by the same, PK : QI :: DK . KE : NI . IO;
theref. by equal. CK . KL : DK . KE :: GI . IH : NI . IO.

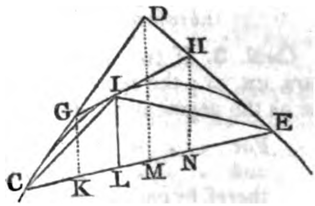
Corol. When one of the pairs of intersecting lines comes into the position of their parallel tangents, meeting and limiting each other, the rectangles of their segments become the squares of their respective tangents. So that the constant ratio of the rectangles, is that of the square of their parallel tangents, namely,

CK . KL : DK . KE : : tang². parallel to CL : tang². parallel to DE.

THEOREM XXI.

If there be three tangents intersecting each other; their segments will be in the same proportion.

That is, GI : IH : : CG : GD : : DH : HE.



For, through the points G, I, D, H, draw the diameters GK, IL, DM, HN; as also the lines CI, EI, which are double ordinates to the diameters GK, HN, by cor. 1 theor. 16; therefore the diameters GK, DM, HN, bisect the lines CL, CE, LE;

hence KM = CM — CK = $\frac{1}{2}$CE — $\frac{1}{2}$CL = $\frac{1}{2}$LE = LN or NE,

and MN = ME — NE = $\frac{1}{2}$CE — $\frac{1}{2}$LE = $\frac{1}{2}$CL = CK or KL,

But, by parallels, GI : IH : : KL : LN,

and - - CG : GD : : CK : KM,

also - - DH : HE : : MN : NE.

But the 3d terms KL, CK, MN are all equal;

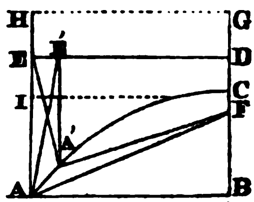
as also the 4th terms LN, KM NE.

Therefore the first and second terms, in all the lines, are proportional, namely, GI : IH : : CG : GD : . DH : HE. Q. E. D.

THEOREM XXII.

The Area or Space of a Parabola, is equal to Two-Thirds of its Circumscribing Parallelogram.

Let ACB be a semi-parabola, CB the axis, F the focus, ED the directrix; then if the line AF be supposed to revolve about F as a centre, while the line AE moves along the directrix perpendicularly to it, the area generated by the motion of AF, will always be equal to double the area generated by FA; and consequently the whole external area AEGD = double the area ACF.

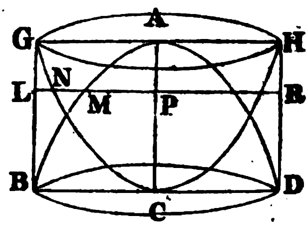


For draw A′ E′ parallel, and indefinitely near, to AE; and draw the diagonals AE′ and A′E; then by th. 6, cor. 4, the angles E′A′A and FA′A are equal, AA′ being considered as part of the tangent at A′; and in the same manner, the angles EAA′ and FAA′ are also equal to each other; and since EA = AF, and E′A′ = A′F; the triangles EAA′ and E′A′A are each equal to the triangle AA′F; but the triangle EAA′ = the triangle EE′A, being on the same base and between the same parallels; therefore the sum of the two triangles EE′A and EA′A, or the quadrilateral space EAA′E′ is double the trilateral space AA′F; and as this is the case in every position of FA′, E′A′, it follows that the whole external area EAOD = double the internal area AFC.

Hence, Take DG = FB, and complete the parallelogram DGHE, which is double the triangle ABF; therefore the area ABC = ½ the area HACG, or ½ of the rectangle AEGH, or ⅔ of the rectangle ABCI, because BC = ¾BG; that is, the area of a parabola = ⅔ of the circumscribing rectangle. Q. E. D.*

THEOREM XXIII.

The Solid Content of a Paraboloid (or Solid generated by the Rotation of a Parabola about its Axis), is equal to Half its Circumscribing Cylinder.



Let GHBD be a cylinder, in which two equal paraboloids are inscribed; one BAD having its base BCD equal to the lower extremity of the cylinder; the other GCH inverted with respect to the former, but of equal base and altitude. Let the plane LR parallel to each end of the cylinder, cut all the three solids, while a vertical plane may be supposed to cut them so as to define the parabolas shown in the figure.

Then, in the semi-parabola ACB, $p \cdot AP = PM^2$,
also, in the semi-parabola ACG, $p \cdot CP = PN^2$,
consequently, by addition, $p \cdot (AP + CP) = p \cdot AC = PM^2 + PN^2$.
But, $p \cdot AC = CB^2 = PL^2$.
Therefore $PL^2 = PM^2 + PN^2$:
That is, since circles are as the squares of their radii, the

* This demonstration was given by Lieut. Drummond of the Royal Engineers, when he was a gentleman Cadet at the Royal Military Academy.

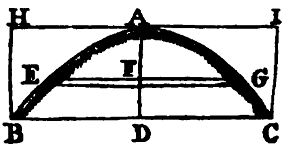
circular section of the cylinder, is equal to the sum of the corresponding sections of the two paraboloids.

The same property evidently obtains for any sections whatever parallel to BD; it therefore holds for the two paraboloids. In other words, the cylinder is equal to the two paraboloids taken together: wherefore, since the two paraboloids. having equal bases and equal altitudes, are equal to one another, it follows that each paraboloid is half of its circumscribing cylinder. Q. E. D.

THEOREM XXIV.

The Solidity of the Frustum BEGC of the Paraboloid, is equal to a Cylinder whose Height is DF, and its Base Half the Sum of the two Circular Bases EG, BC.

Let $c = 3.1416$:



Then, by the last theor. $\frac{1}{2}pc \times AD^2$ = the solid ABC,
and, by the same $\frac{1}{2}pc \times AF^2$ = the solid AEG,
theref. the diff. $\frac{1}{2}pc \times (AD^2 - AF^2)$ = the frust. BEGC.
But $AD^2 - AF^2 = DF \times (AD + AF)$,
theref. $\frac{1}{2}pc \times DF \times (AD + AF)$ = the frust. BEGC.
But, by th. 1, $p \times AD = DC^2$, and $p \times AF = FG^2$;
theref. $\frac{1}{2}c \times DF \times (DC^2 + FG^2)$ = the frust. BEGC.

Q. E. D.

PROBLEMS, &c. FOR EXERCISE IN CONIC SECTIONS.

1. Demonstrate that if a cylinder be cut obliquely the section will be an ellipse.

2. Show how to draw a tangent to an ellipse whose foci are F, *f*, from a given point P.

3. Show how to draw a tangent to a given parabola from a given point P.

4. The diameters of an ellipse are 16 and 12. Required the parameter and the area.

5. The base and altitude of a parabola are 12 and 9. Required the parameter, and the semi-ordinates corresponding to the abscissæ 2, 3, and 4.

6. In the actual formation of arches, the voussoirs or arch-stones are so cut as to have their faces always *perpendicular*

to the respective points of the curve upon which they stand. By what constructions may this be effected for the parabola and the ellipse?

7. Construct accurately on paper, a parabola whose base shall be 12 and altitude 9.

8. A cone, the diameter of whose base is 10 inches, and whose altitude is 12, is cut obliquely by a plane, which enters at 3 inches from the vertex on one slant slide, and comes out at 3 inches from the base on the opposite slant side. Required the dimensions of the section?

9. Suppose the same cone to be cut by a plane parallel to one of the slant sides, entering the other slant side at 4 inches from the vertex, what will be the dimensions of the section?

10. Let any straight line EFR be drawn through F, one of the foci, of an ellipse, and terminated by the curve in E and R; then it is to be demonstrated that $EF \cdot FR = ER \cdot \frac{1}{4}$ parameter.

11. Demonstrate that, in any conic section, a straight line drawn from a focus to the intersection of two tangents makes equal angles with straight lines drawn from the same focus to the points of contact.

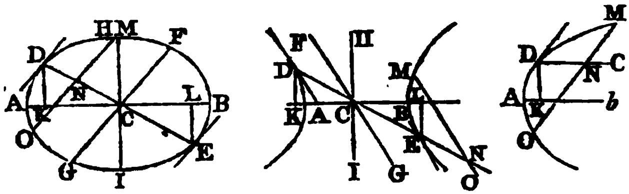
12. In every conic section the radius of curvature at any point is to half the parameter, in the triplicate ratio of the distance of the focus from that point to its distance from the tangent.

Also, in every conic section the radius of curvature is proportional to the cube of the normal.

Also, let PC be the radius of curvature at any point, P, in an ellipse or hyperbola whose tranverse axis is AB, conjugate ab, and foci F and f: then is $PC = \dfrac{(PF \cdot Pf)^{\frac{3}{2}}}{\frac{1}{2}AB \cdot ab}$.

Required demonstrations of these properties.

ON THE CONIC SECTIONS AS EXPRESSED BY ALGEBRAIC EQUATIONS CALLED THE EQUATIONS OF THE CURVE.



1. *For the Ellipse.*

Let t denote AB, the transverse, or any diameter;
$c =$ HI its conjugate;
$x =$ AK, any abscisa, from the extremity of the diam.
$y =$ DK the correspondent ordinate: the two being jointly denominated co-ordinates.

Then, theor. 2, $AB^2 : HI^2 :: AK \cdot KB : DK^2$,
that is, $t^2 : c^2 :: x(t - x) : y^2$, hence $t^2y^2 = c^2(tx - x^2)$,

or $y = \frac{c}{t}\sqrt{(tx - x^2)}$, the equation of the curve.

And from these equations, any one of the four letters or quantities, t, c, x, y, may easily be found, by the reduction of equations, when the other three are given.

Or, if p denote the parameter, $= c^2 \div t$ by its definition; then, by cor. th. 2, $t : p :: x(t - x) : y^2$, or $y^2 = \frac{p}{t}(tx - x^2)$, which is another form of the equation of the curve.

Otherwise.

If $t =$ AC the semiaxis; $c =$ CH the semiconjugate; then $p = c^2 \div t$ the semiparameter; $x =$ CK the abscisa counted from the centre; and $y =$ DK the ordinate as before.
Then is $AK = t - x$, and $KB = t + x$, and $AK \cdot KB = (t - x) \times (t + x) = t^2 - x^2$.

Then, by th. 2, $t^2 : c^2 :: t^2 - x^2 : y^2$, and $t^2y^2 = c^2(t^2 - x^2)$,

or $y = \frac{c}{t}\sqrt{(t^2 - x^2)}$, the equation of the curve.

Or, $t : p :: t^2 - x^2 : y^2$, and $y^2 = \frac{p}{t}(t^2 - x^2)$, another form of the equation to the curve; from which any one of the quantities may be found, when the rest are given.

2. *For the Hyperbola.*

Because the general property of the opposite hyperbolas, with respect to their abscisses and ordinates, is the same as that of the ellipse, therefore the process here is the very same as in the former case for the ellipse; and the equation to the curve must come out the same also, with sometimes only the change of the sign of a letter or term, from + to —, or from — to +, because here the abscisses lie beyond or without the transverse diameter, whereas they lie between or upon them in the ellipse. Thus, making the same notation for the whole diameter, conjugate, absciss, and ordinate, as at first in the ellipse; then, the one absciss AK being x, the other BK will be $t + x$, which in the ellipse was $t - x$; so the sign of x must be changed in the general property and equation, by which it becomes $t^2 : c^2 :: x(t + x) : y^2$; hence $t^2 y^2 = c^2 (tx + x^2)$ and $y = \frac{c}{t} \sqrt{(tx + x^2)}$, the equation of the curve.

Or using p the parameter, as before, it is, $t : p :: x(t+x) : y^2$ or $y^2 = \frac{p}{t} (tx + x^2)$, another form of the equation to the curve.

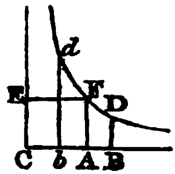
Otherwise, by using the same letters t, c, p, for the halves of the diameters and parameter, and x for the absciss CK counted from the centre; then is AK $= x - t$, and BK $= x + t$, and the property $t^2 : c^2 :: (x - t) \times (x + t) : y^2$, gives $t^2 y^2 = c^2 (x^2 - t^2)$, or $y = \frac{c}{t} \sqrt{(x^2 - t^2)}$, where the signs of t^2 and x^2 are changed from what they were in the ellipse.

Or again, using the semiparameter, $t : p :: x^2 - t^2 : y^2$, and $y^2 = \frac{p}{t} (x^2 - t^2)$ the equation of the curve.

But for the conjugate hyperbola, as in the figure to theorem 3, the signs of both x^2 and t^2 will be positive; for the property in that theorem being CA^2 : ca^2 :: CD^2 + CA^2 : De^2, it is $t^2 : c^2 :: x^2 + t^2 : y^2 =$ De^2, or $t^2 y^2 = c^2(x^2 + t^2)$, and $y = \frac{c}{t} \sqrt{(x^2 + t^2)}$, the equation to the conjugate hyperbola.

Or, as $t : p :: x^2 + c^2 : y^2$, and $y^2 = \frac{p}{t} (x^2 + t^2)$ also the equation to the same curve.

On the Equation to the Hyperbola between the Asymptotes.



Let CE and CB be the two asymptotes to the hyperbola dFD, its vertex being F, and EF, bd, AF, BD ordinates parallel to the asymptotes. Put AF or EF $= a$, CB $= x$, and BD $= y$. Then, by theor. 28, AF . EF $=$ CB . BD, or $a^2 = xy$, the equation to the hyperbola, when the abscisses and ordinates are taken parallel to the asymptotes. If the hyperbola be not rectangular AF . EF. sin. F will be equal to a given square.

3. *For the Parabola.*

If x denote any absciss beginning at the vertex, and y its ordinate, also p the parameter. Then, by cor. theorem 1, AK : KD :: KD : p, or $x : y :: y : p$; hence $px = y^2$ is the equation to the parabola. Or, if $a =$ abscissa and b the corresponding semiordinate, then $\frac{b^2}{a} x = y^2$, is the equation.

4. *For the Circle.*

Because the circle is only a species of the ellipse, in which the two axes are equal to each other; therefore, making the two diameters t and c equal each to d in the foregoing equations to the ellipse, they become $y^2 = dx - x^2$, when the absciss x begins at the vertex of the diameter: and $y^2 = \frac{1}{4}d^2 - x^2$, when the absciss begins at the centre. Or $y = \sqrt{(2rx - x^2)}$, and $y = \sqrt{(r^2 - x^2)}$, respectively, when r is the radius.

Scholium.

In every one of these equations, we perceive that they rise to the 2d or quadratic degree, or to two dimensions; which is also the number of points in which any one of these curves may be cut by a right line. Hence also it is that these four curves are said to be lines of the 2d order. And these four are all the lines that are of that order, every other curve having some higher equation, or may be cut in more points by a right line.

We may here add an important observation with regard to all curves expressed by equations: viz. that the origin of the co-ordinates is necessarily on a point of the curve itself when *all* the terms of its equation are affected by one of the variable quantities x or y; and when, on the contrary, there is in the equation one term entirely known, then the origin of the co-ordinates *cannot* be on a point of the curve. In proof of this, let the general equation of a curve be $ax^m + bx^p y^q + cy^n = 0$; then, it is evident that if we take $x = 0$, we shall likewise have $cy^n = 0$, or $y = 0$; and consequently the origin of the co-ordinates is a point in the curve. So again, if, in the same equation, we take $y = 0$, it will result that $ax = 0$, and $x = 0$, which brings us to the same thing as before. But, if the equation of the curve include one known term, as, for example, $ax^m + bx^p y^q + cy^s - g^n = 0$; then taking $x = o$, we shall have $cy^s - g^n = 0$, or $y = \sqrt[s]{\frac{g^n}{c}}$, which proves that the corresponding point P, of the curve, is distant from the origin of the x's by the quantity $\sqrt[s]{\frac{g^n}{c}}$. A similar truth will flow from making $y = 0$, when the same equation will give $x = \sqrt[m]{\frac{g^n}{a}}$.

ELEMENTS OF ISOPERIMETRY.

Def. 1. When a variable quantity has its mutations regulated by a certain law, or confined within certain limits, it is called a *maximum* when it has reached the greatest magnitude it can possibly attain; and, on the contrary, when it has arrived at the least possible magnitude, it is called a *minimum*.

Def. 2. *Isoperimeters*, or *Isoperimetrical Figures*, are those which have equal perimeters.

Def. 3. The *Locus* of any point, or intersection, &c. is the right line or curve in which these are always situated.

The problem in which it is required to find, among figures of the same or of different kinds, those which, within equal perimeters, shall comprehend the greatest surfaces, has long engaged the attention of mathematicians. Since the admirable invention of the method of Fluxions, this problem has been elegantly treated by some of the writers on that branch of analysis; especially by Maclaurin and Simpson. A much more extensive problem was investigated at the time of "the war of problems," between the two brothers John and James Bernoulli: namely, "To find, among all the isoperimetrical curves between given limits, such a curve, that, constructing a second curve, the ordinates of which shall be functions of the ordinates or arcs of the former, the area of the second curve shall be a maximum or a minimum." While, however, the attention of mathematicians was drawn to the most abstruse inquiries connected with isoperimetry, the *elements* of the subject were lost sight of. Simpson was the first who called them back to this interesting branch of research, by giving in his neat little book of Geometry a chapter on the maxima and minima of geometrical quantities, and some of the simplest problems concerning isoperimeters. The next who treated this subject in an elementary manner was Simon Lhuillier, of Geneva, who, in 1782, published his treatise *De Relatione mutua Capacitatis et Terminorum Figurarum*, &c. His principal object in the composition of that work was to supply the deficiency in this respect which he found in most of the Elementary Courses; and to determine, with regard to both the most usual surfaces and solids, those which possessed the minimum of contour with the same capacity; and, reciprocally, the maximum of capacity with the same boundary. M. Legendre has also considered the same subject, in a manner somewhat different from either Simpson or Lhuillier, in his *Elémens de Géométrie*. An elegant geometrical tract, on the same subject, was also given by Dr. Horsley, in the Philos. Trans. vol. 75, for 1775; contained also in the New Abridgement, vol. 13, page 653*. The chief propositions deduced by these four geometers, together with a few additional propositions, are reduced into one system in the following theorems.

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* Another work on the same general subject, containing many valuable theorems, has been published since the first edition of this volume, by Dr. *Creswell* of Trinity College, Cambridge.

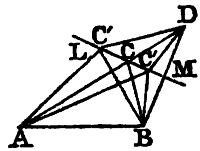
SECTION I.

SURFACES.

THEOREM I.

Of all triangles of the same base, and whose vertices fall in a right line given in position, the one whose perimeter is a minimum is that whose sides are equally inclined to that line.

Let AB be the common base of a series of triangles ABC′, ABC, &c. whose vertices C′, C, fall in the right line LM, given in position, then is the triangle of least perimeter that whose sides AC, BC, are inclined to the line LM in equal angles.



For, let BM be drawn from B, perpendicularly to LM, and produced till DM = BM : join AD, and from the point C where AD cuts LM draw BC : also, from any other point C′, assumed in LM, draw C′A, C′B, C′D. Then the triangles DMC, BMC, having the angle DCM = angle ACL (th. 7 Geom.) = MCB (by hyp.), DMC = BMC, and DM = BM, and MC common to both, have also DC = BC (th. 1 Geom.).

So also, we have C′D = C′B. Hence AC + CB = AC + CD = AD, is less than AC′ + C′D (theor. 10 Geom.), or than its equal AC′ + C′B. And consequently, AB + BC + AC is less than AB + BC′ + AC′. Q. E. D.

Cor. 1. Of all triangles of the same base and the same altitude, or of all equal triangles of the same base, the isosceles triangle has the smallest perimeter.

For, the locus of the vertices of all triangles of the same altitude will be a right line LM *parallel* to the base; and when LM in the above figure becomes parallel to AB, since MCB = ACL, MCB = CBA (th. 12 Geom.), ACL = CAB; it follows that CAB = CBA, and consequently AC = CB (th. 4 Geom.)

Cor. 2. Of all triangles of the same surface, that which has the minimum perimeter is equilateral.

For the triangle of the smallest perimeter, with the same surface, must be isosceles, whichever of the sides be considered as base : therefore, the triangle of smallest perimeter has each two or each pair of its sides equal, and consequently it is equilateral.

Cor. 3. Of all rectilinear figures, with a given magnitude and a given number of sides, that which has the smallest perimeter is equilateral.

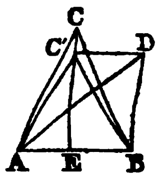
For so long as any two adjacent sides are not equal, we may draw a diagonal to become a base to those two sides, and then draw an isosceles triangle equal to the triangle so cut off, but of less perimeter: whence the corollary is manifest.

Scholium.

To illustrate the second corollary above, the student may proceed thus: assuming an isosceles triangle whose base is *not* equal to either of the two sides, and then, taking for a new base one of those sides of that triangle, he may construct another isosceles triangle equal to it, but of a smaller perimeter. Afterwards, if the base and sides of this second isosceles triangle are not respectively equal, he may construct a *third* isosceles triangle equal to it, but of a still smaller perimeter; and so on. In performing these successive operations, he will find that the new triangles will approach nearer and nearer to an equilateral triangle.

THEOREM II.

Of all triangles of the same base, and of equal perimeters, the isosceles triangle has the greatest surface.



Let ABC, ABD, be two triangles of the same base AB and with equal perimeters, of which the one ABC is isosceles, the other is not: then the triangle ABC has a surface (or an altitude) greater than the surface (or than the altitude) of the triangle ABD.

Draw C′D through D, parallel to AB, to cut CE (drawn perpendicular to AB) in C′: then it is to be demonstrated that CE is greater than C′E.

The triangles AC′B, ADB, are equal both in base and altitude; but the triangle AC′B is isosceles, while ADB is scalene: therefore the triangle AC′B has a smaller perimeter than the triangle ADB (th. 1 cor. 1), or than ACB (by hyp.). Consequently AC′ $<$ AC; and in the right-angled triangles AEC′, AEC, having AE common, we have C′E $<$ CE *. Q. E. D.

* When two mathematical quantities are separated by the character $<$,

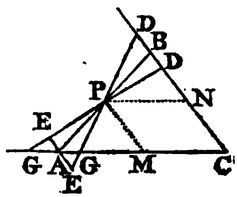
Cor. Of all isoperimetrical figures, of which the number of sides is given, that which is the greatest has all its sides equal. And in particular, of all isoperimetrical triangles, that whose surface is a maximum, is equilateral.

For, so long as any two adjacent sides are not equal, the surface may be augmented without increasing the perimeter.

Remark. Nearly as in this theorem may it be proved that, of all triangles of equal heights, and of which the sum of the two sides is equal, that which is isosceles has the greatest base. And, of all triangles standing on the same base and having equal vertical angles, the isosceles one is the greatest.

THEOREM III.

Of all right lines that can be drawn through a given point, between two right lines given in position, that which is bisected by the given point forms with the other two lines the least triangle.



Of all right lines GD, AB, GD, that can be drawn through a given point P to cut the right lines CA, CD, given in position, that, AB, which is bisected by the given point P, forms with CA, CD, the least triangle, ABC.

For, let EE be drawn through A parallel to CD, meeting DG (produced if necessary) in E; then the triangles PED, PAE, are manifestly equiangular; and, since the corresponding sides PB, PA are equal, the triangles are equal also. Hence PBD will be less or greater than PAG, according as CG is greater or less than CA. In the former case, let PACD, which is common, be added to both; then will BAC be less than DGC (ax. 4 Geom.). In the latter case, if PGCB be added, DCG will be greater than BAC; and consequently in this case also BAC is less than DCG. Q. E. D.

Cor. If PM and PN be drawn parallel to CB and CA respectively, the two triangles PAM, PBN, will be equal, and these two taken together (since AM = PN = MC) will be equal to the parallelogram PMCN: and consequently the parallelogram PMCN is equal to half ABC, but less than half DGC. From which it follows (consistently with both the algebraical and geometrical solution of prob. 8, Application of

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it denotes that the preceding quantity is *less than* the succeeding one: when, on the contrary, the separating character is $>$, it denotes that the preceding quantity is *greater than* the succeeding one.

Algebra to Geometry), that a parallelogram is always less than half a triangle in which it is inscribed, except when the base of the one is half the base of the other, or the height of the former half the height of the latter; in which case the parallelogram is just half the triangle: this being the maximum parallelogram inscribed in the triangle.

Scholium.

From the preceding corollary it might easily be shown, that the least triangle which can possibly be described *about*, and the greatest parallelogram which can be inscribed in, any curve concave to its axis, will be when the subtangent is equal to half the base of the triangle, or to the whole base of the parallelogram: and that the two figures will be in the ratio of 2 to 1. But this is foreign to the present inquiry.

THEOREM IV.

Of all triangles in which two sides are given in magnitude, the greatest is that in which the two given sides are perpendicular to each other.

For, assuming for base one of the given sides, the surface is proportional to the perpendicular let fall upon that side from the opposite extremity of the other given side: therefore, the surface is the greatest, when that perpendicular is the greatest; that is to say, when the other side is not inclined to that perpendicular, but *coincides* with it: hence the surface is a maximum when the two given sides are perpendicular to each other.

Otherwise. Since the surface of a triangle, in which two sides are given, is proportional to the sine of the angle *included* between those two sides; it follows, that the triangle is the greatest when that sine is the greatest: but the greatest sine is the sine total, or the sine of a quadrant; therefore the two sides given make a quadrantal angle, or are perpendicular to each other. Q. E. D.

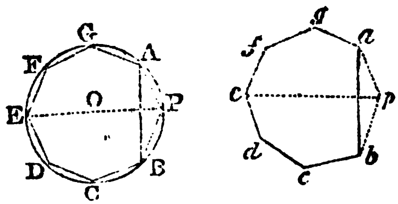
THEOREM V.

Of all rectilinear figures in which all the sides except one are known, the greatest is that which may be inscribed in a semicircle whose diameter is that unknown side.

For, if you suppose the contrary to be the case, then whenever the figure made with the sides given, and the side unknown, is not inscribable in a semicircle of which this latter i the diameter, viz. whenever any one of the angles, formed by lines drawn from the extremities of the unknown side to one of the summits of the figure, is not a right angle; we may make a figure greater than it, in which that angle shall be right, and which shall only differ from it in that respect: therefore, whenever all the angles, formed by right lines drawn from the several vertices of the figure to the extremities of the unknown line, are not right angles, or do not fall in the circumference of a semicircle, the figure is not in its maximum state. Q. E. D.

THEOREM VI.

Of all figures made with sides given in number and magnitude, that which may be inscribed in a circle is the greatest.



Let ABCDEFG be the polygon inscribed, and *abcdefg* a polygon with equal sides, but not inscribable in a circle; so that AB = *ab*, BC = *bc*, &c.; it is affirmed that the polygon ABCDEFG is greater than the polygon *abcdefg*.

Draw the diameter EP; join AP, PB; upon *ab* = AB make the triangle *abp*, equal in all respects to ABP; and join *ep*. Then, of the two figures *edcbp*, *pag fe*, one at least is not (by hyp.) inscribable, in the semicircle of which *ep* is the diameter. Consequently, one at least of these two figures is smaller than the corresponding part of the figure APBCDEFG (th. 5). Therefore the figure APBCDEFG is greater than the figure *apbc'efg*: and if from these there be taken away the respective triangles APB, *apb*, which are equal by construction, there will remain (ax. 5 Geom.) the polygon ABCDEFG greater than the polygon *abcdefg*. Q. E. D.

THEOREM VII.

The magnitude of the greatest polygon which can be contained under any number of unequal sides, does not at all depend on the order in which those lines are connected with each other.

For, since the polygon is a maximum under given sides, it is inscribable in a circle (th. 6). And this inscribed polygon is constituted of as many isosceles triangles as it has sides, those sides forming the bases of the respective triangles, the other sides of all the triangles being radii of the circle, and their common summit the centre of the circle. Consequently the magnitude of the polygon, that is, of the assemblage of these triangles, does not at all depend on their disposition, or arrangement around the common centre. Q. E. D.

THEOREM VIII.

If a polygon inscribed in a circle have all its sides equal, all its angles are likewise equal, or it is a regular polygon.

For, if lines be drawn from the several angles of the polygon, to the centre of the circumscribing circle, they will divide the polygon into as many isosceles triangles as it has sides; and each of these isosceles triangles will be equal to either of the others in all respects, and of course they will have the angles at their bases all equal: consequently, the angles of the polygon, which are each made up of two angles at the bases of two contiguous isosceles triangles, will be equal to one another. Q. E. D.

THEOREM IX.

Of all figures having the same number of sides and equal perimeters, the greatest is regular.

For, the greatest figure under the given conditions has all its sides equal (th. 2. cor.). But since the sum of the sides and the number of them are given, each of them is given: therefore (th. 6), the figure is inscribable in a circle: and consequently (th. 8) all its angles are equal; that is, it is regular. Q. E. D.

Cor. Hence we see that regular polygons possess the property of a maximum of surface, when compared with any other figures of the same name and with equal perimeters.

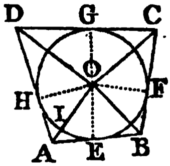
THEOREM X.

A regular polygon has a smaller perimeter than an irregular one equal to it in surface, and having the same number of sides.

This is the converse of the preceding theorem, and may be demonstrated thus: Let R and I be two figures equal in surface, and having the same number of sides, of which R is regular, I irregular: let also R′ be a regular figure similar to R, and having a perimeter equal to that of I. Then (th. 9) $R' > I$; but $I = R$: therefore $R' > R$. But R′ and R are similar; consequently, perimeter of R′ > perimeter of R; while per. R′ = per. I (by hyp.). Hence, per. I > per. R. Q. E. D.

THEOREM XI.

The surfaces of polygons, circumscribed about the same or equal circles, are respectively as their perimeters*.



Let the polygon ABCD be circumscribed about the circle EFGH; and let this polygon be divided into triangles, by lines drawn from its several angles to the centre O of the circle. Then, since each of the tangents AB, BC, &c. is perpendicular to its corresponding radius, OE, OF, &c., drawn to the point of contact (th. 46 Geom.); and since the area of a triangle is equal to the rectangle of the perpendicular and half the base (Mens. of Surfaces, pr. 2); it follows, that the area of each of the triangles ABO, BCO, &c. is equal to the rectangle of the radius of the circle and half the corresponding side AB, BC, &c.; and consequently, the area of the polygon ABCD, circumscribing the circle, will be equal to the rectangle of the radius of the circle and half the perimeter of the polygon. But, the surface of the circle is equal to the rectangle of the radius and half the circumference (th. 94 Geom.). Therefore, the surface of the circle, is to that of the polygon, as half the circumference of the former, to half the perimeter of the latter; or, as the circumference of the former, to the perimeter of the latter. Now, let P and P′ be any two polygons circumscribing a circle C: then, by the foregoing, we have

surf. C : surf. P :: circum. C : perim. P.
surf. C : surf. P′ :: circum. C : perim. P′.

But, since the antecedents of the ratios in both these proportions, are equal, the consequents are proportional: that is, surf. P : surf. P′ :: perim. P : perim. P′. Q. E. D.

Cor. 1. Any one of the triangular portions ABO, of a polygon circumscribing a circle, is to the corresponding circular sector, as the side AB of the polygon, to the arc of the circle included between AO and BO.

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* This theorem, together with the analogous ones respecting bodies circumscribing cylinders and spheres, were given by Emerson in his Geometry, and their use in the theory of Isoperimeters was just suggested: but the full application of them to that theory is due to Simon Lhuillier.

Cor. 2. Every circular arc is greater than its chord, and less than the sum of the two tangents drawn from its extremities and produced till they meet.

The first part of this corollary is evident, because a right line is the shortest distance between two given points. The second part follows at once from this proposition: for EA + AH being to the arch EIH, as the quadrangle AEOH to the circular sector HIEO; and the quadrangle being greater than the sector, because it contains it; it follows that EA + AH is greater than the arch EIH*.

Cor. 3. Hence also, any single tangent EA, is greater than its corresponding arc EI.

THEOREM XII.

If a circle and a polygon, circumscribable about another circle, are isoperimeters, the surface of the circle is a geometrical mean proportional between that polygon and a similar polygon (regular or irregular) circumscribed about that circle.

Let C be a circle, P a polygon isoperimetrical to that circle, and circumscribable about some other circle, and P′ a polygon similar to P and circumscribable about the circle C: it is affirmed that P : C : : C : P′.

For, P : P′ : : perim². P : perim². P′ : : circum². C : perim². P′ by th. 89, geom. and the hypothesis.
But (th. 11) P′ : C : : per. P′ : cir. C : : per². P′ : per. P′ × cir. C.
Therefore P : C : : . . . . cir². C : per. P′ × cir. C.
: : cir. C : per. P′ : : C : P′. Q. E. D.

THEOREM XIII.

If a circle and a polygon, circumscribable about another circle, are equal in surface, the perimeter of that figure is a geometrical mean proportional between the circumference of the first circle and the perimeter of a similar polygon circumscribed about it.

Let C = P, and let P′ be circumscribed about C and similar to C: then it is affirmed that cir. C : per. P : : per. P : per. P′.

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* This second corollary is introduced, not because of its immediate connexion with the subject under discussion, but because, notwithstanding its simplicity, some authors have employed whole pages in attempting its demonstration, and failed at last.

For. cir. C : per. P' :: C : P' :: P : P' :: per^2. P : per^2. P'.
Also, per. P' : per. P . . :: per^2. P' : per. P × per. P'.
Therefore, cir. C : per. P . :: per^2. P : per. P × per P'.
:: per. P :: per. P'. Q. E. D.

THEOREM XIV.

The circle is greater than any rectilinear figure of the same perimeter: and it has a perimeter smaller than any rectilinear figure of the same surface.

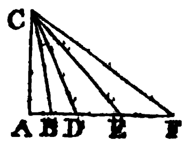
For, in the proportion, P : C :: C : P' (th. 12), since C $<$ P', therefore P $<$ C.
And, in the propor. cir. C : per. P :: per. P : per. P' (th. 13), or, cir. C : per. P' :: cir^2. C : per^2. P,
and cir. C $<$ per. P';
therefore, cir^2. C $<$ per^2. P, or cir. C $<$ per. P. Q. E. D.

Cor. 1. It follows at once, from this and the two preceding theorems, that rectilinear figures which are isoperimeters, and each circumscribable about a circle, are respectively in the inverse ratio of the perimeters, or of the surfaces, of figures similar to them, and both circumscribed about one and the same circle. And that the perimeters of equal rectilineal figures, each circumscribable about a circle, are respectively in the subduplicate ratio of the perimeters, or of the surfaces, of figures similar to them, and both circumscribed about one and the same circle.

Cor. 2. Therefore, the comparison of the perimeters of equal regular figures, having different numbers of sides, and that of the surfaces of regular isoperimetrical figures, is reduced to the comparison of the perimeters, or of the surfaces of regular figures respectively similar to them, and circumscribable about one and the same circle.

Lemma 1.

If an acute angle of a right-angled triangle be divided into any number of equal parts, the side of the triangle opposite to that acute angle is divided into unequal parts, which are greater as they are more remote from the right angles.



Let the acute angle C, of the right-angled triangle ACF, be divided into equal parts, by the lines BC, CD, CE, drawn from that angle to the opposite side; then shall the parts AB, BD, &c. intercepted by the lines drawn from C, be successively longer as they are more remote from the right angle A.

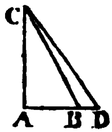
For, the angles ACD, BCE. &c. being bisected by CB, CD, &c. therefore by theor. 83 Geom. AC : CD : : AB : BD. and BC : CE : : BD : DE, and DC : CF : : DE : EF. And by th. 21 Geom. CD > CA, CE > CB, CF > CC, and so on: whence it follows, that DB > AB, DE > DB, and so on. Q. E. D.

Cor. Hence it is obvious that, if the part the most remote from the right angle A, be repeated a number of times equal to that into which the acute angle is divided, there will result a quantity greater than the side opposite to the divided angle.

THEOREM XV.

If two regular figures, circumscribed about the same circle, differ in their number of sides by unity, that which has the greatest number of sides shall have the smallest perimeter.

Let CA be the radius of a circle, and AB, AD, the half sides of two regular polygons circumscribed about that circle, of which the number of sides differ by unity, being respectively $n + 1$ and n. The angles ACB, ACD, therefore are respectively the $\frac{1}{n+1}$ and the $\frac{1}{n}$ th part of two right angles: consequently these angles are as n and $n + 1$: and hence, the angle may be conceived divided into $n + 1$ equal parts, of which BCD is one. Consequently, (cor. to the lemma) $(n + 1)$ BD > AD. Taking, then, unequal quantities from equal quantities, we shall have



$$(n + 1) \text{AD} - (n + 1) \text{BD} < (n + 1) \text{AD} - \text{AD},$$
$$\text{or } (n + 1) \text{AB} < n \,.\, \text{AD}.$$

That is, the semiperimeter of the polygon whose half side is AB, is smaller than the semiperimeter of the polygon whose half side is AD: whence the proposition is manifest.

Cor. Hence, augmenting successively by unity the number of sides, it follows generally, that the perimeters of polygons circumscribed about any proposed circle, become smaller as the number of their sides become greater.

THEOREM XVI.

The surfaces of regular isoperimetrical figures are greater as the number of their sides is greater: and the perimeters of equal regular figures are smaller as the number of their sides is greater.

For, 1st. Regular isoperimetrical figures are (cor. 1. th. 14) in the inverse ratio of figures similar to them circumscribed about the same circle. And (th. 15) these latter are smaller when their number of sides is greater: therefore, on the contrary, the former become greater as they have more sides.

2dly. The perimeters of equal regular figures are (cor. 1 th. 14) in the subduplicate ratio of the perimeters of similar figures circumscribed about the same circle: and (th. 15) these latter are smaller as they have more sides: therefore the perimeters of the former also are smaller when the number of their sides is greater. Q. E. D.

SECTION II.

SOLIDS.

THEOREM XVII.

Of all prisms of the same altitude, whose base is given in magnitude and species, or figure, or shape, the right prism has the smallest surface.

For, the area of each face of the prism is proportional to its height; therefore the area of each face is the smallest when its height is the smallest, that is to say, when it is equal to the altitude of the prism itself: and in that case the prism is evidently a right prism. Q. E. D.

THEOREM XVIII.

Of all prisms whose base is given in magnitude and species, and whose lateral surface is the same, the right prism has the greatest altitude, or the greatest capacity.

This is the converse of the preceding theorem, and may readily be proved after the manner of theorem 2.

THEOREM XIX.

Of all right prisms of the same altitude, whose bases are given in magnitude and of a given number of sides, that whose base is a regular figure has the smallest surface.

For, the surface of a right prism of given altitude, and base given in magnitude, is evidently proportional to the perimeter of its base. But (th. 10) the base being given in magnitude, and having a given number of sides, its perimeter is smallest when it is regular: whence, the truth of the proposition is manifest.

THEOREM XX.

Of two right prisms of the same altitude, and with irregular bases equal in surface, that whose base has the greatest number of sides has the smallest surface; and, in particular, the right cylinder has a smaller surface than any prism of the same altitude and the same capacity.

The demonstration is analogous to that of the preceding theorem, being at once deducible from theorems 16 and 14.

THEOREM XXI.

Of all right prisms whose altitudes and whose whole surfaces are equal, and whose bases have a given number of sides; that whose base is a regular figure is the greatest.

Let P, P′, be two right prisms of the same name, equal in altitude, and equal whole surface, the first of these having a regular, the second an irregular base; then is the base of the prism P′, less than the base of the prism P.

For, let P″ be a prism of equal altitude, and whose base is equal to that of the prism P′ and similar to that of the prism P. Then, the lateral surface of the prism P″ is smaller than the lateral surface of the prism P′ (th. 19): hence, the total surface of P″ is smaller than the total surface of P′, and therefore (by hyp.) smaller than the whole surface of P. But the prisms P″ and P have equal altitudes, and *similar* bases; therefore the dimensions of the base of P″ are smaller than the dimensions of the base of P. Consequently the base of P″, or that of P′, is less than the base of P; or the base of P greater than that of P′. Q. E. D.

THEOREM XXII.

Of two right prisms, having equal altitudes, equal total surfaces, and regular bases, that whose base has the greatest number of sides, has the greatest capacity. And, in particular, a right cylinder is greater than any right prism of equal altitude and equal total surface.

The demonstration of this is similar to that of the preceding theorem, and flows from th. 20.

THEOREM XXIII.

The greatest parallelopiped which can be contained under the three parts of a given line, any way taken, will be that constituted of equal length, breadth, and depth.

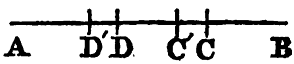
For, let AB be the given line, and, if possible, let two parts AE, ED, be unequal. Bisect AD and C, then will the rectangle under AE (= AC + CE) and ED (= AC − CE), be less than AC^2, or than AC . CD, by the square of CE (th. 33 Geom.). Consequently, the solid AE . ED . DB, will be less than the solid AC . CD . DB; which is repugnant to the hypothesis.

Cor. Hence, of all the rectangular parallelopipeds, having the sum of their three dimensions the same, the cube is the greatest.

THEOREM XXIV.

The greatest parallelopiped that can possibly be contained under the square of one part of a given line, and the other part, any way taken, will be when the former part is the double of the latter.

Let AB be a given line, and AC = 2CB, then is AC^2 . CB the greatest possible.



For, let AC′ and C′B be any other parts into which the given line AB may be divided; and let AC, AC′ be bisected in DD′, respectively. Then shall AC^2 . CB = 4AD . DC . CB (cor. to theor. 31 Geom.) > 4AD′ . D′C . CB, or greater than its equal $C'A^2$. C′B, by the preceding theorem.

THEOREM XXV.

Of all right parallelopipeds given in magnitude, that which has the smallest surface has all its faces squares, or is a cube. And reciprocally, of all parallelopipeds of equal surface, the greatest is a cube.

For, by theorems 19 and 21, the right parallelopiped having the smallest surface with the same capacity, or the greatest capacity with the same surface, has a square for its base. But, any face whatever may be taken for base: therefore, in the parallelopiped whose surface is the smallest with the same capacity, or whose capacity is the greatest with the same surface, any two opposite faces whatever are squares: consequently, this parallelopiped is a cube.

THEOREM XXVI.

The capacities, of prisms circumscribing the same right cylinder, are respectively as their surfaces, whether total or lateral.

For, the capacities are respectively as the bases of the prisms; that is to say (th. 11), as the perimeters of their bases; and these are manifestly as the lateral surfaces: whence the proposition is evident.

Cor. The surface of a right prism circumscribing a cylinder, is to the surface of that cylinder, as the capacity of the former, to the capacity of the latter.

Def. The Archimedean cylinder is that which circumscribes a sphere, or whose altitude is equal to the diameter of its base.

THEOREM XXVII.

The Archimedean cylinder has a smaller surface than any other right cylinder of equal capacity; and it is greater than any other right cylinder of equal surface.

Let C and C′ denote two right cylinders, of which the first is Archimedean, the other not: then,

1st, If . . . $C = C'$, surf. $C <$ surf. C':
2dly, if surf. $C =$ surf. C', $C > C'$.

For, having circumscribed about the cylinders C, C′, the right prisms P, P′, with square bases, the former will be a cube, the second not: and the following series of equal ratios will obtain, viz. C : P : : surf. C : surf. P : : base C : base P : : base C′ : base P′ : : C′ : P′ : : surf. C′ : surf. P′.

Then, 1st: when $C = C'$. Since C : P : : C′ : P′, it follows that $P = P'$; and therefore (th. 25) surf. $P <$ surf. P'. But, surf. C : surf. P : : surf. C′ : surf. P′; consequently surf. $C <$ surf. C'. Q. E. 1D.

2dly: when surf. $C =$ surf. C'. Then, since surf. C : surf. P : : surf. C′ : surf. P′, it follows that surf. $P =$ surf. P'; and therefore (th. 25) $P > P'$. But C : P : : C′ : P′; consequently $C > C'$. Q. E. 2D.

THEOREM XXVIII.

Of all right prisms whose bases are circumscribable about circles, and given in species, that whose altitude is double the radius of the circle inscribed in the base, has the smallest surface with the same capacity, and the greatest capacity with the same surface.

This may be demonstrated exactly as the preceding theorem, by supposing cylinders inscribed in the prisms.

Scholium.

If the base cannot be circumscribed about a circle, the right prism which has the minimum surface, or the maximum capacity, is that whose lateral surface is quadruple of the surface of one end, or that whose lateral surface is two-thirds of the total surface. This is manifestly the case with the Archimedean cylinder; and the extension of the property depends solely on the mutual connexion subsisting between the properties of the cylinder, and those of circumscribing prisms.

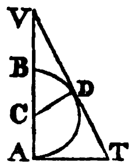
THEOREM XXIX.

The surfaces of right cones circumscribed about a sphere, are as their solidities.

For, it may be demonstrated, in a manner analogous to the demonstrations of theorems 11 and 26, that these cones are equal to right cones whose altitude is equal to the radius of the inscribed sphere, and whose bases are equal to the total surfaces of the cones: therefore the surfaces and solidities are proportional.

THEOREM XXX.

The surface or the solidity of a right cone circumscribed about a sphere is directly as the square of the cone's altitude, and inversely as the excess of that altitude over the diameter of the sphere.



Let VAT be a right-angled triangle which, by its rotation upon VA as an axis, generates a right cone; and BDA the semicircle which by a like rotation upon VA forms the inscribed sphere: then, the surface or the solidity of the cone varies as $\frac{VA^2}{VB}$.

For, draw the radius CD to the point of contact of the semicircle and VT. Then, because the triangles VAT, VDC, are similar, it is AT : VT : : CD : VC.

And, by compos. AT : AT + VT : : CD : CD + CV = VA;

Therefore $AT^2 : (AT + VT)\ AT :: CD : VA$, by multiplying the terms of the first ratio by AT.
But, because VB, VD, VA, are continued proportionals, it is $VB : VA :: VD^2 : VA^2 :: CD^2 : AT^2$ by sim. triangles.
But $CD : VA :: AT^2 : (AT + VT)\ AT$ by the last: and these mult. give $CD \cdot VB : VA^2 :: CD^2 : (AT + VT)\ AT$,

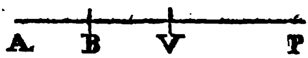
$$\text{or } VB : CD :: VA^2 : (AT + VT)\ AT = CD \cdot \frac{VA^2}{VB}.$$

But the surface of the cone, which is denoted by $\pi \cdot AT^2 + \pi \cdot AT \cdot VT$*, is manifestly proportional to the first member of this equation, is also proportional to the second member, or, since CD is constant, it is proportional to $\frac{AV^2}{VB}$, or to a third proportional to BV and AV. And, since the capacities of these circumscribing cones are as their surfaces (th. 29), the truth of the whole proposition is evident.

Lemma 2.

The difference of two right lines being given, the third proportional to the less and the greater of them is a minimum when the greater of those lines is double the other.

Let AV and BV be two right lines, whose difference AB is given, and let AP be a third proportional to BV and AV; then is AP a minimum when $AV = 2BV$.



For, since $AP : AV :: AV : BV$;
By division $AP : AP - AV :: AV : AV - BV$;
That is, $AP : VP :: AV : AB$.
Hence, $VP \cdot AV = AP \cdot AB$.

But $VP \cdot AV$ is either $=$ or $< \frac{1}{4}AP^2$ (cor. to th. 31 Geom. and th. 23 of this chapter).

Therefore $AP \cdot AB \leqq \frac{1}{4}AP^2$: whence $4AB \leqq AP$, or $AP \geqq 4AB$. Consequently the minimum value of AP is the quadruple of AB; and in that case $PV = VA = 2AB$. Q. E. D†.

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* π being $= 3{\cdot}141593$. See Vol. i. p. 422.

† Though the evidence of a single demonstration, conducted on sound mathematical principles, is really irresistible, and therefore needs no corroboration; yet it is frequently conducive as well to mental improvement, as to mental delight, to obtain like results from different processes. In this view it will be advantageous to the student, to confirm the truth of several of the propositions in this chapter by means of the fluxional analysis. Let the truth enunciated in the above lemma be taken for an

THEOREM XXXI.

Of all right cones circumscribed about the same sphere, the smallest is that whose altitude is double the diameter of the sphere.

For, by th. 30, the solidity varies as $\frac{VA^2}{VB}$ (see the fig. to that theorem): and, by lemma 2, since VA − VB is given, the third proportional $\frac{VA^2}{VB}$ is a minimum when VA = 2AB. Q. E. D.

Cor. 1. Hence, the distance from the centre of the sphere to the vertex of the least circumscribing cone, is triple the radius of the sphere.

Cor. 2. Hence also, the side of such cone is triple the radius of its base.

THEOREM XXXII.

The whole surface of a right cone being given, the inscribed sphere is the greatest when the slant side of the cone is triple the radius of its base.

For, let C and C′ be two right cones of equal whole surface, the radii of their respective inscribed spheres being denoted by R and R′; let the side of the cone C be triple the radius of its base, the same ratio not obtaining in C; and let C″ be a cone similar to C, and circumscribed about the same sphere with C′. Then, (by th. 31) surf. C″ < surf. C′: therefore surf. C″ < surf. C. But C″ and C are similar, therefore all the dimensions of C″ are less than the corresponding dimensions of C: and consequently the radius R′ of the sphere inscribed in C″ or in C′, is less than the radius R of the sphere inscribed in C, or R > R′. Q. E. D.

Cor. The capacity of a right cone being given, the inscribed sphere is the greatest when the side of the cone is triple the radius of its base.

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example; and let AB be denoted by a, AV by x, BV by $x - a$. Then we shall have $x - a : x :: x : \frac{x^2}{x-a}$, the third proportional; which is to be a minimum. Hence, the fluxion of this fraction will be equal to zero (Flux. art. 57). That is, (Flux. arts. 19 and 35), $\frac{x^2\dot{x} - 2ax\dot{x}}{(x-a)^2} = 0$. Consequently $x^2 - 2ax = 0$, and $x = 2a$, or AV = 2AB, as above.

For the capacities of such cones vary as their surfaces (th. 29).

THEOREM XXXIII.

Of all right cones of equal whole surface, the greatest is that whose side is triple the radius of its base: and reciprocally, of all right cones of equal capacity, that whose side is triple the radius of its base has the least surface.

For, by th. 29, the capacity of a right cone is in the compound ratio of its whole surface and the radius of its inscribed sphere. Therefore, the whole surface being given, the capacity is proportional to the radius of the inscribed sphere: and consequently is a maximum when the radius of the inscribed sphere is such; that is, (th. 32) when the side of the cone is triple the radius of the base*.

Again, reciprocally, the capacity being given, the surface is in the inverse ratio of the sphere inscribed: therefore, it

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* Here again a similar result may easily be deduced from the method of fluxions. Let the radius of the base be denoted by x, the slant side of the cone by z, its whole surface by a^2, and 3·14159 by π. Then the circumference of the cone's base will be $2\pi x$, its area πx^2, and the convex surface πxz. The whole surface is, therefore, $= \pi x^2 + \pi xz$: and this being $= a^2$, we have $z = \frac{a^2}{\pi x} - x$. But the altitude of the cone is equal to the square root of the difference of the squares of the side and of the radius of the base; that is, it is $= \sqrt{(\frac{a^4}{\pi^2 x^2} - \frac{2a^2}{\pi})}$. And this multiplied into $\frac{1}{3}$ of the area of the base, viz. by $\frac{1}{3}\pi x^2$, gives $\frac{1}{3}\pi x^2 \sqrt{(\frac{a^4}{\pi^2 x^2} - \frac{2a^2}{\pi})}$, for the capacity of the cone. Now, this being a maximum, its square must be so likewise (Flux. art. 58), that is, $\frac{a^4x^2 - 2\pi a^2 x^4}{9}$, or rejecting the denominator, as constant, $a^4x^2 - 2\pi a^2 x^4$ must be a maximum. This, in fluxions, is $2a^4x\dot{x} - 8\pi a^2 x^3\dot{x} = 0$; whence we have $a^2 - 4\pi x^2 = 0$, and consequently $x = \sqrt{\frac{a^2}{4\pi}}$; and $a^2 = 4\pi x^2$. Substituting this value of a^2 for it, in the value of z above given, there results $z = \frac{a^2}{\pi x} - x = \frac{4\pi x^2}{\pi x} - x = 4x - x = 3x$. Therefore, the side of the cone is triple the radius of its base. Or, the square of the altitude is to the square of the radius of the base, as 8 to 1, or, to the square of the diameter of the base, as 2 to 1.

is the smallest when that radius is the greatest; that is (th. 32) when the side of the cone is triple the radius of its base.

Q. E. D.

THEOREM XXXIV.

The surfaces, whether total or lateral, of pyramids circumscribed about the same right cone, are respectively as their solidities. And, in particular, the surface of a pyramid circumscribed about a cone, is to the surface of that cone, as the solidity of the pyramid is to the solidity of the cone; and these ratios are equal to those of the surfaces or the perimeters of the bases.

For, the capacities of the several solids are respectively as their bases; and their surfaces are as the perimeters of those bases: so that the proposition may manifestly be demonstrated by a chain of reasoning exactly like that adopted in theorem 11.

THEOREM XXXV.

The base of a right pyramid being given in species, the capacity of that pyramid is a maximum with the same surface, and, on the contrary, the surface is a minimum with the same capacity, when the height of one face is triple the radius of the circle inscribed in the base.

Let P and P′ be two right pyramids with similar bases, the height of one lateral face of P being triple the radius of the circle inscribed in the base, but this proportion not obtaining with regard to P′: then

1st. If surf. P = surf. P′, P > P′.

2dly. If . . P = . . P′, surf. P < surf. P′.

For, let C and C′ be right comes inscribed within the pyramids P and P′: then, in the cone C, the slant side is triple the radius of its base, while this is not the case with respect to the cone C′. Therefore, if C = C′, surf. C < surf. C′; and, if surf. C = surf. C′, C > C′ (th. 33).

But, 1st. surf. P : surf. C : : surf. P′ : surf. C′;
whence, if surf. P = surf. P′, surf. C = surf. C′;
therefore C > C′. But P : C : : P′ : C′. Therefore P > P′.

2dly. P : C : : P′ : C′. Theref. if P = P′, C = C′: consequently surf. C < surf. C′. But surf. P : surf. C : : surf. P′ : surf. C′. Whence, surf. P < surf. P′.

Cor. The regular tetraedron possesses the property of the minimum surface with the same capacity, and of the maximum capacity with the same surface, relatively to all right pyramids with equilateral triangular bases, and, *a fortiori*, relatively to every other triangular pyramid.

THEOREM XXXVI.

A sphere is to any circumscribing solid, bounded by plane surfaces, as the surface of the sphere to that of the circumscribing solid.

For, since all the planes touch the sphere, the radius drawn to each point of contact will be perpendicular to each respective plane. So that, if planes be drawn through the centre of the sphere and through all the edges of the body, the body will be divided into pyramids whose bases are the respective planes, and their common altitude the radius of the sphere. Hence, the sum of all these pyramids, or the whole circumscribing solid, is equal to a pyramid or a cone whose base is equal to the whole surface of that solid, and altitude equal to the radius of the sphere. But the capacity of the sphere is equal to that of a cone whose base is equal to the surface of the sphere, and altitude equal to its radius. Consequently, the capacity of the sphere, is to that of the circumscribing solid, as the surface of the former to the surface of the latter: both having, in this mode of considering them, a common altitude. Q. E. D.

Cor. 1. All circumscribing cylinders, cones, &c. are to the sphere they circumscribe, as their respective surfaces.

For the same proportion will subsist between their indefinitely small corresponding segments, and therefore between their wholes.

Cor. 2. All bodies circumscribing the same sphere, are respectively as their surfaces.

THEOREM XXXVII.

The sphere is greater than any polyedron of equal surface.

For, first it may be demonstrated, by a process similar to that adopted in theorem 9, that a *regular* polyedron has a greater capacity than any other polyedron of equal surface. Let P, therefore, be a regular polyedron of equal surface to a sphere S. Then P must either circumscribe S, or fall partly within it and partly without it, or fall entirely within it. The first of these suppositions is contrary to the hypothesis of the proposition, because in that case the surface of P could not be *equal* to that of S. Either the 2d or 3d supposition therefore must obtain; and then each plane of the surface of P must fall either partly or wholly within the sphere S: whichever of these be the case, the perpendiculars demitted from the centre of S upon the planes, will be each less than the radius of that sphere: and consequently the polyedron P must be less than the sphere S, because it has an equal base, but a less altitude. Q. E. D.

Cor. If a prism, a cylinder, a pyramid, or a cone, be equal to a sphere either in capacity, or in surface; in the first case, the surface of the sphere is less than the surface of any of those solids; in the second, the capacity of the sphere is greater than that of either of those solids.

The theorems in this chapter will suggest a variety of practical examples to exercise the student in computation. A few such are given below.

EXERCISES.

Ex. 1. Find the areas of an equilateral triangle, a square, a hexagon, a dodecadon, and a circle, the perimeter of each being 36.

Ex. 2. Find the difference between the area of a triangle whose sides are 3, 4, and 5, and of an equilateral triangle of equal perimeter.

Ex. 3. What is the area of the greatest triangle which can be constituted with two given sides 8 and 11; and what will be the length of its third side?

Ex. 4. The circumference of a circle is 12, and the perimeter of an irregular polygon which circumscribes it is 15: what are their respective areas?

Ex. 5. Required the surface and the solidity of the greatest parallelopiped, whose length, breadth, and depth, together make 18?

Ex. 6. The surface of a square prism is 546: what is its solidity when a maximum?

Ex. 7. The content of a cylinder is 169·645968: what is its surface when a minimum?

Ex. 8. The whole surface of a right cone is 201·061952: what is its solidity when a maximum?

Ex. 9. The surface of a triangular pyramid is 43·30127: what is its capacity when a maximum?

Ex. 10. The radius of a sphere is 10. Required the solidities of this sphere, of its circumscribed equilateral cone, and of its circumscribed cylinder.

Ex. 11. The surface of a sphere is 28·274337, and of an irregular polyedron circumscribed about it 35 : what are their respective solidities?

Ex. 12. The solidity of a sphere, equilateral cone, and Archimedean cylinder, are each 500 : what are the surfaces and respective dimensions of each?

Ex. 13. If the surface of a sphere be represented by the number 4, the circumscribed cylinder's convex surface and whole surface will be 4 and 6, and the circumscribed equilateral cone's convex and whole surface, 6 and 9 respectively. Show how these numbers are deduced.

Ex. 14. The solidity of a sphere, circumscribed cylinder, and circumscribed equilateral cone, are as the numbers 4, 6, and 9. Required the proof.

PRACTICAL EXERCISES IN MENSURATION.

QUEST. 1. WHAT difference is there between a floor 28 feet long by 20 broad, and two others, each of half the dimensions; and what do all three come to at 45*s*. per square, or 100 square feet?

Ans. dif. 280 sq. feet. Amount 18 guineas.

QUEST. 2. An elm plank is 14 feet 3 inches long, and I would have just a square yard slit off it; at what distance from the edge must the line be struck? Ans. $7\frac{11}{19}$ inches.

QUEST. 3. A ceiling contains 114 yards 6 feet of plastering, and the room is 28 feet broad; what is the length of it?

Ans. $36\frac{6}{7}$ feet.

QUEST. 4. A common joist is 7 inches deep, and $2\frac{1}{2}$ thick; but I want a scantling just as big again, that shall be 3 inches thick; what will the other dimension be?

Ans. $11\frac{2}{3}$ inches.

QUEST. 5. A wooden trough cost me 3*s*. 2*d*. painting within, at 6*d*. per yard; the length of it was 102 inches, and the depth 21 inches; what was the width?

Ans. $27\frac{1}{4}$ inches.

QUEST. 6. If my court-yard be 47 feet 9 inches square, and I have laid a foot-path with Purbeck-stone, of 4 feet wide, along one side of it; what will paving the rest with flints come to, at 6*d*. per square yard? Ans. 5*l*. 16*s*. $0\frac{1}{4}$*d*.

QUEST. 7. A ladder, 36 feet long, may be so planted, that it shall reach a window 30·7 feet from the ground on one side of the street; and, by only turning it over, without moving the foot out of its place, it will do the same by a window 18·9 feet high on the other side: what is the breadth of the street? Ans. 50·984 feet.

QUEST. 8. The paving of a triangular court, at 18*d.* per foot, came to 100*l.*; the longest of the three sides was 88 feet; required the sum of the other two equal sides? Ans. 10·685 feet.

QUEST. 9. The perambulator, or surveying wheel, is so contrived, as to turn just twice in the length of a pole, or 16½ feet; required the diameter? Ans. 2·626 feet.

QUEST. 10. In turning a one-horse chaise within a ring of a certain diameter, it was observed, that the outer wheel made two turns, while the inner made but one: the wheels were both 4 feet high; and, supposing them fixed at the statutable distance of 5 feet asunder on the axle-tree, what was the circumference of the track described by the outer wheel? Ans. 62·832 feet.

QUEST. 11. What is the side of that equilateral triangle, whose area cost as much paving at 8*d.* a foot, as the pallisading the three sides did at a guinea a yard? Ans. 72·746 feet.

QUEST. 12. A roof, which is 24 feet 8 inches by 14 feet 6 inches, is to be covered with lead at 8lb. per square foot: what will it come to at 18*s.* per cwt.? Ans. 22*l.* 19*s.* 10¼*d.*

QUEST. 13. Having a rectangular marble slab, 58 inches by 27, I would have a square foot cut off parallel to the shorter edge; I would then have the like quantity divided from the remainder parallel to the longer side; and this alternately repeated, till there shall not be the quantity of a foot left: what will be the dimensions of the remaining piece? Ans. 20·7 inches by 6·086.

N. B. This question may be solved neatly by an algebraical process, as may be seen in the Ladies' Diary for 1823.

QUEST. 14. Given two sides of an obtuse-angled triangle, which are 20 and 40 poles; required the third side, that the triangle may contain just an acre of land? Ans. 58·876 or 23·099.

QUEST. 15. How many bricks will it take to build a wall, 10 feet high, and 500 feet long, of a brick and half thick; reckoning the brick 10 inches long, and 4 courses to the foot in height? Ans. 72000.

QUEST. 16. How many bricks will build a square pyramid of 100 feet on each side at the base, and also 100 feet perpendicular height: the dimensions of a brick being supposed 10 inches long, 5 inches broad, and 3 inches thick?

Ans. 3840000.

Quest. 17. If, from a right-angled triangle, whose base is 12, and perpendicular 16 feet, a line be drawn parallel to the perpendicular, cutting off a triangle whose area is 24 square feet; required the sides of this triangle?

Ans. 6, 8, and 10.

Quest. 18. If a round pillar, 7 inches across, have 4 feet of stone in it; of what diameter is the column, of equal length, that contains 10 times as much?

Ans. 22·136 inches.

Quest. 19. A circular fish-pond is to be made in a garden, that shall take up just half an acre; what must be the length of the cord that strikes the circle? Ans. $27\frac{3}{4}$ yards.

Quest. 20. When a roof is of a true pitch, the rafters are $\frac{3}{4}$ of the breadth of the building: now supposing the eaves-boards to project 10 inches on a side, what will the new ripping a house cost, that measures 32 feet 9 inches long, by 22 feet 9 inches broad on the flat, at 15*s.* per square? Ans. 8*l.* 15*s.* $9\frac{1}{4}$*d.*

Quest. 21. A cable, which is 3 feet long, and 9 inches in compass, weighs 22lb.; what will a fathom of that cable weigh, which measures a foot round? Ans. $78\frac{2}{9}$lb.

Quest. 22. A plumber has put 28lb. per square foot into a cistern, 74 inches and twice the thickness of the lead long, 26 inches broad, and 40 deep; he has also put three stays across it within, 16 inches deep, of the same strength, and reckons 22*s.* per cwt. for work and materials. A mason has in return paved him a workshop, 22 feet 10 inches broad, with Purbeck stone, at 7*d.* per foot; and upon the balance finds there is 3*s.* 6*d.* due to the plumber; what was the length of the workshop, supposing sheet lead $\frac{1}{10}$ of an inch thick to weigh 5·899lbs. per foot? Ans. 32·2825 feet.

Quest. 23. The distance of the centres of two circles, whose diameters are each 50, being given, equal to 30; what is the area of the space inclosed by their circumferences?

Ans. 559·119.

Quest. 24. If 20 feet of iron railing weigh half a ton, when the bars are an inch and quarter square; what will 50 feet come to at $3\frac{1}{2}$*d.* per lb., the bars being but $\frac{7}{8}$ of an inch square? Ans. 20*l.* 0*s.* 2*d.*

Quest. 25. It is required to find the thickness of the lead in a pipe, of an inch and quarter bore, which weighs

14lb. per yard in length; the cubic foot of lead weighing 11325 ounces? Ans. ·20737 inches.

QUEST. 26. Supposing the expense of paving a semicircular plot, at 2*s*. 4*d*. per foot, come to 10*l*.; what is the diameter of it? Ans. 14·7737 feet.

QUEST. 27. What is the length of a chord which cuts off $\frac{1}{8}$ of the area from a circle whose diameter is 289?

Ans. 278·6716.

QUEST. 28. My plumber has set me up a cistern, his shop-book being burnt, he has no means of bringing in the charge, and I do not choose to take it down to have it weighed; but by measure he finds it contains $64\frac{3}{16}$ square feet, and that it is precisely $\frac{1}{8}$ of an inch in thickness. Lead was then wrought at 2*l*. per fother of $19\frac{1}{2}$ cwt. It is required from these items to make out the bill, allowing $6\frac{5}{9}$ oz. for the weight of a cubic inch of lead?

Ans. 4*l*. 11*s*. 2*d*.

QUEST. 29. What will the diameter of a globe be, when the solidity and superficial content are expressed by the same number? Ans. 6.

QUEST. 30. A sack, that would hold 3 bushels of corn, is $22\frac{1}{2}$ inches broad when empty; what will another sack contain, which, being of the same length, has twice its breadth or circumference? Ans. 12 bushels.

QUEST. 31. A carpenter is to put an oaken curb to a round well, at 8*d*. per foot square: the breadth of the curb is to be 8 inches, and the diameter within $3\frac{1}{2}$ feet: what will be the expense? Ans. 6*s*. $6\frac{3}{4}$*d*.

QUEST. 32. A gentleman has a garden 100 feet long, and 80 feet broad; and a gravel walk is to be made of an equal width half round it: what must the breadth of the walk be, to take up just half the ground? Ans. 25·968 feet.

QUEST. 33. The top of a may-pole, being broken off by a blast of wind, struck the ground at 15 feet distance from the foot of the pole; what was the height of the whole may-pole, supposing the length of the broken piece to be 39 feet? Ans. 75 feet.

QUEST. 34. Seven men bought a grinding-stone of 60 inches diameter, each paying $\frac{1}{7}$ part of the expense; what part of the diameter must each grind down for his share?

Ans. the 1st 4·4508, 2d. 4·8400, 3d 5·3535, 4th 6·0765, 5th 7·2079, 6th 9·3935, 7th 22·6778 inches.

QUEST. 35. A maltster has a kiln, that is 16 feet 6 inches square: but he wants to pull it down, and build a new one, that may dry three times as much at once as the old one; what must be the length of its side? Ans. 28 feet, 7 inches.

QUEST. 36. How many 3 inch cubes may be cut out of a 12 inch cube? Ans. 64.

QUEST. 37. How long must the tether of a horse be, that will allow him to graze, quite around, just an acre of ground? Ans. 39¼ yards.

QUEST. 38. What will the painting of a conical spire come to at 8*d.* per yard; supposing the height to be 118 feet, and the circumference of the base 64 feet?

Ans. 11*l.* 0*s* 8¾*d.*

QUEST. 39. The diameter of a standard corn bushel is 18½ inches, and its depth 8 inches; then what must the diameter of that bushel be, whose depth is 7½ inches?

Ans. 19·1067 inches.

QUEST. 40. Suppose the ball on the top of St. Paul's church is 6 feet in diameter; what did the gilding of it cost at 3½ per square inch? Ans. 237*l.* 10*s.* 1*d.*

QUEST. 41. What will a frustum of a marble cone come to at 12*s.* per solid foot; the diameter of the greater end being 4 feet, that of the less end 1½, and the length of the slant side 8 feet? Ans. 30*l.* 1*s.* 10¼*d.*

QUEST. 42. To divide a cone into three equal parts by sections parallel to the base, and to find the altitudes of the three parts, the height of the whole cone being 20 inches?

Ans. the upper part 13·867
the middle part 3·605
the lower part 2·528

QUEST. 43. A gentleman has a bowling-green, 300 feet long, and 200 feet broad, which he would raise 1 foot higher, by means of the earth to be dug out of a ditch that goes round it: to what depth must the ditch be dug, supposing its breadth to be every where 8 feet? Ans. 7$\frac{23}{86}$ feet.

QUEST. 44. How high above the earth must a person be raised, that he may see $\frac{1}{3}$ of its surface?

Ans. to the height of the earth's diameter.

QUEST. 45. A cubic foot of brass is to be drawn into wire of $\frac{1}{40}$ of an inch in diameter; what will the length of the wire be, allowing no loss in the metal?

Ans. 97784·797 yards, or 55 miles 984·797 yards.

QUEST. 46. Of what diameter must the bore of a cannon be, which is cast for a ball of 24lb. weight, so that the diameter of the bore may be $\frac{1}{10}$ of an inch more than that of the ball? Ans. 5·647 inches.

QUEST. 47. Supposing the diameter of an iron 9lb. ball to be 4 inches, as it is very nearly; it is required to find the diameters of the several balls weighing 1, 2, 3, 4, 6, 12, 18, 24, 32, 36, and 42lb. and the calibre of their guns, allowing $\frac{1}{52}$ of the calibre, or $\frac{1}{50}$ of the ball's diameter, for windage.

Answer.

Wt. ball.	Diameter ball.	Calibre gun.
1	1·9230	1·9622
2	2·4228	2·4723
3	2·7734	2·8301
4	3·0526	3·1149
6	3·4943	3·5656
9	4·0000	4·0816
12	4·4026	4·4924
18	5·0397	5·1425
24	5·5469	5·6601
32	6·1051	6·2297
36	6·3496	6·4792
42	6·6844	6·8208

QUEST. 48. Supposing the windage of all mortars to be $\frac{1}{52}$ of the calibre, and the diameter of the hollow part of the shell to be $\frac{7}{10}$ of the calibre of the mortar: it is required to determine the diameter and weight of the shell, and the quantity or weight of powder requisite to fill it, for each of the several sorts of mortars, namely, the 13, 10, 8, 5·8, and 4·6 inch mortar.

Answer.

Calib. mort.	Diameter ball.	Wt. shell empty.	Wt. of powder.	Wt. shell filled.
4·6	4·523	7·320	0·583	8·903
5·8	5·703	16·677	1·168	17·845
8	7·867	43·734	3·065	46·829
10	9·833	85·476	5·986	91·462
13	12·783	187·791	13·151	200·942

QUEST. 49. If a heavy sphere, whose diameter is 4 inches, be let fall into a conical glass, full of water, whose diameter

is 5, and altitude 6 inches; it is required to determine how much water will run over?

Ans. 26·272 cubic inches, or near $\frac{43}{47}$ parts of a pint.

Quest. 50. The dimensions of a sphere and cone being the same as in the last question, and the cone only $\frac{1}{4}$ full of water; required what part of the axis of the sphere is immersed in the water? Ans. ·546 parts of an inch.

Quest. 51. The cone being still the same, and $\frac{1}{2}$ full of water; required the diameter of a sphere which shall be just all covered by the water? Ans. 2·445996 inches.

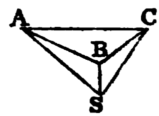
Quest. 52. If a person, with an air balloon, ascend vertically from London, to such height that he can just see Oxford appear in the horizon; it is required to determine his height above the earth, supposing its circumference to be 25000 miles, and the distance between London and Oxford 49·5933 miles?

Ans. $\frac{311}{1000}$ of a mile, or 547 yards 1 foot.

Quest. 53. In a garrison there are three remarkable objects, A, B, C, the distances of which from one to another are known to be, AB 213, AC 424, and BC 262 yards; I am desirous of knowing my position and distance at a place or station S, from whence I observed the angle ASB 13° 30′, and the angle CSB 29° 50′, both by geometry and trigonometry.

Answer.

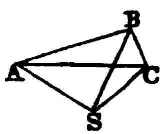
AS	605·7122,
BS	429·6814,
CS	524·2365.



Quest. 54. Required the same as in the last question, when the point B is on the other side of AC, supposing AB 9, AC 12, and BC 6 furlongs; also the angle ASB 33° 45′, and the angle BSC 22° 30′.

Answer.

AS	10·64,
BS	15·64,
CS	14·01.



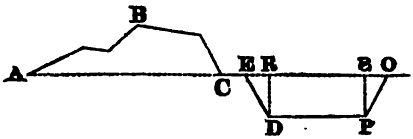
Quest. 55. It is required to determine the magnitude of a cube of gold, of the standard fineness, which shall be equal to a sum of 960 millions pounds sterling; supposing a guinea to weigh 5 dwts $9\frac{1}{2}$ grains. Ans. 23·549 feet.

Quest. 56. The ditch of a fortification is 1000 feet long, 9 feet deep, 29 feet broad at bottom, and 22 at top; how much water will fill the ditch?

Ans. 1158127 gallons nearly.

QUEST. 57. If the diameter of the earth be 7930 miles, and that of the moon 2160 miles: required the ratio of their surfaces, and also of their solidities: supposing them both to be globular, as they are very nearly?

Ans. the surfaces are as $13\frac{1}{2}$ to 1 nearly; and the solidities as $49\frac{1}{2}$ to 1 nearly.



QUEST. 58. Let ABC be the profile, or perpendicular section of a breast-work, and EP that of a ditch. Now, suppose the area of the section ABC is 88 feet, the depth of the ditch RD 6 feet, ER = SO = 3 feet; what is the breadth of the ditch at top when the sections of the ditch and the breast-work are equal; that is, when the earth thrown out of the ditch is sufficient to make the breast-work?

LOGARITHMS

OF THE

NUMBERS

FROM

1 to 1000.

N.	Log.	N.	Log.	N.	Log.	N.	Log.
1	0.000000	26	1·414973	51	1·707570	76	1.880814
2	0·301030	27	1·431364	52	1·716003	77	1·886491
3	0·477121	28	1·447158	53	1·724276	78	1·892095
4	0·602060	29	1·462398	54	1·732394	79	1·897627
5	0·698970	30	1·477121	55	1·740363	80	1·903090
6	0·778151	31	1·491362	56	1·748188	81	1·908485
7	0·845098	32	1·505150	57	1·755875	82	1·913814
8	0·903090	33	1·518514	58	1·763428	83	1·919078
9	0·954243	34	1·531479	59	1·770852	84	1·924279
10	1·000000	35	1·544068	60	1·778151	85	1·929419
11	1·041393	36	1·556303	61	1·785330	86	1·934498
12	1·079181	37	1·568202	62	1·792392	87	1·939519
13	1·113943	38	1·579784	63	1·799341	88	1·944483
14	1·146128	39	1·591065	64	1·806180	89	1·949390
15	1·176091	40	1·602060	65	1·812913	90	1·954243
16	1·204120	41	1·612784	66	1·819544	91	1·959041
17	1·230449	42	1·623249	67	1·826075	92	1·963788
18	1·255273	43	1·633468	68	1·832509	93	1·968483
19	1·278754	44	1·643453	69	1·838849	94	1·973128
20	1·301030	45	1·653213	70	1·845098	95	1·977724
21	1·322219	46	1·662758	71	1·851258	96	1·982271
22	1·342423	47	1·672098	72	1·857333	97	1·986772
23	1·361728	48	1·681241	73	1·863323	98	1·991226
24	1·380211	49	1·690196	74	1·869232	99	1·995635
25	1·397940	50	1·698970	75	1·875061	100	2·000000

N. B. In the following table, in the last nine columns of each page, where the first or leading figures change from 9's to 0's, points or dots are now introduced instead of the 0's through the rest of the line, to catch the eye, and to indicate that from thence the corresponding natural number in the first column stands in the next lower line, and its annexed first two figures of the Logarithm in the second column.

N.	0	1	2	3	4	5	6	7	8	9
100	000000	0434	0868	1301	1734	2166	2598	3029	3461	3891
101	4321	4751	5181	5609	6038	6466	6894	7321	7748	8174
102	8600	9026	9451	9876	.300	.724	1147	1570	1993	2415
103	012837	3259	3680	4100	4521	4940	5360	5779	6197	6616
104	7033	7451	7868	8284	8700	9116	9532	9947	.361	.775
105	021189	1603	2016	2428	2841	3252	3664	4075	4486	4896
106	5306	5715	6125	6533	6942	7350	7757	8164	8571	8978
107	9384	9789	.195	.600	1004	1408	1812	2216	2619	3021
108	033424	3826	4227	4628	5029	5430	5830	6230	6629	7028
109	7426	7825	8223	8620	9017	9414	9811	.207	.602	.998
110	041393	1787	2182	2576	2969	3362	3755	4148	4540	4932
111	5323	5714	6105	6495	6885	7275	7664	8053	8442	8830
112	9218	9606	9993	.380	.766	1153	1538	1924	2309	2694
113	053078	3463	3846	4230	4613	4996	5378	5760	6142	6524
114	6905	7286	7666	8046	8426	8805	9185	9563	9942	.320
115	060698	1075	1452	1829	2206	2582	2958	3333	3709	4083
116	4458	4832	5206	5580	5953	6326	6699	7071	7443	7815
117	8186	8557	8928	9298	9668	..38	.407	.776	1145	1514
118	071882	2250	2617	2985	3352	3718	4085	4451	4816	5182
119	5547	5912	6276	6640	7004	7368	7731	8094	8457	8819
120	9181	9543	9904	.266	.626	.987	1347	1707	2067	2426
121	082785	3144	3503	3861	4219	4576	4934	5291	5647	6004
122	6360	6716	7071	7426	7781	8136	8490	8845	9198	9552
123	9905	.258	.611	.963	1315	1667	2018	2370	2721	3071
124	093422	3772	4122	4471	4820	5169	5518	5866	6215	6562
125	6910	7257	7604	7951	8298	8644	8990	9335	9681	.026
126	100371	0715	1059	1403	1747	2091	2434	2777	3119	3462
127	3804	4146	4487	4828	5169	5510	5851	6191	6531	6871
128	7210	7549	7888	8227	8565	8903	9241	9579	9916	.253
129	110590	0926	1263	1599	1934	2270	2605	2940	3275	3609
130	3943	4277	4611	4944	5278	5611	5943	6276	6608	6940
131	7271	7603	7934	8265	8595	8926	9256	9586	9915	.245
132	120574	0903	1231	1560	1888	2216	2544	2871	3198	3525
133	3852	4178	4504	4830	5156	5481	5806	6131	6456	6781
134	7105	7429	7753	8076	8399	8722	9045	9368	9690	..12
135	130334	0655	0977	1298	1619	1939	2260	2580	2900	3219
136	3539	3858	4177	4496	4814	5133	5451	5769	6086	6403
137	6721	7037	7354	7671	7987	8303	8618	8934	9249	9564
138	9879	.194	.508	.822	1136	1450	1763	2076	2389	2702
139	143015	3327	3639	3951	4263	4574	4885	5196	5507	5818
140	6128	6438	6748	7058	7367	7676	7985	8294	8603	8911
141	9219	9527	9835	.142	.449	.756	1063	1370	1676	1982
142	152288	2594	2900	3205	3510	3815	4120	4424	4728	5032
143	5336	5640	5943	6246	6549	6852	7154	7457	7759	8061
144	8362	8664	8965	9266	9567	9868	.168	.469	.769	1068
145	161368	1667	1967	2266	2564	2863	3161	3460	3758	4055
146	4353	4650	4947	5244	5541	5838	6134	6430	6726	7022
147	7317	7613	7908	8203	8497	8792	9086	9380	9674	9968
148	170262	0555	0848	1141	1434	1726	2019	2311	2603	2895
149	3186	3478	3769	4060	4351	4641	4932	5222	5512	5802

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150	176091	6381	6670	6959	7248	7536	7825	8113	8401	8689
151	8977	9264	9552	9839	.126	.413	.699	.986	1272	1558
152	181844	2129	2415	2700	2985	3270	3555	3839	4123	4407
153	4691	4975	5259	5542	5825	6108	6391	6674	6956	7239
154	7521	7803	8084	8366	8647	8928	9209	9490	9771	..51
155	190332	0612	0892	1171	1451	1730	2010	2289	2567	2846
156	3125	3403	3681	3959	4237	4514	4792	5069	5346	5623
157	5899	6176	6453	6729	7005	7281	7556	7832	8107	8382
158	8657	8932	9206	9481	9755	..29	.303	.577	.850	1124
159	201397	1670	1948	2216	2488	2761	3033	3305	3577	3848
160	4120	4391	4663	4934	5204	5475	5746	6016	6286	6556
161	6826	7096	7365	7634	7904	8173	8441	8710	8979	9247
162	9515	9783	..51	.319	.586	.853	1121	1388	1654	1921
163	212188	2454	2720	2986	3252	3518	3783	4049	4314	4579
164	4844	5109	5373	5638	5902	6166	6430	6694	6957	7221
165	7484	7747	8010	8273	8536	8798	9060	9323	9585	9846
166	220108	0370	0631	0892	1153	1414	1675	1936	2196	2456
167	2716	2976	3236	3496	3755	4015	4274	4533	4792	5051
168	5309	5568	5826	6084	6342	[illegible]	6858	7115	7372	7630
169	7887	8144	8400	8657	8913	[illegible]	9426	9682	9938	.193
170	230449	0704	0960	1215	1470	1724	1979	2234	2488	2742
171	2996	3250	3504	3757	4011	4264	4517	4770	5023	5276
172	5528	5781	6033	6285	6537	6789	7041	7292	7544	7795
173	8046	8297	8548	8799	9049	9299	9550	9800	..50	.300
174	240549	0799	1048	1297	1546	1795	2044	2293	2541	2790
175	3038	3286	3534	3782	4030	4277	4525	4772	5019	5266
176	5513	5759	6006	6252	6499	6745	6991	7237	7482	7728
177	7973	8219	8464	8709	8954	9198	9443	9687	9932	.176
178	250420	0664	0908	1151	1395	1638	1881	2125	2368	2610
179	2853	3096	3338	3580	3822	4064	4306	4548	4790	5031
180	5273	5514	5755	5996	6237	6477	6718	6958	7198	7439
181	7679	7918	8158	8398	8637	8877	9116	9355	9594	9833
182	260071	0310	0548	0787	1025	1263	1501	1739	1976	2214
183	2451	2688	2925	3162	3399	3636	3873	4109	4346	4582
184	4818	5054	5290	5525	5761	5996	6232	6467	6702	6937
185	7172	7406	7641	7875	8110	8344	8580	8812	9046	9279
186	9513	9746	9980	.213	.446	.679	.912	1144	1377	1609
187	271842	2074	2306	2538	2770	3001	3233	3464	3696	3927
188	4158	4389	4620	4850	5081	5311	5542	5772	6002	6232
189	6462	6692	6921	7151	7380	7609	7838	8067	8296	8525
190	8754	8982	9211	9439	9667	9895	.123	.351	.578	.806
191	281033	1261	1488	1715	1942	2169	2396	2622	2849	3075
192	3301	3527	3753	3979	4205	4431	4656	4882	5107	5332
193	5557	5782	6007	6232	6456	6681	6905	7130	7354	7578
194	7802	8026	8249	8473	8696	8920	9143	9366	9589	9812
195	290035	0257	0480	0702	0925	1147	1369	1591	1813	2034
196	2256	2478	2699	2920	3141	3363	3584	3804	4025	4246
197	4466	4687	4907	5127	5347	5567	5787	6007	6226	6446
198	6665	6884	7104	7323	7542	7761	7979	8198	8416	8635
199	8853	9071	9289	9507	9725	9943	.161	.378	.595	.813

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200	301030	1247	1464	1681	1898	2114	2331	2547	2764	2980
201	3196	3412	3628	3844	4059	4275	4491	4706	4921	5136
202	5351	5566	5781	5996	6211	6425	6639	6854	7068	7282
203	7496	7710	7924	8137	8351	8564	8778	8991	9204	9417
204	9630	9843	..56	.268	.481	.693	.906	1118	1330	1542
205	311754	1966	2177	2389	2600	2812	3023	3234	3445	3656
206	3867	4078	4289	4499	4710	4920	5130	5340	5551	5760
207	5970	6180	6390	6599	6809	7018	7227	7436	7646	7854
208	8063	8272	8481	8689	8898	9106	9314	9522	9730	9938
209	320146	0354	0562	0769	0977	1184	1391	1598	1805	2012
210	2219	2426	2633	2839	3046	3252	3458	3665	3871	4077
211	4282	4488	4694	4899	5105	5310	5516	5721	5926	6131
212	6336	6541	6745	6950	7155	7359	7563	7767	7972	8176
213	8380	8583	8787	8991	9194	9398	9601	9805	...8	.211
214	330414	0617	0819	1022	1225	1427	1630	1832	2034	2236
215	2438	2640	2842	3044	3246	3447	3649	3850	4051	4253
216	4454	4655	4856	5057	5257	5458	5658	5859	6059	6260
217	6460	6660	6860	7060	7260	7459	7659	7858	8058	8257
218	8456	8656	8855	9054	9253	9451	9650	9849	..47	.246
219	340444	0642	0841	1[illegible]9	1237	1435	1632	1830	2028	2225
220	2423	2620	2817	3014	3212	3409	3606	3802	3999	4196
221	4392	4589	4785	4981	5178	5374	5570	5766	5962	6157
222	6353	6549	6744	6939	7135	7330	7525	7720	7915	8110
223	8305	8500	8694	8889	9083	9278	9472	9666	9860	..54
224	350248	0442	0636	0829	1023	1216	1410	1603	1796	1989
225	2183	2375	2568	2761	2954	3147	3339	3532	3724	3916
226	4108	4301	4493	4685	4876	5068	5268	5452	5643	5834
227	6026	6217	6408	6599	6790	6981	7172	7363	7554	7744
228	7935	8125	8316	8506	8696	8886	9076	9266	9456	9646
229	9835	..25	.215	.404	.593	.783	.972	1161	1350	1539
230	361728	1017	2105	2294	2482	2671	2859	3048	3236	3424
231	3612	3800	3988	4176	4363	4551	4739	4926	5113	5301
232	5488	5675	5862	6049	6236	6423	6610	6796	6983	7169
233	7356	7542	7729	7915	8101	8287	8473	8659	8845	9030
234	9216	9401	9587	9772	9958	.143	.328	.513	.698	.883
235	371068	1253	1437	1622	1806	1991	2175	2360	2544	2728
236	2912	3096	3280	3464	3647	3831	4015	4198	4382	4565
237	4748	4932	5115	5298	5481	5664	5846	6029	6212	6394
238	6577	6759	6942	7124	7306	7488	7670	7852	8034	8216
239	8398	8580	8761	8943	9124	9306	9487	9668	9849	..30
240	380211	0392	0573	0754	0934	1115	1296	1476	1656	1837
241	2017	2197	2377	2557	2737	2917	3097	3277	3456	3636
242	3815	3995	4174	4353	4533	4712	4891	5070	5249	5428
243	5606	5785	5964	6142	6321	6499	6677	6856	7034	7212
244	7390	7568	7746	7023	8101	8279	8456	8634	8811	8989
245	9166	9343	9520	9698	9875	..51	.228	.405	.582	.759
246	390935	1112	1288	1464	1641	1817	1993	2169	2345	2521
247	2697	2873	3048	3224	3400	3575	3751	3926	4101	4277
248	4452	4627	4802	4977	5152	5326	5501	5676	5850	6025
249	6199	6374	6548	6722	6896	7071	7245	7419	7592	7766

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250	397940	8114	8287	8461	8634	8808	8981	9154	9328	9501
251	9674	9847	..20	.192	.365	.538	.711	.883	1056	1228
252	401401	1573	1745	1917	2089	2261	2433	2605	2777	2949
253	3121	3292	3464	3635	3807	3978	4149	4320	4492	4663
254	4834	5005	5176	5346	5517	5688	5858	6029	6199	6370
255	6540	6710	6881	7051	7221	7391	7561	7731	7901	8070
256	8240	8410	8579	8749	8918	9087	9257	9426	9595	9764
257	9933	.102	.271	.440	.609	.777	.946	1114	1283	1451
258	411620	1788	1956	2124	2293	2461	2629	2796	2964	3132
259	3300	3467	3635	3803	3970	4137	4305	4472	4639	4806
260	4973	5140	5307	5474	5641	5808	5974	6141	6308	6474
261	6641	6807	6973	7139	7306	7472	7638	7804	7970	8135
262	8301	8467	8633	8798	8964	9129	9295	9460	9625	9791
263	9956	.121	.286	.451	.616	.781	.945	1110	1275	1439
264	421604	1768	1933	2097	2261	2426	2590	2754	2918	3082
265	3246	3410	3574	3737	3901	4065	4228	4392	4555	4718
266	4882	5045	5208	5371	5534	5697	5860	6023	6186	6349
267	6511	6674	6836	6999	7161	7324	7486	7648	7811	7973
268	8135	8297	8459	8621	8783	8944	9106	9268	9429	9591
269	9752	9914	..75	.236	.398	.559	.720	.881	1042	1203
270	431364	1525	1685	1846	2007	2167	2328	2488	2649	2809
271	2969	3130	3290	3450	3610	3770	3930	4090	4249	4409
272	4569	4729	4888	5048	5207	5367	5526	5685	5844	6004
273	6163	6322	6481	6640	6800	6957	7116	7275	7433	7592
274	7751	7909	8067	8226	8384	8542	8701	8859	9017	9175
275	9333	9491	9648	9806	9964	.122	.279	.437	.594	.752
276	440909	1066	1224	1381	1538	1695	1852	2009	2166	2323
277	2480	2637	2793	2950	3106	3263	3419	3576	3732	3889
278	4045	4201	4357	4513	4669	4825	4981	5137	5293	5449
279	5604	5760	5915	6071	6226	6382	6537	6692	6848	7003
280	7158	7313	7468	7623	7778	7933	8088	8242	8397	8552
281	8706	8861	9015	9170	9324	9478	9633	9787	9941	..95
282	450249	0403	0557	0711	0865	1018	1172	1326	1479	1633
283	1786	1940	2093	2247	2400	2553	2706	2859	3012	3165
284	3318	3471	3624	3777	3930	4082	4235	4387	4540	4692
285	4845	4997	5150	5302	5454	5606	5758	5910	6062	6214
286	6366	6518	6670	6821	6973	7125	7276	7428	7579	7731
287	7882	8033	8184	8336	8487	8638	8789	8940	9091	9242
288	9392	9543	9694	9845	9995	.146	.296	.447	.597	.748
289	460898	1048	1198	1348	1499	1649	1799	1948	2098	2248
290	2398	2548	2697	2847	2997	3146	3296	3445	3594	3744
291	3893	4042	4191	4340	4490	4639	4788	4936	5085	5234
292	5383	5532	5680	5829	5977	6126	6274	6423	6571	6719
293	6868	7016	7164	7312	7460	7608	7756	7904	8052	8200
294	8347	8495	8643	8790	8938	9085	9233	9380	9527	9675
295	9822	9969	.116	.263	.410	.557	.704	.851	.998	1145
296	471292	1438	1585	1732	1878	2025	2171	2318	2464	2610
297	2756	2903	3049	3195	3341	3487	3633	3779	3925	4071
298	4216	4362	4508	4653	4799	4944	5090	5235	5381	5526
299	5671	5816	5962	6107	6252	6397	6542	6687	6832	6976

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300	477121	7266	7411	7555	7700	7844	7989	8133	8278	8422
301	8566	8711	8855	8999	9143	9287	9431	9575	9719	9863
302	480007	0151	0294	0438	0582	0725	0869	1012	1156	1299
303	1443	1586	1729	1872	2016	2159	2302	2445	2588	2731
304	2874	3016	3159	3302	3445	3587	3730	3872	4015	4157
305	4300	4442	4585	4727	4869	5011	5153	5295	5437	5579
306	5721	5863	6005	6147	6289	6430	6572	6714	6855	6997
307	7138	7280	7421	7563	7704	7845	7986	8127	8269	8410
308	8551	8692	8833	8974	9114	9255	9396	9537	9677	9818
309	9958	..99	.239	.380	.520	.661	.801	.941	1081	1222
310	491362	1502	1642	1782	1922	2062	2201	2341	2481	2621
311	2760	2900	3040	3179	3319	3458	3597	3737	3876	4015
312	4155	4294	4433	4572	4711	4850	4989	5128	5267	5406
313	5544	5683	5822	5960	6099	6238	6376	6515	6653	6791
314	6930	7068	7206	7344	7483	7621	7759	7897	8035	8173
315	8311	8448	8586	8724	8862	8999	9137	9275	9412	9550
316	9687	9824	9962	..99	.236	.374	.511	.648	.785	.922
317	501059	1196	1333	1470	1607	1744	1880	2017	2154	2291
318	2427	2564	2700	2837	2973	3109	3246	3382	3518	3655
319	3791	3927	4063	4199	4335	4471	4607	4743	4878	5014
320	5150	5286	5421	5557	5693	5828	5964	6099	6234	6370
321	6505	6640	6776	6911	7046	7181	7316	7451	7586	7721
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323	9203	9337	9471	9606	9740	9874	...9	.143	.277	.411
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327	4548	4681	4813	4946	5079	5211	5344	5476	5609	5741
328	5874	6006	6139	6271	6403	6535	6668	6800	6932	7064
329	7196	7328	7460	7592	7724	7855	7987	8119	8251	8382
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331	9828	9959	..90	.221	.353	.484	.615	.745	.876	1007
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334	3746	3876	4006	4136	4266	4396	4526	4656	4785	4915
335	5045	5174	5304	5434	5563	5693	5822	5951	6081	6210
336	6339	6469	6598	6727	6856	6985	7114	7243	7372	7501
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339	530200	0328	0456	0584	0712	0840	0968	1096	1223	1351
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352	6543	6666	6789	6913	7036	7159	7282	7405	7529	7652
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358	3883	4004	4126	4247	4368	4489	4610	4731	4852	4973
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361	7507	7627	7748	7868	7988	8108	8228	8349	8469	8589
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366	3481	3600	3718	3837	3955	4074	4192	4311	4429	4548
367	4666	4784	4903	5021	5139	5257	5376	5494	5612	5730
368	5848	5966	6084	6202	6320	6437	6555	6673	6791	6909
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383	3199	3312	3426	3539	3652	3765	3879	3992	4105	4218
384	4331	4444	4557	4670	4783	4896	5009	5122	5235	5348
385	5461	5574	5686	5799	5912	6024	6137	6250	6362	6475
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387	7711	7823	7935	8047	8160	8272	8384	8496	8608	8720
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389	9950	..61	.173	.284	.396	.507	.619	.730	.842	.953
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392	3286	3397	3508	3618	3729	3840	3950	4061	4171	4282
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394	5496	5606	5717	5827	5937	6047	6157	6267	6377	6487
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396	7695	7805	7914	8024	8134	8243	8353	8462	8572	8681
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398	9883	9992	.101	.210	.319	.428	.537	.646	.755	.864
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403	5305	5413	5521	5628	5736	5844	5951	6059	6166	6274
404	6381	6489	6596	6704	6811	6919	7026	7133	7241	7348
405	7455	7562	7669	7777	7884	7991	8098	8205	8312	8419
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411	3842	3947	4053	4159	4264	4370	4475	4581	4686	4792
412	4897	5003	5108	5213	5319	5424	5529	5634	5740	5845
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414	7000	7105	7210	7315	7420	7525	7629	7734	7839	7943
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419	2214	2318	2421	2525	2628	2732	2835	2939	3042	3146
420	3249	3353	3456	3559	3663	3766	3869	3973	4076	4179
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424	7366	7468	7571	7673	7775	7878	7980	8082	8185	8287
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431	4477	4578	4679	4779	4880	4981	5081	5182	5283	5383
432	5484	5584	5685	5785	5886	5986	6087	6187	6287	6388
433	6488	6588	6688	6789	6889	6989	7089	7189	7290	7390
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443	6404	6502	6600	6698	6796	6894	6992	7089	7187	7285
444	7383	7481	7579	7676	7774	7872	7969	8067	8165	8262
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452	5138	5235	5331	5427	5526	5619	5715	5810	5906	6002
453	6098	6194	6290	6386	6482	6577	6673	6769	6864	6960
454	7056	7152	7247	7343	7438	7534	7629	7725	7820	7916
455	8011	8107	8202	8298	8393	8488	8584	8679	8774	8870
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472	3942	4034	4126	4218	4310	4402	4494	4586	4677	4769
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481	2145	2235	2326	2416	2506	2596	2686	2777	2867	2957
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487	7529	7618	7707	7796	7886	7975	8064	8153	8242	8331
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494	3727	3815	3903	3991	4078	4166	4254	4342	4430	4517
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497	6356	6444	6531	6618	6706	6793	6880	6968	7055	7142
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505	3291	3377	3463	3549	3635	3721	3807	3893	3979	4065
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507	5008	5094	5179	5265	5350	5436	5522	5607	5693	5778
508	5864	5949	6035	6120	6206	6291	6376	6462	6547	6632
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527	1811	1893	1975	2058	2140	2222	2305	2387	2469	2552
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535	8354	8435	8516	8597	8678	8759	8841	8922	9003	9084
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539	1589	1669	1750	1830	1911	1991	2072	2152	2233	2313
540	2394	2474	2555	2635	2715	2796	2876	2956	3037	3117
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543	4800	4880	4960	5040	5120	5200	5279	5359	5439	5519
544	5599	5679	5759	5838	5918	5998	6078	6157	6237	6317
545	6397	6476	6556	6635	6715	6795	6874	6954	7034	7113
546	7193	7272	7352	7431	7511	7590	7670	7749	7829	7908
547	7987	8067	8146	8225	8305	8384	8463	8543	8622	8701
548	8781	8860	8939	9018	9097	9177	9256	9335	9414	9493
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552	1939	2018	2096	2175	2254	2332	2411	2489	2568	2646
553	2725	2804	2882	2961	3039	3118	3196	3275	3353	3431
554	3510	3588	3667	3745	3823	3902	3980	4058	4136	4215
555	4293	4371	4449	4528	4606	4684	4762	4840	4919	4997
556	5075	5153	5231	5309	5387	5465	5543	5621	5699	5777
557	5855	5933	6011	6089	6167	6245	6323	6401	6479	6556
558	6634	6712	6790	6868	6945	7023	7101	7179	7256	7334
559	7412	7489	7567	7645	7722	7800	7878	7955	8033	8110
560	8188	8266	8343	8421	8498	8576	8653	8731	8808	8885
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565	2048	2125	2202	2279	2356	2433	2509	2586	2663	2740
566	2816	2893	2970	3047	3123	3200	3277	3353	3430	3506
567	3583	3660	3736	3813	3889	3966	4042	4119	4195	4272
568	4348	4425	4501	4578	4654	4730	4807	4883	4960	5036
569	5112	5189	5265	5341	5417	5494	5570	5646	5722	5799
570	5875	5951	6027	6103	6180	6256	6332	6408	6484	6560
571	6636	6712	6788	6864	6940	7016	7092	7168	7244	7320
572	7396	7472	7548	7624	7700	7775	7851	7927	8003	8079
573	8155	8230	8306	8382	8458	8533	8609	8685	8761	8836
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580	3428	3503	3578	3653	3727	3802	3877	3952	4027	4101
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585	7156	7230	7304	7379	7453	7527	7601	7675	7749	7823
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670	6075	6140	6204	6269	6334	6399	6464	6528	6593	6658
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694	1359	1422	1485	1547	1610	1672	1735	1797	1860	1922
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719	6729	6789	6850	6910	6970	7031	7091	7152	7212	7272
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768	5361	5418	5474	5531	5587	5644	5700	5757	5813	5870
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770	6491	6547	6604	6660	6716	6773	6829	6885	6942	6998
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792	8725	8780	8835	8890	8944	8999	9054	9109	9164	9218
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797	1453	1513	1567	1622	1676	1731	1785	1840	1894	1948
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802	4174	4229	4283	4337	4391	4445	4499	4553	4607	4661
803	4716	4770	4824	4878	4932	4986	5040	5094	5148	5202
804	5256	5310	5364	5418	5472	5526	5580	5634	5688	5742
805	5796	5850	5904	5958	6012	6066	6119	6173	6227	6281
806	6335	6389	6443	6497	6551	6604	6658	6712	6766	6820
807	6874	6927	6981	7035	7089	7143	7196	7250	7304	7358
808	7411	7465	7519	7573	7626	7680	7734	7787	7841	7895
809	7949	8002	8056	8110	8163	8217	8270	8324	8378	8431
810	8485	8539	8592	8646	8699	8753	8807	8860	8914	8967
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812	9556	9610	9663	9716	9770	9823	9877	9930	9984	..37
813	910091	0144	0197	0251	0304	0358	0411	0464	0518	0571
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815	1158	1211	1264	1317	1371	1424	1477	1530	1584	1637
816	1690	1743	1797	1850	1903	1956	2009	2063	2116	2169
817	2222	2275	2328	2381	2435	2488	2541	2594	2647	2700
818	2753	2806	2859	2913	2966	3019	3072	3125	3178	3231
819	3284	3337	3390	3443	3496	3549	3602	3655	3708	3761
820	3814	3867	3920	3973	4026	4079	4132	4184	4237	4290
821	4343	4396	4449	4502	4555	4608	4660	4713	4766	4819
822	4872	4925	4977	5030	5083	5136	5189	5241	5294	5347
823	5400	5453	5505	5558	5611	5664	5716	5769	5822	5875
824	5927	5980	6033	6085	6138	6191	6243	6296	6349	6401
825	6454	6507	6559	6612	6664	6717	6770	6822	6875	6927
826	6980	7033	7085	7138	7190	7243	7295	7348	7400	7453
827	7506	7558	7611	7663	7716	7768	7820	7873	7925	7978
828	8030	8083	8135	8188	8240	8293	8345	8397	8450	8502
829	8555	8607	8659	8712	8764	8816	8869	8921	8973	9026
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834	1166	1218	1270	1322	1374	1426	1478	1530	1582	1634
835	1686	1738	1790	1842	1894	1946	1998	2050	2102	2154
836	2206	2258	2310	2362	2414	2466	2518	2570	2622	2674
837	2725	2777	2829	2881	2933	2985	3037	3089	3140	3192
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840	4279	4331	4383	4434	4486	4538	4589	4641	4693	4744
841	4796	4848	4899	4951	5003	5054	5106	5157	5209	5261
842	5312	5364	5415	5467	5518	5570	5621	5673	5725	5776
843	5828	5879	5931	5982	6034	6085	6137	6188	6240	6291
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845	6857	6908	6959	7011	7062	7114	7165	7216	7268	7319
846	7370	7422	7473	7524	7576	7627	7678	7730	7781	7832
847	7883	7935	7986	8037	8088	8140	8191	8242	8293	8345
848	8396	8447	8498	8549	8601	8652	8703	8754	8805	8857
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854	1458	1509	1560	1610	1661	1712	1763	1814	1865	1915
855	1966	2017	2068	2118	2169	2220	2271	2322	2372	2423
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857	2981	3031	3082	3133	3183	3234	3285	3335	3386	3437
858	3487	3538	3589	3639	3690	3740	3791	3841	3892	3943
859	3993	4044	4094	4145	4195	4246	4296	4347	4394	4448
860	4498	4549	4599	4650	4700	4751	4801	4852	4902	4953
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862	5507	5558	5608	5658	5709	5759	5809	5860	5910	5960
863	6011	6061	6111	6162	6212	6262	6313	6363	6413	6463
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871	940018	0068	0118	0168	0218	0267	0317	0367	0417	0467
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873	1014	1064	1114	1163	1213	1263	1313	1362	1412	1462
874	1511	1561	1611	1660	1710	1760	1809	1859	1909	1958
875	2008	2058	2107	2157	2207	2256	2306	2355	2405	2455
876	2504	2554	2603	2653	2702	2752	2801	2851	2901	2950
877	3000	3049	3099	3148	3198	3247	3297	3346	3396	3445
878	3495	3544	3593	3643	3692	3742	3791	3841	3890	3939
879	3989	4038	4088	4137	4186	4236	4285	4335	4384	4433
880	4483	4532	4581	4631	4680	4729	4779	4828	4877	4927
881	4976	5025	5074	5124	5173	5222	5272	5321	5370	5419
882	5469	5518	5567	5616	5665	5715	5764	5813	5862	5912
883	5961	6010	6059	6108	6157	6207	6256	6305	6354	6403
884	6452	6501	6551	6600	6649	6698	6747	6796	6845	6894
885	6943	6992	7041	7090	7140	7189	7238	7287	7336	7385
886	7434	7483	7532	7581	7630	7679	7728	7777	7826	7875
887	7924	7973	8022	8070	8119	8168	8217	8266	8315	8364
888	8413	8462	8511	8560	8609	8657	8706	8755	8804	8853
889	8902	8951	8999	9048	9097	9146	9195	9244	9292	9341
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892	950365	0414	0462	0511	0560	0608	0657	0706	0754	0803
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894	1338	1386	1435	1483	1532	1580	1629	1677	1726	1775
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896	2308	2356	2405	2453	2502	2550	2599	2647	2696	2744
897	2792	1841	2889	2938	2986	3034	3083	3131	3180	3228
898	3276	3325	3373	3421	3470	3518	3566	3615	3663	3711
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902	5207	5255	5303	5351	5399	5447	5495	5543	5592	5640
903	5688	5736	5784	5832	5880	5928	5976	6024	6072	6120
904	6168	6216	6265	6313	6361	6409	6457	6505	6553	6601
905	6649	6697	6745	6793	6840	6888	6936	6984	7032	7080
906	7128	7176	7224	7272	7320	7368	7416	7464	7512	7559
907	7607	7655	7703	7751	7799	7847	7894	7942	7990	8038
908	8086	8134	8181	8229	8277	8325	8373	8421	8468	8516
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912	9995	..42	..90	.138	.185	.233	.280	.328	.376	.423
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915	1421	1469	1516	1563	1611	1658	1706	1753	1801	1848
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917	2369	2417	2464	2511	2559	2606	2653	2701	2748	2795
918	2843	2890	2937	2985	3032	3079	3126	3174	3221	3268
919	3316	3363	3410	3457	3504	3552	3599	3646	3693	3741
920	3788	3835	3882	3929	3977	4024	4071	4118	4165	4212
921	4260	4307	4354	4401	4448	4495	4542	4590	4637	4684
922	4731	4778	4825	4872	4919	4966	5013	5061	5108	5155
923	5202	5249	5296	5343	5390	5437	5484	5531	5578	5625
924	5672	5719	5766	5813	5860	5907	5954	6001	6048	6095
925	6142	6189	6236	6283	6329	6376	6423	6470	6517	6564
926	6611	6658	6705	6752	6799	6845	6892	6939	6986	7033
927	7080	7127	7173	7220	7267	7314	7361	7408	7454	7501
928	7548	7595	7642	7688	7735	7782	7829	7875	7922	7969
929	8016	8062	8109	8156	8203	8249	8296	8343	8390	8436
930	8483	8530	8576	8623	8670	8716	8763	8810	8856	8903
931	8950	8996	9043	9090	9136	9183	9229	9276	9323	9369
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933	9882	9928	9975	..21	..68	.114	.161	.207	.254	.300
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935	0812	0858	0904	0951	0997	1044	1090	1137	1183	1229
936	1276	1322	1369	1415	1461	1508	1554	1601	1647	1693
937	1740	1786	1832	1879	1925	1971	2018	2064	2110	2157
938	2203	2249	2295	2342	2388	2434	2481	2527	2573	2619
939	2666	2712	2758	2804	2851	2897	2943	2989	3035	3082
940	3128	3174	3220	3266	3313	3359	3405	3451	3497	3543
941	3590	3636	3682	3728	3774	3820	3866	3913	3959	4005
942	4051	4097	4143	4189	4235	4281	4327	4374	4420	4466
943	4512	4558	4604	4650	4696	4742	4788	4834	4880	4926
944	4972	5018	5064	5110	5156	5202	5248	5294	5340	5386
945	5432	5478	5524	5570	5616	5662	5707	5753	5799	5845
946	5891	5937	5983	6029	6075	6121	6167	6212	6258	6304
947	6350	6396	6442	6488	6533	6579	6625	6671	6717	6763
948	6808	6854	6900	6946	6992	7037	7083	7129	7175	7220
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922	8637	8683	8728	8774	8819	8865	8911	8956	9002	9047
953	9093	9138	9184	9230	9275	9321	9366	9412	9457	9503
954	9548	9594	9639	9685	9730	9776	9821	9867	9912	9958
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956	0458	0503	0549	0594	0640	0685	0730	0776	0821	0867
957	0912	0957	1003	1048	1093	1139	1184	1229	1275	1320
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959	1819	1864	1909	1954	2000	2045	2090	2135	2181	2226
960	2271	2316	2362	2407	2452	2497	2543	2588	2633	2678
961	2723	2769	2814	2859	2904	2949	2994	3040	3085	3130
962	3175	3220	3265	3310	3356	3401	3446	3491	3536	3581
963	3626	3671	3716	3762	3807	3852	3897	3942	3987	4032
964	4077	4122	4167	4212	4257	4302	4347	4392	4437	4482
965	4527	4572	4617	4662	4707	4752	4797	4842	4887	4932
966	4977	5022	5067	5112	5157	5202	5247	5292	5337	5382
967	5426	5471	5516	5561	5606	5651	5699	5741	5786	5830
968	5875	5920	5965	6010	6055	6100	6144	6189	6234	6279
969	6324	6369	6413	6458	6503	6548	6593	6637	6682	6727
970	6772	6817	6861	6906	6951	6996	7040	7085	7130	7175
971	7219	7264	7309	7353	7398	7443	7488	7532	7577	7622
972	7666	7711	7756	7800	7845	7890	7934	7979	8024	8068
973	8113	8157	8202	8247	8291	8336	8381	8425	8470	8514
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975	9005	9049	9049	9138	9183	9227	9272	9316	9361	9405
976	9450	9494	9539	9583	9628	9672	9717	9761	9806	9850
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978	990339	0383	0428	0472	0516	0561	0605	0650	0694	0738
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980	1226	1270	1315	1359	1403	1448	1492	1536	1580	1625
981	1669	1713	1758	1802	1846	1890	1935	1979	2023	2067
982	2111	2156	2200	2244	2288	2333	2377	2421	2465	2509
983	2554	2598	2642	2686	2730	2774	2819	2863	2907	2951
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985	3436	3480	3524	3568	3613	3657	3701	3745	3789	3833
986	3877	3921	3965	4009	4053	4097	4141	4185	4229	4273
987	4317	4361	4405	4449	4493	4337	4581	4625	4669	4713
988	4757	4801	4845	4889	4933	4977	5021	5065	5108	5152
989	5196	5240	5284	5328	5372	5416	5460	5504	5547	5591
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991	6074	6117	6161	6205	6249	6293	6337	6380	6424	6468
992	6512	6555	6599	6643	6687	6731	6774	6818	6862	6906
993	6949	6993	7037	7080	7124	7168	7212	7255	7299	7343
994	7386	7430	7474	7517	7561	7605	7648	7692	7736	7779
995	7823	7867	7910	7954	7998	8041	8085	8129	8172	8216
996	8259	8303	8347	8390	8434	8477	8521	8564	8608	8652
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998	9131	9174	9218	9261	9305	9348	9392	9435	9479	9522
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	0 Deg.				1 Deg.				
	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	
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3	6.940847	10.000000	6.940847	13.059153	8.263042	9.999927	8.263115	11.736885	57
4	7.065786	10.000000	7.065786	12.934214	8.269881	9.999925	8.269956	11.730044	56
5	7.162696	10.000000	7.162696	12.837304	8.276614	9.999922	8.276691	11.723309	55
6	7.241877	9.999999	7.241878	12.758122	8.283243	9.999920	8.283323	11.716677	54
7	7.308824	9.999999	7.308825	12.691175	8.289773	9.999918	8.289856	11.710144	53
8	7.366816	9.999999	7.366817	12.633183	8.296207	9.999915	8.296292	11.703708	52
9	7.417968	9.999999	7.417970	12.582030	8.302546	9.999913	8.302634	11.697366	51
10	7.463726	9.999998	7.463727	12.536273	8.308794	9.999910	8.318884	11.691116	50
11	7.505118	9.999998	7.505120	12.494880	8.314954	9.999907	8.315046	11.684954	49
12	7.542906	9.999997	7.542909	12.457091	8.321027	9.999905	8.321122	11.678878	48
13	7.577668	9.999997	7.577672	12.422328	8.327016	9.999902	8.327114	11.672886	47
14	7.609853	9.999996	7.609857	12.390143	8.332924	9.999899	8.333025	11.666975	46
15	7.639816	9.999996	7.639820	12.361080	8.338755	9.999897	8.338856	11.661144	45
16	7.667845	9.999995	7.667849	12.332151	8.344504	9.999894	8.344610	11.655390	44
17	7.694173	9.999995	7.694179	12.305821	8.350181	9.999891	8.350289	11.649711	43
18	7.718997	9.999994	7.719003	12.280997	8.355783	9.999888	8.355895	11.644105	42
19	7.742478	9.999993	7.742484	12.257516	8.361315	9.999885	8.361430	11.638570	41
20	7.764754	9.999993	7.764761	12.235239	8.366777	9.999882	8.366895	11.633105	40
21	7.785943	9.999992	7.785951	12.214049	8.372171	9.999879	8.372292	11.627708	39
22	7.806146	9.999991	7.806155	12.193845	8.377499	9.999876	8.377622	11.622378	38
23	7.825451	9.999990	7.825460	12.174540	8.382762	9.999873	8.382889	11.617111	37
24	7.843934	9.999989	7.843944	12.156056	8.387962	9.999870	8.388092	11.611908	36
25	7.861662	9.999989	7.861674	12.138326	8.393101	9.999867	8.393234	11.606766	35
26	7.878695	9.999988	7.870708	12.121292	8.398179	9.999864	8.398315	11.601685	34
27	7.895085	9.999987	7.895099	12.104901	8.403199	9.999861	8.403338	11.596662	33
28	7.910879	9.999986	7.910894	12.089106	8.408161	9.999858	8.408304	11.591696	32
29	7.926119	9.999985	7.926134	12.073866	8.413068	9.999854	8.413213	11.586787	31
30	7.940842	9.999983	7.940858	12.059142	8.417919	9.999851	8.418068	11.581932	30
31	7.955082	9.999982	7.955100	12.044900	8.422717	9.999848	8.422869	11.577131	29
32	7.968870	9.999981	7.968889	12.031111	8.427462	9.999845	8.427618	11.572382	28
33	7.982233	9.999980	7.982253	12.017747	8.432156	9.999841	8.432315	11.567685	27
34	7.995198	9.999979	7.995219	12.004881	8.436800	9.999838	8.436962	11.563038	26
35	8.007787	9.999977	8.007809	11.992191	8.441394	9.999834	8.441560	11.558440	25
36	8.020021	9.999976	8.020044	11.979956	8.445941	9.999831	8.446110	11.553890	24
37	8.031919	9.999975	8.031945	11.968055	8.450440	9.999827	8.450613	11.549387	23
38	8.043501	9.999973	8.043527	11.956473	8.454893	9.999824	8.455070	11.544930	22
39	8.054781	9.999972	8.054809	11.945191	8.459301	9.999820	8.459481	11.540519	21
40	8.065776	9.999971	8.065806	11.934194	8.463665	9.999816	8.463849	11.536151	20
41	8.076500	9.999969	8.076531	11.923469	8.467985	9.999813	8.468172	11.531828	19
42	8.086965	9.999968	8.086997	11.913003	8.472263	9.999809	8.472454	11.527546	18
43	8.097183	9.999966	8.097217	11.902783	8.476498	9.999805	8.476693	11.523307	17
44	8.107167	9.999964	8.107203	11.892797	8.480693	9.999801	8.480892	11.519108	16
45	8.116926	9.999963	8.116963	11.883037	8.484848	9.999797	8.485050	11.514950	15
46	8.126471	9.999961	8.126510	11.873490	8.488963	9.999794	8.489170	11.510830	14
47	8.135810	9.999959	8.135851	11.864149	8.493040	9.999790	8.493250	11.506750	13
48	8.144953	9.999958	8.144996	11.855004	8.497078	9.999786	8.497293	11.502707	12
49	8.153907	9.999956	8.153952	11.846048	8.501080	9.999782	8.501298	11.498702	11
50	8.162681	9.999954	8.162727	11.837273	8.505045	9.999778	8.505267	11.494733	10
51	8.171280	9.999952	8.171328	11.828672	8.508974	9.999774	8.509200	11.490800	9
52	8.179713	9.999950	8.179763	11.820237	8.512867	9.999769	8.513098	11.486902	8
53	8.187985	9.999948	8.188036	11.811964	8.516726	9.999765	8.516961	11.483039	7
54	8.196102	9.999946	8.196156	11.803844	8.520551	9.999761	8.520790	11.479210	6
55	8.204070	9.999944	8.204126	11.765874	8.524343	9.999757	8.524586	11.475414	5
56	8.211895	9.999942	8.211953	11.788047	8.528102	9.999753	8.528349	11.471651	4
57	8.219581	9.999940	8.219641	11.780359	8.531828	9.999748	8.532080	11.467920	3
58	8.227134	9.999938	8.227195	11.772805	8.535523	9.999744	8.535779	11.464221	2
59	8.234557	9.999936	8.234621	11.765379	8.539186	9.999740	8.539447	11.460553	1
60	8.241855	9.999934	8.241921	11.758079	8.542819	9.999735	8.543084	11.456916	0
	Cosine.	Sine.	Cotan.	Tang.	Cosine.	Sine.	Cotan.	Tang.	
	89 Deg.				88 Deg.				

	2 Deg.				3 Deg.				
	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	
0	8.542819	9.999735	8.543084	11.456916	8.718800	9.999404	8.719396	11.280604	60
1	8.546422	9.999731	8.546691	11.453309	8.721204	9.999398	8.721806	11.278194	59
2	8.549995	9.999726	9.550268	11.449732	8.723595	9.999391	8.724204	11.275796	58
3	8.553539	9.999722	8.553817	11.446183	8.725972	9.999384	8.726588	11.273412	57
4	8.557054	9.999717	8.557316	11.442664	8.728337	3.999378	8.728959	11.271041	56
5	8.560540	9.999713	8.560828	11.439179	8.730688	9.999371	8.731317	11.268683	55
6	8.563999	9.999708	9.564291	11.435709	8.733027	9.999364	8.733663	11.266337	54
7	8.567431	9.999704	8.567727	11.432273	8.735354	9.999357	8.735996	11.264004	53
8	8.570836	9.999699	8.571137	11.428863	8.737667	9.999350	3.738317	11.261683	52
9	8.574214	9.999694	8.574530	11.425480	8.739969	9.999343	8.740626	11.259374	51
10	8.577566	9.999689	8.577877	11.422[illegible]	8.742259	9.999336	8.742922	11.257078	50
11	8.580892	9.999685	8.581208	11.418792	8.744536	9.999329	8.745207	11.254793	49
12	8.584193	9.999680	8.584514	11.415486	8.746802	9.999322	8.747479	11.252521	48
13	8.587469	9.999675	8.587795	11.412205	8.749055	9.999315	8.749740	11.250260	47
14	8.590721	9.999670	8.591051	11.408949	8.751297	9.999308	8.751989	11.248011	46
15	8.593948	9.999665	8.594283	11.405717	8.753528	9.999301	8.754227	11.245773	45
16	8.597152	9.999660	8.597492	11.402508	8.755747	9.999284	8.756453	11.243547	44
17	8.600332	9.999655	8.600677	11.399323	8.757955	9.999287	8.758668	11.241332	43
18	8.603489	9.999650	8.603839	11.396161	8.760151	9.999279	8.760872	11.239128	42
19	8.606623	9.999645	8.606978	11.393022	8.762337	9.999272	8.763065	11.236935	41
20	8.609734	9.999640	8.610094	11.389909	8.764511	9.999265	8.765246	11.234754	40
21	8.612823	9.999635	8.613189	11.386811	8.766675	9.999257	8.767417	11.232583	39
22	8.615891	9.999629	8.616262	11.383738	8.768828	9.999250	8.769578	11.230422	38
23	8.618937	9.999624	8.619313	11.380687	8.770970	9.999242	8.771727	11.228273	37
24	8.621962	9.999619	8.622343	11.377657	8.773101	9.999235	8.773866	11.226134	36
25	8.624965	9.999614	8.625352	11.374648	8.775223	9.999527	8.775995	11.224005	35
26	8.627948	9.999608	8.628340	11.371660	8.777333	9.999220	8.778114	11.221886	34
27	8.630911	9.999603	8.631308	11.368692	8.779434	9.999212	8.780222	11.219778	33
28	8.633854	9.999597	8.634256	11.365744	8.781524	9.999205	8.782320	11.217680	32
29	8.636776	9.999592	8.637184	11.362816	8.783605	9.999197	8.784408	11.215592	31
30	8.639680	9.999586	8.640093	11.359907	8.785675	9.999189	8.786486	11.213514	30
31	8.642563	9.999581	8.642982	11.357018	8.787736	9.999181	8.788554	11.211446	29
32	8.645428	9.999575	8.645[illegible]	11.354147	[illegible]	9.999174	8.790613	11.209337	28
33	8.648274	9.999570	8.648704	11.351296	[illegible]	9.999166	[illegible]	[illegible]	27
34	8.651102	9.999564	8.651537	11.348463	8.793859	9.999158	8.794701	11.205299	26
35	8.653911	9.999558	8.654352	11.345648	8.795881	9.999150	8.796731	11.203269	25
36	8.656702	9.999553	8.657149	11.342851	8.797894	9.999142	8.798752	11.201248	24
37	8.659475	9.999547	8.659928	11.340072	8.799897	9.999134	8.800763	11.199237	23
38	8.662230	9.999541	8.662689	11.337311	8.801892	9.999126	8.802765	11.197235	22
39	8.664968	9.999535	8.665433	11.334567	8.803876	9.999118	8.804758	11.195242	21
40	8.667689	9.999529	8.668160	11.331840	8.805852	9.999110	8.806742	11.193258	20
41	8.670393	9.999524	8.670870	11.329130	8.807819	9.999102	8.858717	11.191283	19
42	8.673080	9.999518	8.673563	11.326437	8.809777	9.999094	8.810683	11.189317	18
43	8.675751	9.999512	8.676239	11.323761	8.811726	9.999086	8.812641	11.187359	17
44	8.678405	9.999506	8.678900	11.321100	8.813667	9.999077	8.814589	11.185411	16
45	8.681043	9.999500	8.681544	11.318456	8.815599	9.999069	8.816529	11.183471	15
46	8.683665	9.999493	8.684172	11.315828	8.817522	9.999061	8.818461	11.181539	14
47	8.686272	9.999487	8.686784	11.313216	8.819436	9.999053	8.820384	11.179616	13
48	8.688863	9.999481	8.689391	11.310619	8.821343	9.999044	8.822298	11.177702	12
49	8.691438	9.999475	8.691963	11.308037	8.823240	9.999036	8.824205	11.175795	11
50	8.693998	9.999469	8.694529	11.305471	8.825130	9.999027	8.826103	11.173897	10
51	8.696543	9.999463	8.697081	11.302919	8.827011	9.999019	8.827992	11.172008	9
52	8.699073	9.999456	8.699617	11.300383	8.828884	9.999010	8.829874	11.170126	8
53	8.701589	9.999450	8.702139	11.297861	8.830749	9.999002	8.831748	11.165252	7
54	8.704090	9.999445	8.704646	11.295354	8.832607	9.998993	8.833613	11.166387	6
55	8.706577	9.999437	8.707140	11.292860	8.834456	9.998984	8.835471	11.164529	5
56	8.709049	9.999431	8.709618	11.290382	8.836297	9.998976	8.837321	11.162679	4
57	8.711507	9.999424	8.712083	11.287917	8.838130	9.998967	8.839163	11.160837	3
58	8.713952	9.999418	8.714534	11.285466	8.839956	9.998958	8.840998	11.159002	2
59	8.716383	9.999411	8.716972	11.283028	8.841774	9.998950	8.842825	11.157175	1
60	8.718800	9.999404	8.719396	11.280604	8.843585	9.998941	8.844644	11.155536	0
	Cosine.	Sine.	Cotan.	Tang.	Cosine.	Sine.	Cotan.	Tang.	
	87 Deg.				86 Deg.				

	4 Deg.				5 Deg.				
	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	
0	8.843585	9.998941	8.844644	11.155356	8.940296	9.998344	8.941952	11.058048	60
1	8.845387	9.998932	8.846455	11.153545	8.941738	9.998333	8.943404	11.056596	59
2	8.847183	9.998923	8.848260	11.151740	8.943174	9.998322	8.944852	11.055148	58
3	8.848971	9.998914	8.850057	11.149943	8.944606	9.998311	8.946295	11.053705	57
4	8.850751	9.998905	8.851846	11.148154	8.946034	9.998300	8.947734	11.052266	56
5	8.852525	9.99[illegible]	8.853628	11.146372	8.947456	9.998289	8.949168	11.050832	55
6	8.854291	9.998887	8.855403	11.144597	8.948874	9.998277	8.950597	11.049403	54
7	8.856049	9.998878	8.857171	11.142829	8.950287	9.998266	8.952021	11.047979	53
8	8.857801	9.998869	8.858932	11.141068	8.951696	9.998255	8.953441	11.046559	52
9	8.859546	9.998860	8.860686	11.139314	8.953100	9.998243	8.954856	11.045144	51
10	8.861283	9·998851	8.862433	11.137567	8.954499	9.998232	8.956267	11.043733	50
11	8.863014	9.998841	8.864173	11.135827	8.955894	9.998220	8.957674	11.042326	49
12	8.864738	9.998832	8.865906	11.134094	8.957284	9.998209	8.959075	11.040925	48
13	8.866455	9.998823	8.867632	11.132368	8.958670	9.998197	8.960473	11.039527	47
14	8.868165	9.998813	8.869351	11.130649	8.960052	9.998186	8.961866	11.038134	46
15	8.869868	9.998804	8.871064	11.128936	8.961429	9.998174	8.963255	11.036745	45
16	8.871565	9.998795	8.872770	11.127230	8.962801	9.998163	8.964639	11.035361	44
17	8.873255	9,998785	8.874469	11.125531	8.964170	9.998151	8.966019	11.033981	43
18	8.874938	9.998776	8.870162	11.123838	8.965534	9.998139	8.967394	11.032606	42
19	8.876615	9.998766	8.877849	11.122151	8.966893	9.998128	8.968766	11.031234	41
20	8.878285	9.998757	8.879529	11.120471	8.968249	9.998116	8.970133	11.029867	40
21	8.879949	9.998747	8.881202	11.118798	8.969600	9.998104	8.971496	11.028504	39
22	8.881607	9.998738	8.882869	11.117131	8.970947	9.998092	8.972855	11.027145	38
23	8.883258	9.998728	8.884530	11.115470	8.972289	9.998080	8.974209	11.025791	37
24	8.884903	9.998718	8.886185	11.113815	8.973628	9.998068	8.975560	11.024440	36
25	8.886542	9,998708	8.887833	11.112167	8.974962	9.998056	8.976906	11.023094	35
26	8.888174	9,998699	8.889476	11.110524	8.976293	9.998044	8.978248	11.021752	34
27	8.889801	9.998689	8.891112	11.108888	8.977619	9.998032	8.979586	11.020414	33
28	8·891421	9.998679	8.892742	11.107258	8.978941	9.998020	8.980921	11.019079	32
29	8.893035	9.998669	8.894366	11.105634	8.980259	9.998008	8.982251	11.017749	31
30	8.894643	9.998659	8.895984	11.104016	8.981573	9.997990	8.983577	11.016423	30
31	8.896246	9.998649	8.897596	11.102404	8.982883	9.997984	8.984899	11.015101	29
32	8.897842	9.998639	8.899203	11.100797	8.984189	9.997972	8.986217	11.013783	28
33	8,899432	9.998629	8.900803	11.099197	8.985491	9.997959	8.987532	11.012468	27
34	8.901017	9.998619	8.902398	11.097602	8.986789	9.997947	8.988842	11.011158	26
35	8.902596	9.998609	8.903987	11.096013	8.988085	9.997935	8.990149	11.009851	25
36	8.904169	9.998599	8.905570	11.094430	8.989374	9.997922	8.991451	11.008549	24
37	8.905736	9.998589	8.907147	11.092853	8.990660	9.997910	8.992750	11.007250	23
38	8.907297	9.998578	8.908719	11.091281	8.991943	9.997897	8.994045	11.005955	22
39	8.908853	9.998568	8.910285	11.089715	8.993222	9.997885	8.995337	11.004663	21
40	8.910404	9.998558	8.911846	11.088154	8.994497	9.997872	8.996624	11.003376	20
41	8.911949	9.998548	8.913401	11.086599	8.995768	9.997860	8.997908	11.002092	19
42	8,913488	9.998537	8.914951	11.085049	8.997036	9.997847	8.999188	11.000812	18
43	3.915022	9.998527	8.916495	11.083505	8.998299	9.997835	9.000465	10.999535	17
44	8.916550	9.998516	8.918034	11.581966	8.999560	9.997822	9.001758	10.998262	16
45	8.918073	9.998506	8.919568	11.0804[illegible]	9.000816	9.997809	9.003007	10.996993	15
46	8.919591	9.998495	8.921096	11.078904	9.002069	9.997797	9.004272	10.995728	14
47	8.929103	9.998485	8.922619	11.077381	9.003318	9.997784	9.005534	10.994466	13
48	8.922610	9.998474	8.924136	11.075864	9.004563	9.997771	9.006792	10.993208	12
49	8.924112	9.998464	8.925649	11.074351	9.005805	9.997758	9.008047	10.991953	11
50	8.925609	9.998453	8.927156	11,072844	9.007044	9.997745	9.009298	10.990702	10
51	8.927100	9.998442	8.928658	11.071342	9.008278	9.997732	9,010546	10.989454	9
52	8.928587	9.998431	8.930155	11.069845	9.009510	9.997719	9,011790	10.988210	8
53	8.930068	9,998421	8.931647	11.068353	9.010757	9.997706	9.013031	10.986969	7
54	8.931544	9.998410	8.933134	11.066866	9.011962	9.997695	9.014268	10.985732	6
55	8.933015	9.998399	8.934616	11.065384	9.013182	9.997680	9.015502	10.984498	5
56	8.934481	9.998388	8.936093	11.063907	9.014400	9.997667	9.016732	10.983268	4
57	8.935942	9.998377	8.937565	11.062435	9.015613	9.997654	9.017959	10.982041	3
58	8.937398	9.998366	8.939032	11.060968	9.016824	9.997641	9.019183	10.980817	2
59	8.938850	9.998355	8.940494	11.059506	9.018031	9.997628	9.020403	10.979579	1
60	8.940296	9.998344	8.941952	11.058048	9.019235	9.997614	9.021620	10.978380	0
	Cosine.	Sine.	Cotan.	Tang.	Cosine.	Sine.	Cotan.	Tang.	
	85 Deg.				84 Deg.				

	6 Deg.				7 Deg.				
	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine	Tang.	Cotang.	
0	9.019235	9.997614	9.021620	10.978380	9.085894	9.996751	9.089144	10.910856	60
1	9.020435	9.997601	9.022834	10.977166	9.086922	9.996735	9.090187	10.909813	59
2	9.021632	9.997588	9.024044	10.975956	9.087947	9.996720	9.091228	10.908772	58
3	9.022825	9.997574	9.025251	10.974749	9.088970	9.996704	9.092262	10.907734	57
4	9.024016	9.997561	9.026455	10.973545	9.089990	9.996688	9.093302	10.906698	56
5	9.025203	9.997547	9.027655	10.972345	9.091006	9.996673	9.094336	10.905664	55
6	9.026386	9.997534	9.028852	10.971148	9.092024	9.996657	9.095367	10.904633	54
7	9.027567	9.997520	9.030046	10.969954	9.093037	9.996641	9.096395	10.903605	53
8	9.028744	9.997507	9.031237	10.968763	9.094047	9.996625	9.097422	10.902578	52
9	9.029918	9.997493	9.032425	10.967575	9.095056	9.996610	9.098448	10.901554	51
10	9.031089	9.997480	9.033609	10.966391	9.096062	9.996594	9.099468	10.900532	50
11	9.032257	9.997466	9.034791	10.965209	9.097065	9.996578	9.100487	10.899513	49
12	9.033421	9.997452	9.035969	10.964031	9.098066	9.996562	9.101501	10.898496	48
13	9.034582	9.997439	9.037144	10.962856	9.099065	9.996546	9.102519	10.897481	47
14	9.035741	9.997425	9.038316	10.961684	9.100062	9.996530	9.103532	10.896468	46
15	9.036896	9.997411	9.039485	10.960515	9.101056	9.996514	9.104542	10.895458	45
16	9.038048	9.997397	9.040651	10.959349	9.102048	9.996498	9.105550	10.894450	44
17	9.039197	9.997383	9.041813	10.958187	9.103037	9.996482	9.106556	10.89344[illegible]	43
18	9.040342	9.997369	9.042973	10.957027	9.104025	9.996465	9.107559	10.8924[illegible]	42
19	9.041485	9.997355	9.044130	10.955870	9.105010	9.996449	9.108560	10.891440	41
20	9.042625	9.997341	9.045284	10.954716	9.105992	9.995433	9.109559	10.890441	40
21	9.043762	9.997327	9.046434	10.953566	9.106973	9.996417	9.110556	10.889444	39
22	9.044895	9.997313	9.047582	10.952418	9.107951	9.996400	9.111551	10.888449	38
23	9.046026	9.997299	9.048727	10.951273	9.108927	9.996384	9.112543	10.887457	37
24	9.047154	9.997285	9.049869	10.950131	9.109901	9.996368	9.113533	10.886467	36
25	9.048277	9.997271	9.051008	10.948992	9.110873	9.996351	9.114521	10.885479	35
26	9.049400	9.997257	9.052144	10.947856	9.111842	9.996335	9.115597	10.884498	34
27	9.050519	9.997242	9.053277	10.946723	9.112809	9.996318	9.116491	10.883509	33
28	9.051635	9.997228	9.054407	10.945593	9.113774	9.996302	9.117472	10.882528	32
29	9.052749	9.997214	9.055535	10.944465	9.114737	9.996285	9.118452	10.881548	31
30	9.053859	9.997199	9.056659	10.943341	9.115698	9.996269	9.119429	10.880571	30
31	9.054966	9.997185	9.057781	10.942219	9.116656	9.996252	9.120404	10.879596	29
32	9.056071	9.997170	9.058900	10.941100	9.117613	9.996235	9.121377	10.878623	28
33	9.057172	9.997156	9.060016	10.939984	9.118567	9.996219	9.122348	10.877652	27
34	9.058271	9.997141	9.061130	10.938870	9.119519	9.996202	9.123317	10.876683	26
35	9.059367	9.997127	9.062240	10.937760	9.120469	9.996185	9.124284	10.875716	25
36	9.060460	9.997112	9.063348	10.936652	9.121417	9.996168	9.125249	10.874751	24
37	9.061551	9.997098	9.064453	10.935547	9.122362	9.996151	9.126211	10.873789	23
38	9.062639	9.997083	9.065556	10.934444	9.123306	9.996134	9.127172	10.872828	22
39	9.063724	9.997068	9.066655	10.933345	9.124248	9.996117	9.128130	10.871870	21
40	9.094806	9.997053	9.067752	10.932248	9.125187	9.996110	9.129087	10.870913	20
41	9.065885	9.997039	9.068846	10.931154	9.126125	9.996083	9.130041	10.869959	19
42	9.066962	9.997024	9.069938	10.930062	9.127068	9.996066	9.130994	10.869006	18
43	9.068036	9.997099	9.071027	10.928973	9.127993	9.996049	9.131944	10.868056	17
44	9.069107	9.996994	9.072113	10.927887	9.128925	9.996032	9.132893	10.867107	16
45	9.070176	9.996979	9.073197	10.926803	9.129854	9.996015	9.133839	10.866161	15
46	9.071242	9.996964	9.074278	10.925722	9.130781	9.995998	9.134784	10.865216	14
47	9.072306	9.996949	9.075356	10.924644	9.131706	9.995980	9.135726	10.864274	13
48	9.073366	9.996934	9.076432	10.923563	9.132630	9.995963	9.136667	10.863333	12
49	9.074424	9.996919	9.077505	10.922495	9.133551	9.995946	9.137605	10.862395	11
50	9.075480	9.996904	9.078576	10.921424	9.134470	9.995928	9.138542	10.861458	10
51	9.076533	9.996889	9.079644	10.920356	9.135387	9.995911	9.139476	10.860524	9
52	9.077583	9.996874	9.080710	10.919290	9.136303	9.995894	9.140409	10.859591	8
53	9.078631	9.996858	9.081773	10.918227	9.137216	9.995876	9.141340	10.858660	7
54	9.079676	9.996843	9.082833	10.917167	9.138128	9.995859	9.142269	10.857731	6
55	9.080719	9.996828	9.083891	10.916109	9.139037	9.995841	9.143196	10.856804	5
56	9.081759	9.996812	9.084947	10.915053	9.139944	9.995823	9.144121	10.855879	4
57	9.082797	9.996797	9.086000	10.914000	9.140850	9.995806	9.145044	10.854956	3
58	9.083832	9.996782	9.087050	10.912950	9.141754	9.995788	9.145966	10.854034	2
59	9.084864	9.996766	9.088098	10.911902	9.142655	9.995771	9.146885	10.853115	1
60	9.085894	9.996751	9.089144	10.910856	9.143555	9.995753	9.147808	10.852197	0
	Cosine.	Sine.	Cotan.	Tang.	Cosine.	Sine.	Cotan.	Tang.	
	83 Deg.				82 Deg.				

	8 Deg.				9 Deg.				
	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	
0	9.143555	9.995753	9.147803	10.852197	9.194332	9.994620	9.199713	10.800287	60
1	9.144453	9.995735	9.148718	10.851282	9.195129	9.994600	9.200529	10.799471	59
2	9.145349	9.995717	9.149632	10.850368	9.195925	9.994580	9.201345	10.798655	58
3	9.146243	9.995699	9.150544	10.849456	9.196719	9.994560	9.202159	10.797841	57
4	9.147136	9.995681	9.151454	10.848546	9.197511	9.994540	9.202951	10.797029	56
5	9.148026	9.695664	9.152363	10.847637	9.198402	9.994519	9.203782	10.796218	55
6	9.148915	9.995646	9.153269	10.846731	9.199091	9.994499	9.204592	10.795408	54
7	9.149802	9.995628	9.154174	10.845826	9.199879	9.994479	9.205400	10.794400	53
8	9.150686	9.995610	9.155077	10.844923	9.200666	9.994459	9.206207	10.793793	52
9	9.151569	9.995591	9.155978	10.844022	9.201451	9.994438	9.207013	10.792987	51
10	9.152451	9.995573	9.156877	10.843123	9.202234	9.994418	9.207817	10.792183	50
11	9.153330	9.995555	9.157775	10.842225	9.203017	9.994398	9.208619	10.791381	49
12	9.154208	9.995537	9.158671	10.841329	9.203797	9.994377	9.209420	10.790580	48
13	9.155083	9.995519	9.159565	10.840435	9.204577	9.994357	9.210220	10.789780	47
14	9.155957	9.995501	9.160457	10.839543	9.205354	9.994336	9.211018	10.788982	46
15	9.156830	9.995482	9.161347	10.838653	9.206131	9.994316	9.211815	10.788185	45
16	9.157700	9.995464	9.162236	10.837764	9.206906	9.994295	9.212611	10.787389	44
17	9.158569	9.995446	9.163123	10.836877	9.207679	9.994274	9.213405	10.786595	43
18	9.159435	9.995427	9.164008	10.835992	9.208452	9.994254	9.214198	10.785802	42
19	9.160301	9.995409	9.164892	10.835108	9.209222	9.994233	9.214989	10.785011	41
20	9.161164	9.995300	9.165774	10.834226	9.209992	9.994212	9.215780	10.784220	40
21	·162025	9.995372	9.166654	10.833346	9.210760	9.994191	9.216568	10.783432	39
22	9.162885	9.995353	9.167532	10.832468	9.211526	9.994171	9.217356	10.782644	38
23	9.163743	9.995334	9.168409	10.831591	9.212291	9.994150	9.218142	10.781858	37
24	9.164600	9.995316	9.169284	10.830716	9.213055	9.994129	9.218926	10.781074	36
25	9.165454	9.995297	9.170157	10.829843	9.213818	9.994108	9.219710	10.780290	35
26	9.166307	9.995278	9.171029	10.828971	9.214579	9.994087	9.220492	10.789508	34
27	9.167159	9.995260	9.171899	10.828101	9.215338	9.994066	9.221272	10.778728	33
28	9.168008	9.995241	9.172767	10.827233	9.216097	9.994045	9.222052	10.777948	32
29	9.168856	9.995222	9.173634	10.826366	9.216854	9.994024	9.222830	10.777170	31
30	9.169702	9.995203	9.174499	10.825501	9.217609	9.994003	9.223607	10.776393	30
31	9.170547	9.995184	9.175362	10.824638	9.218363	9.993982	9.224382	10.775618	29
32	9.171389	9.995165	9.176224	10.823776	9.219116	9.993960	9.225156	10.774844	28
33	9,172230	9.995146	9.177084	10.822916	9.219868	9.993939	9.225929	10.774071	27
34	9.173070	9.995127	9.177942	10.822058	9.220618	9.993918	9.226700	10 773300	26
35	9.173908	9.995108	9.178799	10.821201	9.221367	9.993897	9.227471	10.772529	25
36	9.174744	9.995089	9.179655	10.820345	9.222115	9.993875	9.228239	10.771761	24
37	9.175578	9.995070	9.180508	10.819492	9.222861	9.993854	9.229007	10.770993	23
38	9.176411	9.995051	9.181360	10.818640	9.223606	9.993832	9.229773	10.770227	22
39	9.177242	9.995032	9.182211	10 817789	9.224349	9.993811	9.230539	10.769461	21
40	9.178072	9.995013	9.183059	10.816941	9.225092	9.993789	9.231302	10.768698	20
41	9.178900	9.994993	9.183907	10.816093	9.225835	9.993768	9.232065	10.767935	19
42	9.179726	9.994974	9.184752	10.815248	9.226573	9.993746	9.232826	10.767174	18
43	9.180551	9.994955	9.185597	10 814403	9.227311	9.993725	9.233586	10.766414	17
44	9.181374	9.994935	9.186439	10.813561	9.228048	9.993703	9.234345	10.765655	16
45	9.182196	9.994916	9.187280	10.812720	9.228784	9.993681	9.235103	10.764897	15
46	9.183016	9.994896	9.188120	10.811880	9.229518	9.993660	9.235859	10.764141	14
47	9.183834	9.994877	9.189958	10.811042	9.230252	9.993638	9.236614	10.763386	13
48	9.184651	9.994857	9.189794	10.810206	9.230984	9.993616	9.237368	10.762632	12
49	9.185466	9.994838	9.190629	10.809371	9.231715	9.993594	9.238120	10.761880	11
50	9.186280	9.994818	9.191492	10 808558	9.232444	9.993572	9.238872	10.761128	10
51	9.187092	9.994798	9.192294	10.807706	9.233172	9.993550	9.239622	10.760378	9
52	9.187903	9.994779	9.193124	10.809876	9.233899	9.993528	9.240371	10.759629	8
53	9.188712	9.994759	9.193953	10.806047	9.234625	9.993506	9.241118	10.758882	7
54	9.189519	9.994739	9.194780	10.805220	9.235349	9.993484	9.241865	10.758135	6
55	9.190325	9.994720	9.195606	10.804394	9.236073	9.993462	9.242610	10.757390	5
56	9.191130	9.994700	9.196430	10.803570	9.236795	9.993440	9.243454	10.756646	4
57	9.191933	9 994680	9.197253	10.802747	8.237515	9.993418	9.244097	10.755903	3
58	9.192734	9.994660	9.198074	10.801926	9.238235	9.993396	9.244839	10.755161	2
59	9.193534	9.904640	9.198894	10.801106	9.238953	9.993374	9.245579	10.754421	1
60	9.194332	9.994620	9.190713	10.800287	9.239670	9.993351	9.246319	10.753682	0
	Cosine.	Sine.	Cotan.	Tang.	Cosine.	Sine.	Cotan.	Tang.	
	81 Deg.				80 Deg.				

	10 Deg.				11 Deg.				
	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	
0	9.239670	9.993351	9.246319	10.753681	9.280599	9.991947	9.288652	10.711348	60
1	9.240386	9.993329	9.247057	10.752943	9.281248	9.991922	9.289326	10.710674	59
2	9.241101	9.993307	9.247794	10.752206	9.281897	9.991897	9.289999	10.710001	58
3	9.241814	9.993284	9.248530	10.751470	9.282544	9.991873	9.290671	10.709329	57
4	9.242526	9.993262	9.249264	10.750736	9.283190	9.991848	9.291342	10.708658	56
5	9.243237	9.993240	9.249998	10.750002	9.283836	9.991823	9.292013	10.707987	55
6	9.243947	9.993217	9.250730	10.749270	9.284480	9.991799	9.292682	10.707318	54
7	9.244656	9.993195	9.251461	10.748539	9.285124	9.991774	9.293350	10.706650	53
8	9.245363	9.993172	9.252191	10.747809	9.285766	9.991749	9.294017	10.704983	52
9	9.246069	9.993149	9.252920	10.747080	9.286408	9.991724	9.294684	10.705316	51
10	9.246775	9.993127	9.253648	10.746352	9.287048	9.991699	9.295349	10.704651	50
11	9.247478	9.993104	9.254374	10.745626	9.287688	9.991674	9.296013	10.703987	49
12	9.248181	9.993081	9.255100	10.744900	9.288326	9.991649	9.296677	10.703323	48
13	9.248883	9.993059	9.255824	10.744176	9.288964	9.991624	9.297339	10.702661	47
14	9.249583	9.993036	9.256547	10.743456	9.289600	9.991599	9.298001	10.701999	46
15	9.250282	9.993013	9.257269	10.742731	9.290236	9.991574	9.298662	10.701338	45
16	9.250980	9.992990	9.257990	10.742010	9.290870	9.991549	9.299322	10.700678	44
17	9.251677	9.992967	9.258710	10.741290	9.291504	9.991524	9.299980	10.700020	43
18	9.252373	9.992944	9.259429	10.740571	9.292137	9.991498	9.300638	10.699362	42
19	9.253067	9.992921	9.260146	10.739584	9.292768	9.991473	9.301295	10.698705	41
20	9.253761	9.992898	9.260863	10.789137	9.293399	9.991448	9.301951	10.698049	40
21	9.254453	9.992875	9.261578	10.738422	9.294029	9.991422	9.302607	10.697393	39
22	9.255144	9.992852	9.262292	10.737408	9.294658	9.991397	9.307261	10.696739	38
23	9.255834	9.992829	9.263005	10.736995	8.295286	9.991372	9.303914	10.696086	37
24	9.256523	9.992906	9.263717	10.736283	9.295913	9.991346	9.304567	10.695433	36
25	9.257211	9.992783	9.264428	10.735572	9.296539	9.991321	9.305218	10.694782	35
26	9.257898	9.992759	9.265138	10.734862	9.297164	9.991295	9.305869	10.694131	34
27	9.258583	9.992736	9.265847	10.734153	9.297788	9.991270	9.306519	10.693481	33
28	9.259268	9.992713	5.266555	10.733445	9.298412	9.991244	9.307168	10.692832	32
29	9.259951	9.992690	9.267261	10.732739	9.299034	9.991218	9.307816	10.692184	31
30	9.260633	9.992666	9.267967	10.732033	9.299655	9.991193	9.308463	10.691537	30
31	9.261314	9.992643	9.268671	10.731329	9.300276	9.991167	9.309109	10.690891	29
32	9.261994	9.992619	9.269375	10.730625	9.300896	9.991141	9.309754	10.690246	28
33	9.262673	9.992596	9.270077	10.729923	9.301514	9.991115	9.300399	10.689601	27
34	9.263351	9.992572	9.270779	10.729221	9.302132	9.991090	9.311042	10.688958	26
35	9.264027	9.992549	9.271479	10.728521	9.302748	9.991064	9.311685	10.688315	25
36	9.264703	9.992525	9.272178	10.727822	9.303364	9.991038	9.312327	10.687673	24
37	9.265377	9.992501	9.272876	10.727124	9.303979	9.991012	9.312968	10.687032	23
38	9.266051	9.992478	9.273573	10.726427	9.304593	9.990986	9.313608	10.686392	22
39	9.266723	9.992454	9.274269	10.725731	9.305207	9.990960	9.314247	10.685753	21
40	9.267395	9.992430	9.274964	10.725036	9.305819	9.990934	9.314885	10.685115	20
41	9.268065	9.992406	9.275658	10.724342	9.306430	9.990908	9.315523	10.684477	19
42	9.268734	9.992382	9.276351	10.723649	9.307041	9.990882	9.316159	10.683841	18
43	9.269402	9.992359	9.277043	10.722957	9.307650	9.990852	9.316795	10.683205	17
44	9.270069	9.992335	9.277734	10.722266	9.308259	9.990829	9.317430	10.682570	16
45	9.270735	9.992311	9.278424	10.721576	9.308867	9.990803	9.318064	10.681936	15
46	9.271400	9.992287	9.279113	10.720887	9.309474	9.990777	9.318697	10.681303	14
47	9.272064	9.992263	9.279801	10.720199	9.310080	9.990750	9.319330	10.680670	13
48	9.272726	9.992239	9.280488	10.719512	9.310685	9.990724	9.319961	10.680039	12
49	9.273388	9.992214	9.281174	10.718826	9.311284	9.990697	9.320592	10.679408	11
50	9.274049	9.992190	9.281858	10.718142	9.311893	9.990671	9.321222	10.678717	10
51	9.274708	9.992166	9.282542	10.717458	9.312495	9.990645	9.321851	10.678178	9
52	9.275367	9.992142	9.283225	10.716775	9.313097	9.990618	9.322479	10.677549	8
53	9.276025	9.992118	9.283907	10.716093	9.313698	9.990591	4.323106	10.676891	7
54	9.276681	9.992093	9.284588	10.715412	9.314297	9.990565	9.323733	10.676294	6
55	9.277337	9.992069	9.285268	10.717432	9.314897	9.990538	9.324358	10.675667	5
56	9.277991	9.992044	9.285947	10.714053	9.315495	9.990511	9.324983	10.675042	4
57	9.278645	9.992020	9.286624	10.713376	9.316092	9.990485	9.325607	10.674393	3
58	9.279297	9.991996	9.287301	10.712699	9.316689	9.990458	9.326231	10.673769	2
59	9.279948	9.991971	9.287977	10.712023	9.317284	9.990431	9.326853	10.673147	1
60	9.280599	9.991941	9.288652	10.711348	9.317879	9.990404	9.327475	10.672525	0
	Cosine.	Sine.	Cotan.	Tang.	Cosine.	Sine.	Cotan.	Tang.	
	79 Deg.				78 Deg.				

	12 Deg.				13 Deg.				
	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	
0	9.317879	9.990404	9.327475	10.672525	9.352088	9.988724	9.363364	10.636636	60
1	9.318473	9.990378	9.328095	10.671905	9.352635	9.988695	9.363940	10.636060	59
2	9.319066	9.990351	9.328715	10.671265	9.353181	9.988666	9.364515	10.635485	58
3	9.319658	9.990324	9.329334	10.670666	9.353726	9.988636	9.365090	10.634910	57
4	9.320249	9.990297	9.329953	10.670047	9.354271	9.988607	9.365664	10.634336	56
5	9.320840	9.990270	9.330570	10.669430	9.354815	9.988578	9.366257	10.633763	55
6	9.321430	9.990243	9.331187	10.668813	9.355358	9.988548	9.366810	10.633190	54
7	9.322019	9.990215	9.331803	10.668197	9.355901	9.988519	9.367382	10.632618	53
8	9.322607	9.990188	9.332418	10.667582	9.356443	9.988489	9.367953	10.632047	52
9	9.323194	9.990161	9.333033	10.666967	9.356984	9.988460	9.368524	10.631476	51
10	9.323780	9.990934	9.333646	10.666354	9.357524	9.988430	9.369094	10.630906	50
11	9.324366	9.990107	9.334259	10.665741	9.358064	9.988401	9.369663	10.630337	49
12	9.324950	9.990079	9.334871	10.665129	9.358603	9.988371	9.370232	10.629768	48
13	9.325534	9.990052	9.335482	10.664518	9.359141	9.988342	9.370799	10.629201	47
14	9.326111	9.990025	9.336093	10.663907	9.359078	9.988312	9.371367	10.628663	46
15	9.326700	9.989997	9.336702	10.663298	9.360215	9.988282	9.371933	10.628067	45
16	9.327281	9.989970	9.337311	10.662689	9.360752	9.988252	9.372499	10.627501	44
17	9.327862	9.989942	9.337919	10.662081	9.361287	9.988223	9.373064	10.626936	43
18	9.328442	9.989915	9.338527	10.661473	9.361822	9.988193	9.373629	10.626371	42
19	9.329021	9.989887	9.339133	10.660867	9.362356	9.988163	9.374193	10.625807	41
20	9.329599	9.989860	9.339739	10.660261	9.362889	9.988133	9.374756	10.625244	40
21	9.330176	9.989832	9.340344	10.659656	9.363422	9.988103	9.375319	10.624681	39
22	9.330753	9.989804	9.340948	10.659052	9.363954	9.988073	9.375881	10.624119	38
23	9.331329	9.989777	9.341552	10.658448	9.364485	9.988043	9.376442	10.623558	37
24	9.331093	9.989749	9.342155	10.657845	9.365016	9.988013	9.377003	10.622997	36
25	9.332478	9.989721	9.342757	10.657243	9.365546	9.987983	9.377563	10.622437	35
26	9.333051	9.989693	9.343358	10.656642	9.366075	9.987953	9.[illegible]8122	10.621873	34
27	9.333624	9.989665	9.343959	10.656042	9.366604	9.987922	9.378681	10.621319	33
28	9.334195	9.989637	9.344558	10.652442	9.367131	9.987892	9.379239	10.620761	32
29	9.334767	9.989610	9.345157	10.654843	9.367659	9.987862	9.379797	10.620203	31
30	9.335337	9.989582	9.345755	10.654245	9.368185	9.987832	9.380354	10.619646	30
31	9.335906	9.989553	9.346353	10.653647	9.368711	9.987801	9.380910	10.619090	29
32	9.336475	9.989525	9.346949	10.653051	9.369236	9.987771	9.381466	10.618534	28
33	9.337043	9.989497	9.347545	10.6524[illegible]	9.369761	9.987740	9.382020	10.617980	27
34	9.337610	9.989469	9.348141	10.651859	9.370285	9.987710	9.382575	10.617425	26
35	9.338176	9.989441	9.348735	10.651265	9.370808	9.987679	9.383129	10.616871	25
36	9.338742	9.989413	9.349329	10.650671	9.371330	9.987649	9.383682	10.616318	24
37	9.339307	9.989385	9.349922	10.650078	9.371852	9.987618	9.384234	10.615766	23
38	9.339871	9.989356	9.350514	10.649486	9.372373	9.987583	9.384786	10.615214	22
39	9.340434	9.989328	9.531106	10.648894	9.372894	9.987557	9.385337	10.614663	21
40	9.340996	9.989300	9.351697	10.648303	9.373414	9.987526	9.385888	10.614112	20
41	9.341558	9.989271	9.352287	10.647713	9.373933	9.987496	9.386438	10.613562	19
42	9.342119	9.989243	9.352876	10.647124	9.374452	9.987465	9.386987	10.613013	18
43	9.342679	9.989214	9.353465	10.646535	9.374970	9.987434	9.387536	10.612464	17
44	9.343239	9.989186	9.354053	10.645947	9.375487	9.987400	9.388084	10.611916	16
45	9.343797	9.989157	9.354640	10.645360	9.376003	9.987372	9.388631	10.611369	15
46	9.344355	9.989128	9.355227	10.644773	9.376519	9.987341	9.389178	10.610822	14
47	9.344912	9.989100	9.355813	10.644187	9.377035	9.987310	9.389124	10.610276	13
48	9.345469	9.989017	9.356398	10.643602	9.377549	9.987279	9.390270	10.619730	12
49	9.346024	9.989042	9.356982	10.643018	9.378063	9.987248	9.390815	10.609185	11
50	9.346579	9.989014	9.357566	10.642434	9.378577	9.987217	9.391360	10.608640	10
51	9.347134	9.988985	9.358149	10.641851	9.379089	9.987186	9.391903	10.608097	9
52	9 346877	9.988956	9.358731	10.641269	9.379601	9.988155	9.392447	10.607553	8
53	9 348240	9.988927	9.359313	10.640687	9.380113	9.987124	9.392989	10.607011	7
54	9.348792	9.988898	9.359893	10.640107	9.380624	9.987092	9.393531	10.606469	6
55	9.349343	9.988869	9.360474	10.639526	9.381134	9.987061	9.394073	10.605927	5
56	9.349893	9.988840	9.361053	10.638947	9.381643	9.987030	9.394614	10.605386	4
57	9 350443	9.988811	9.361632	10.638368	9.382152	9.986998	9.395154	10.604846	3
58	9.350992	9.998782	9.362210	10.637790	9.382661	9.986967	9.395694	10.604306	2
59	9.351540	9.988753	9.362787	10.637213	9.383168	9.986936	9.396233	10.603767	1
60	9.352088	9.988724	9.363364	10.636636	9.383675	9.986904	9.396771	10.603229	0
	Cosine.	Sine.	Cotan.	Tang.	Cosine.	Sine.	Cotan.	Tang.	
	77 Deg.				76 Deg.				

	14 Deg.				15 Deg.				
	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	
0	9·383675	9·986904	9·396771	10·603229	9·412996	9·984944	9·428052	10·571948	60
1	9·384182	9·986873	9·397309	10·602691	9·413467	9·984910	9·428558	10·571442	59
2	9·384687	9·986841	9·397846	10·602154	9·413938	9·984876	9·429062	10·570938	58
3	9·385192	9·986809	9·398383	10·601617	9·414408	9·984842	9·429566	10·570434	57
4	9·385697	9·986778	9·398919	10·601081	9·414878	9·984808	9·430070	10·569930	56
5	9·386201	9·986746	9·399455	10·600545	9·415347	9·984774	9·430573	10·569427	55
6	9·386704	9·986714	9·399990	10·600010	9·415815	9·984740	9·431075	10·568925	54
7	9·387207	9·986683	9·400524	10·599476	9·416283	9·984706	9·431577	10·568423	53
8	9·387709	9·986651	9·401058	10·598942	9·416751	9·984672	9·432079	10·567921	52
9	9·388210	9·986619	9·401591	10·598409	9·417217	9·984638	9·432580	10·567420	51
10	9·388711	9·986587	9·402124	10·597876	9·417684	9·984603	9·433080	10·566920	50
11	9·389211	9·986555	9·402656	10·597344	9·418150	9·984569	9·433580	10·566420	49
12	9·389711	9·986523	9·403187	10·596813	9·418615	9·984535	9·434080	10·565920	48
13	9·390210	9·986491	9·403718	10·596282	9·419079	9·984500	9·434579	10·565421	47
14	9·390708	9·986459	9·404249	10·595751	9·419544	9·984466	9·435078	10·564922	46
15	9·391206	9·986427	9·404778	10·595222	9·420007	9·984432	9·435576	10·564424	45
16	9·391703	9·986395	9·405308	10·594692	9·420470	9·984397	9·436073	10·563927	44
17	9·392199	9·986363	9·405836	10·594164	9·420933	9·984363	9·436570	10·563430	43
18	9·392695	9·986331	9·406364	10·593636	9·421395	9·984328	9·437067	10·562933	42
19	9·393191	9·986299	9·406892	10·593108	9·421857	9·984294	9·437563	10·562437	41
20	9·393685	9·986266	9·407419	10·592581	9·422318	9·984259	9·438059	10·561941	40
21	9·394179	9·986234	9·407945	10·592055	9·422778	9·984224	9·438554	10·561446	39
22	9·394673	9·986202	9·408471	10·591529	9·423238	9·984190	9·439048	10·560952	38
23	9·395166	9·986169	9·408996	10·591004	9·423697	9·984155	9·439563	10·560457	37
24	9·395658	9·986137	9·409521	10·590479	9·424156	9·984120	9·440036	10·559964	36
25	9·396150	9·986104	9·410045	10·589955	9·424615	9·984085	9·440529	10·559471	35
26	9·396641	9·986072	9·410569	10·589431	9·425073	9·984050	9·441022	10·558978	34
27	9·397132	9·986039	9·411092	10·588908	9·425530	9·984015	9·441514	10·558486	33
28	9·397621	9·986007	9·411615	10·588385	9·425987	9·983981	9·442006	10·557994	32
29	9·398111	9·985974	9·412137	10·587863	9·426443	9·983946	9·442497	10·557503	31
30	9·398600	9·985942	9·412658	10·587342	9·426899	9·983911	9·442988	10·557012	30
31	9·399088	9·985909	9·413179	10·586821	9·427354	9·983875	9·443479	10·556521	29
32	9·399575	9·985876	9·413699	10·586301	9·427809	9·983840	9·443968	10·556032	28
33	9·400062	9·985843	9·414219	10·585781	9·428263	9·983805	9·444458	10·555542	27
34	9·400549	9·985811	9·414738	10·585262	9·428717	9·983770	9·444947	10·555053	26
35	9·401035	9·985778	9·415257	10·584743	9·429170	9·983735	9·445435	10·554565	25
36	9·401520	9·985745	9·415775	10·584225	9·429623	9·983700	9·445923	10·554077	24
37	9·402005	9·985712	9·416293	10·583707	9·430075	9·983664	9·446411	10·553589	23
38	9·402489	9·985679	9·416810	10·583190	9·430527	9·983629	9·446898	10·553102	22
39	9·402972	9·985646	9·417326	10·582674	9·430978	9·983594	9·447384	10·552616	21
40	9·403455	9·985613	9·417842	10·582158	9·431429	9·983558	9·447870	10·552130	20
41	9·403938	9·985580	9·418358	10·581642	9·431879	9·983523	9·448356	10·551644	19
42	9·404420	9·985547	9·418873	10·581127	9·432329	9·983487	9·448841	10·551159	18
43	9·404901	9·985514	9·419387	10·580613	9·432778	9·983452	9·449326	10·550674	17
44	9·405382	9·985480	9·419901	10·580099	9·433226	9·983416	9·449810	10·550190	16
45	9·405862	9·985447	9·420415	10·579585	9·433675	9·983381	9·450294	10·549706	15
46	9·406341	9·985414	9·420927	10·579073	9·434122	9·983345	9·450777	10·549223	14
47	9·406820	9·985381	9·421440	10·578560	9·434569	9·983309	9·451260	10·548740	13
48	6·407299	9·985347	9·421952	10·578048	9·435016	9·983273	9·451743	10·548257	12
49	9·407777	9·985314	9·422463	10·577537	9·435462	9·983238	9·452225	10·547775	11
50	9·408254	9·985280	9·422974	10·577026	9·435908	9·983202	9·452706	10·547294	10
51	9·408731	9·985247	9·423484	10·576516	9·436353	9·983166	9·453187	10·546813	9
52	9·409207	9·985213	9·423993	10·576007	9·436798	9·983130	9·453668	10·546332	8
53	9·409682	9·985180	9·424503	10·575497	9·437242	9·983094	9·454148	10·545852	7
54	9·410157	9·985146	9·425011	10·574989	9·437686	9·983058	9·454628	10·545372	6
55	9·410632	9·985113	9·425519	10·574481	9·438129	9·983022	9·455107	10·544893	5
56	9·411106	9·985079	9·426027	10·573973	9·438572	9·982986	9·455586	10·544414	4
57	9·411579	9·985045	9·426534	10·573469	9·439014	9·982950	9·456064	10·543936	3
58	9·412052	9·985011	9·427041	10·572959	9·439456	9·982914	9·456542	10·543458	2
59	9·412524	9·984978	9·427547	10·572453	9·439897	9·982878	9·457019	10·542981	1
60	9·412996	9·984944	9·428052	10·571948	9·440338	9·982842	9·457496	10·542504	0
	Cosine.	Sine.	Cotan.	Tang.	Cosine.	Sine.	Cotan.	Tang.	
	75 Deg.				74 Deg.				

	16 Deg.				17 Deg.				
	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	
0	9.440338	9.982842	9.457496	10.542504	9.465935	9.980596	9.485339	10.514661	60
1	9.440778	9.982805	9.457973	10.542027	9.466348	9.980558	9.485791	10.514209	59
2	9.441218	9.982769	9.458449	10.541551	9.466761	9.980519	9.486242	10.513758	58
3	9.441658	9.982733	9.458925	10.541075	9.467173	9.980480	9.486693	10.513307	57
4	9.442096	9.982696	9.459400	10.540600	9.467585	9.980442	9.487143	10.512857	56
5	9.442535	9.982660	9.459875	10.540125	9.467996	9.980403	9.487593	10.512407	55
6	9.442973	9.982624	9.460349	10.539651	9.468407	9.980364	9.488043	10.511957	54
7	9.443410	9.982587	9.460823	10.539177	9.468817	9.980325	9.488492	10.511508	53
8	9.443847	9.982551	9.461297	10.538703	9.469227	9.980286	9.488941	10.511059	52
9	9.444284	9.982514	9.461770	10.538230	9.469637	9.980247	9.489390	10.510610	51
10	9.444720	9.982477	9.462242	10.537758	9.470046	9.980208	9.489838	10.510162	50
11	9.445155	9.982441	9.462715	10.537285	9.470455	9.980169	9.490286	10.509714	49
12	9.445590	9.982404	9.463186	10.536814	9.470863	9.980130	9.490733	10.509267	48
13	9.446025	9.982367	9.463658	10.536342	9.471271	9.980091	9.491180	10.508820	47
14	9.446459	9.982331	9.464128	10.535872	9.471679	9.980052	9.491627	10.508373	46
15	9.446893	9.982294	9.464599	10.535401	9.472086	9.980012	9.492073	10.507927	45
16	9.447326	9.982257	9.465069	10.534931	9.472492	9.979973	9.492519	10.507481	44
17	9.447759	9.982220	9.465539	10.534461	9.472898	9.979934	9.492965	10.507035	43
18	9.448191	9.982183	9.466008	10.533992	9.473304	9.979895	9.493410	10.506590	42
19	9.448623	9.982146	9.466477	10.533523	9.473710	9.979855	9.493854	10.506146	41
20	9.449054	9.982109	9.466945	10.533055	9.474115	9.979816	9.494299	10.505701	40
21	9.449485	9.982072	9.467413	10.532587	9.474519	9.979776	9.494743	10.505257	39
22	9.449915	9.982035	9.467880	10.532120	9.474923	9.979737	9.495186	10.504814	38
23	9.450345	9.981998	9.468347	10.531653	9.475327	9.979697	9.495630	10.504370	37
24	9.450775	9.981961	9.468814	10.531186	9.475730	9.979658	9.496073	10.503927	36
25	9.451204	9.981924	9.469280	10.530720	9.476133	9.979618	9.496515	10.503485	35
26	9.451632	9.981886	9.469746	10.530254	9.476536	9.979579	9.496957	10.503043	34
27	9.452060	9.981849	9.470211	10.529789	9.476938	9.979539	9.497399	10.502601	33
28	9.452488	9.981812	9.470676	10.529324	9.477340	9.979499	9.497841	10.502159	32
29	9.452915	9.981774	9.471141	10.528859	9.477741	9.979459	9.498282	10.501718	31
30	9.453342	9.981737	9.471605	10.528395	9.478142	9.979420	9.498722	10.501278	30
31	9.453768	9.981700	9.472069	10.527931	9.478542	9.979380	9.499163	10.500837	29
32	9.454194	9.981662	9.472532	10.527468	9.478942	9.979340	9.499603	10.500397	28
33	9.454619	9.981626	9.472995	10.527005	9.479342	9.979300	9.500042	10.499958	27
34	9.455044	9.981587	9.473457	10.526543	9.479741	9.979260	9.500481	10.499519	26
35	9.455469	9.981549	9.473919	10.526081	9.480140	9.979220	9.500920	10.499080	25
36	9.455893	9.981512	9.474381	10.525619	9.480539	9.979180	9.501359	10.498641	24
37	9.456316	9.981474	9.474842	10.525158	9.480937	9.979140	9.501797	10.498203	23
38	9.456739	9.981436	9.475303	10.524697	9.481334	9.979100	9.502235	10.197765	22
39	9.457162	9.981399	9.475763	10.524237	9.481731	9.979059	9.502672	10.497328	21
40	9.457584	9.981361	9.476223	10.523777	9.482128	9.979019	9.503109	10.496891	20
41	9.458006	9.981323	9.476683	10.523317	9.482525	9.978979	9.503546	10.496454	19
42	9.458427	9.981285	9.477142	10.522858	9.482921	9.978939	9.503982	10.496018	18
43	9.458848	9.981247	9.477601	10.522399	9.483316	9.978898	9.504418	10.495582	17
44	9.459268	9.981209	9.478059	10.521941	9.483712	9.978858	9.504854	10.495146	16
45	9.459688	9.981171	9.478517	10.521483	9.484107	9.978817	9.505289	10.494711	15
46	9.460108	9.981133	9.478975	10.521025	9.484501	9.978777	9.505724	10.494276	14
47	9.460527	9.981095	9.479432	10.520568	9.484895	9.978737	9.506159	10.493841	13
48	9.460946	9.981057	9.479889	10.520111	9.485289	9.978696	9.506593	10.493407	12
49	9.461364	9.981019	9.480345	10.519655	9.485682	9.978655	9.507027	10.492973	11
50	9.461782	9.980981	9.480801	10.519199	9.486075	9.978615	9.507460	10.492540	10
51	9.462199	9.980942	9.481257	10.518743	9.486467	9.978574	9.507893	10.492107	9
52	9.462616	9.980904	9.481712	10.518288	9.486860	9.978533	9.508326	10.491674	8
53	9.463032	9.980866	9.482167	10.517833	9.487251	9.978493	9.508759	10.491241	7
54	9.463448	9.980827	9.482621	10.517379	9.487643	9.978452	9.509191	10.490809	6
55	9.463864	9.980789	9.483075	10.516925	9.488034	9.978411	9.509622	10.490378	5
56	9.464279	9.980750	9.483529	10.516471	9.488424	9.978370	9.510054	10.489946	4
57	9.464694	9.980712	9.483982	10.516018	9.488814	9.978329	9.510485	10.489515	3
58	9.465108	9.980673	9.484435	10.515565	9.489204	9.978288	9.510916	10.489084	2
59	9.465522	9.980635	9.484887	10.515113	9.489593	9.978247	9.511346	10.488654	1
60	9.465935	9.980596	9.485339	10.514661	9.489982	9.978206	9.511776	10.488224	0
	Cosine.	Sine.	Cotan.	Tang.	Cosine.	Sine.	Cotan.	Tang.	
	73 Deg.				72 Deg.				

	18 Deg.				19 Deg.				
	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	
0	9.489982	9.978206	9.511776	10.488224	9.512642	9.975670	9.536972	10.463028	60
1	9.490371	9.978165	9.512206	10.487794	9.513009	9.975627	9.537382	10.462618	59
2	9.490759	9.978124	9.512635	10.487365	9.513375	9.975583	9.537792	10.462208	58
3	9.491147	9.978083	9.513064	10.486936	9.513741	9.975539	9.538202	10.461798	57
4	9.491535	9.978042	9.513493	10.486507	9.514107	9.975496	9.538611	10.461389	56
5	9.491922	9.978001	9.513921	10.486079	9.514472	9.975452	9.539020	10.460980	55
6	9.492308	9.977959	9.514349	10.485651	9.514837	9.975408	9.539429	10.460571	54
7	9.492695	9.977918	9.514777	10.485225	9.515202	9.975365	9.539837	10.460163	53
8	9.493081	9.977877	9.515204	10.484796	9.515566	9.975321	9.540245	10.459755	52
9	9.493466	9.977835	9.515631	10.484369	9.515930	9.975277	9.540653	10.459347	51
10	9.493851	9.977794	9.516057	10.483943	9.516294	9.975233	9.541061	10.458939	50
11	9.494236	9.977752	9.516484	10.483516	9.516657	9.975189	9.541468	10.458532	49
12	9.494621	9.977711	9.516910	10.483090	9.517020	9.975145	9.541875	10.458125	48
13	9.495005	9.977669	9.517335	10.482665	9.517382	9.975101	9.542281	10.457719	47
14	9.495388	9.977628	9.517761	10.482239	9.517745	9.975057	9.542688	10.457312	46
15	9.495772	9.977586	9.518186	10.481814	9.518107	9.975013	9.543094	10.456906	45
16	9.496154	9.977544	9.518610	10.481390	9.518468	9.974969	9.543499	10.456501	44
17	9.496537	9.977503	9.519034	10.480966	9.518829	9.974925	9.543905	10.456095	43
18	9.496919	9.977461	9.519458	10.480542	9.519190	9.974880	9.544310	10.455690	42
19	9.497301	9.977419	9.519882	10.480118	9.519551	9.974836	9.544715	10.455285	41
20	9.497682	9.977377	9.520305	10.479695	9.519911	9.974792	9.545119	10.454881	40
21	9.498064	9.977335	9.520728	10.479272	9.520271	9.974748	9.545524	10.454476	39
22	9.498444	9.977293	9.521151	10.478849	9.520631	9.974703	9.545928	10.454072	38
23	9.498825	9.977251	9.521573	10.478427	9.520990	9.974659	9.546331	10.453669	37
24	9.499204	9.977209	9.521995	10.478005	9.521349	9.974614	9.546735	10.453265	36
25	9.499584	9.977167	9.522417	10.477583	9.521707	9.974570	9.547138	10.452862	35
26	9.499963	9.977125	9.522838	10.477162	9.522066	9.974525	9.547540	10.452460	34
27	9.500342	9.977083	9.523259	10.476741	9.522424	9.974481	9.547943	10.452057	33
28	9.500721	9.977041	9.523680	10.476320	9.522781	9.974436	9.548345	10.451655	32
29	9.501099	9.976999	9.524100	10.475900	9.523138	9.974391	9.548747	10.451253	31
30	9.501476	9.976957	9.524520	10.475480	9.523495	9.974347	9.549149	10.450851	30
31	9.501854	9.976914	9.524940	10.475060	9.523852	9.974302	9.549550	10.450450	29
32	9.502231	9.976872	9.525360	10.474641	9.524208	9.974257	9.549951	10.450049	28
33	9.502607	9.976830	9.525778	10.474222	9.524564	9.974212	9.550352	10.449648	27
34	9.502984	9.976787	9.526197	10.473803	9.524920	9.974167	9.550752	10.449248	26
35	9.503360	9.976745	9.526615	10.473385	9.525275	9.974122	9.551153	10.448847	25
36	9.503735	9.976702	9.527033	10.472967	9.525630	9.974077	9.551552	10.448448	24
37	9.504110	9.976660	9.527451	10.472549	9.525984	9.974032	9.551952	10.448048	23
38	9.504485	9.976617	9.527868	10.472132	9.526339	9.973987	9.552351	10.447649	22
39	9.504860	9.976574	9.528285	10.471715	9.526693	9.973942	9.552750	10.447250	21
40	9.505234	9.976532	9.528702	10.471298	9.527046	9.973897	9.553149	10.446851	20
41	9.505608	9.976489	9.529119	10.470881	9.527400	9.973852	9.553548	10.416452	19
42	9.505981	9.976446	9.529535	10.470465	9.527753	9.973807	9.553946	10.446054	18
43	9.506354	9.976404	9.529951	10.470049	9.528105	9.973761	9.554344	10.445656	17
44	9.506727	9.976361	9.530366	10.469634	9.528458	9.973716	9.554741	10.445259	16
45	9.507099	9.976318	9.530781	10.469219	9.528810	9.973671	9.555139	10.444861	15
46	9.507471	9.976275	9.531196	10.468804	9.529161	9.973625	9.555536	10.444464	14
47	9.507843	9.976232	9.531611	10.468389	9.529513	9.973580	9.555933	10.444067	13
48	9.508214	9.976189	9.532025	10.467875	9.529864	9.973535	9.556329	10.443671	12
49	9.508585	9.976146	9.532439	10.467561	9.530215	9.973489	9.556725	10.443275	11
50	9.508956	9.976103	9.532853	10.467147	9.530565	9.973444	9.557121	10.442879	10
51	9.509326	9.976060	9.533266	10.466784	9.530915	9.973398	9.557517	10.442483	9
52	9.509696	9.976017	9.533679	10.466321	9.531265	9.973352	9.557913	10.442087	8
53	9.510065	9.975974	9.534092	10.465905	9.531614	9.973307	9.558308	10.441692	7
54	9.510434	9.975930	9.534504	10.465496	9.531963	9.973261	9.558703	10.441297	6
55	9.510803	9.975887	9.534916	10.465084	9.532312	9.973215	9.559097	10.440903	5
56	9.511172	9.975844	9.535328	10.464672	9.532661	9.973169	9.559491	10.440509	4
57	9.511540	9.975800	9.535739	10.464261	9.533009	9.973124	9.559885	10.440115	3
58	9.511907	9.975757	9.536150	10.463850	9.533357	9.973078	9.560279	10.439721	2
59	9.512275	9.975714	9.536561	10.463439	9.533704	9.973032	9.560673	10.439327	1
60	9.512642	9.975670	9.536972	10.463028	9.534052	9.972986	9.561066	10.438934	0
	Cosine.	Sine.	Cotang.	Tang.	Cosine.	Sine.	Cotang.	Tang.	
	71 Deg.				70 Deg.				

	20 Deg.				
	Sine.	Cosine.	Tang.	Cotang.	
0	9.534052	9.972986	9.561066	10.438934	60
1	9.534399	9.972940	9.561459	10.438541	59
2	9.534745	9.972894	9.561851	10.438149	58
3	9.535092	9.972848	9.562244	10.437756	57
4	9.535438	9.972802	9.562636	10.437364	56
5	9.535783	9.972755	9.563028	10.436972	55
6	9.536129	9.972709	9.563419	10.436581	54
7	9.536474	9.972663	9.563811	10.436189	53
8	9.536818	9.972617	9.564202	10.435798	52
9	9.537163	9.972570	9.564593	10.435407	51
10	9.537507	9.972524	9.564983	10.435017	50
11	9.537851	9.972478	9.565373	10.434627	49
12	9.538194	9.972431	9.565763	10.434237	48
13	9.538538	9.972385	9.566153	10.433847	47
14	9.538880	9.972338	9.566542	10.433458	46
15	9.539223	9.972291	9.566932	10.433068	45
16	9.539565	9.972245	9.567320	10.432680	44
17	9.539907	9.972198	9.567709	10.432291	43
18	9.540249	9.972151	9.568098	10.431902	42
19	9.540590	9.972105	9.568486	10.431514	41
20	9.540931	9.972058	9.568873	10.431127	40
21	9.541272	9.972011	9.569261	10.430739	39
22	9.541613	9.971964	9.569648	10.430352	38
23	9.541953	9.971917	9.570035	10.429965	37
24	9.542293	9.971870	9.570422	10.429578	36
25	9.542632	9.972823	9.570809	10.429191	35
26	9.542971	9.971776	9.571195	10.428805	34
27	9.543310	9.971729	9.571581	10.428419	33
28	9.543649	9.971682	9.571967	10.428033	32
29	9.543987	9.971635	9.572352	10.427648	31
30	9.544325	9.971588	9.572738	10.427262	30
31	9.544663	9.971540	9.573123	10.426877	29
32	9.545000	9.971493	9.573507	10.426493	28
33	9.545338	9.971446	9.573892	10.426108	27
34	9.545674	9.971398	9.574276	10.425724	26
35	9.546011	9.971351	9.574660	10.425340	25
36	9.546347	9.971303	9.575044	10.424956	24
37	9.546683	9.971256	9.575427	10.424573	23
38	9.547019	9.971208	9.575810	10.424190	22
39	9.547354	9.971161	9.576193	10.423807	21
40	9.547689	9.971113	9.576576	10.423424	20
41	9.548024	9.971066	9.576959	10.423041	19
42	9.548359	9.971018	9.577341	10.422659	18
43	9.548693	9.970970	9.577723	10.422277	17
44	9.549027	9.970922	9.578104	10.421896	16
45	9.549360	9.970874	9.578486	10.421514	15
46	9.549693	9.970827	9.578867	10.421133	14
47	9.550026	9.970779	9.579248	10.420752	13
48	9.550359	9.970731	9.579629	10.420371	12
49	9.550692	9.970683	9.580009	10.419991	11
50	9.551024	9.970635	9.580389	10.419611	10
51	9.551356	9.970586	9.580769	10.419231	9
52	9.551687	9.970538	9.581149	10.418851	8
53	9.552018	9.970490	9.581528	10.418472	7
54	9.552349	9.970442	9.581907	10.418093	6
55	9.552680	9.970394	9.582286	10.417714	5
56	9.553010	9.970345	9.582665	10.417335	4
57	9.553341	9.970297	9.583044	10.416956	3
58	9.553670	9.970249	9.583422	10.416578	2
59	9.554000	9.970200	9.583800	10.416200	1
60	9.554329	9.970152	9.584177	10.415823	0
	Cosine.	Sine.	Cotang.	Tang.	
	69 Deg.				

	21 Deg.				
	Sine.	Cosine.	Tang.	Cotang.	
0	9.554329	9.970152	9.584177	10.415823	60
1	9.554658	9.970103	9.584555	10.415445	59
2	9.554987	9.970055	9.584932	10.415068	58
3	9.555315	9.970006	9.585309	10.414691	57
4	9.555643	9.969957	9.585686	10.414314	56
5	9.555971	9.969909	9.586062	10.413938	55
6	9.556299	9.969860	9.586439	10.413561	54
7	9.556626	9.969811	9.586815	10.413185	53
8	9.556953	9.969762	9.587190	10.412810	52
9	9.557280	9.969714	9.587566	10.412434	51
10	9.557606	9.969665	9.587941	10.412059	50
11	9.557932	9.969616	9.588316	10.411684	49
12	9.558258	9.969567	9.588691	10.411309	48
13	9.558583	9.969518	9.589066	10.410934	47
14	9.558909	9.969469	9.589440	10.410560	46
15	9.559234	9.969420	9.589814	10.410186	45
16	9.559558	9.969370	9.590188	10.409812	44
17	9.559883	9.969321	9.590562	10.409438	43
18	9.560207	9.969272	9.590935	10.409065	42
19	9.560531	9.969223	9.591308	10.408692	41
20	9.560855	9.969173	9.591681	10.408319	40
21	9.561178	9.969124	9.592054	10.407946	39
22	9.561501	9.969075	9.592426	10.407574	38
23	9.561824	9.969025	9.592799	10.407201	37
24	9.562146	9.968976	9.593471	10.406829	36
25	9.562468	9.968926	9.593542	10.406458	35
26	9.562790	9.968877	9.593914	10.406086	34
27	9.563112	9.968827	9.594285	10.405715	33
28	9.563433	9.968777	9.594656	10.405344	32
29	9.563755	9.968728	9.595027	10.404973	31
30	9.564075	9.968678	9.595398	10.404602	30
31	9.564396	9.968628	9.595768	10.404232	29
32	9.564716	9.968578	9.596138	10.403862	28
33	9.565036	9.968528	9.596508	10.403492	27
34	9.565356	9.968479	9.596878	10.403122	26
35	9.565676	9.968429	9.597347	10.402753	25
36	9.565935	9.968379	9.597616	10.402384	24
37	9.566314	9.968329	9.597985	10.402015	23
38	9.566632	9.968278	9.598354	10.401646	22
39	9.566951	9.968228	9.598722	10.401278	21
40	9.567269	9.968178	9.599091	10.400909	20
41	9.567587	9.968128	9.599459	10.400541	19
42	9.567904	9.968078	9.599827	10.400173	18
43	9.568222	9.968027	9.600194	10.399806	17
44	9.568539	9.967977	9.600562	10.399438	16
45	9.568856	9.967927	9.600929	10.399071	15
46	9.569172	9.967876	9.601296	10.398704	14
47	9.569488	9.967826	9.601663	10.398337	13
48	9.569804	9.967775	9.602029	10.397971	12
49	9.570120	9.967725	9.602395	10.397605	11
50	9.570435	9.967674	9.602761	10.397239	10
51	9.570751	9.967624	9.603127	10.396873	9
52	9.571066	9.967573	9.603493	10.396507	8
53	9.571380	9.967522	9.603858	10.396142	7
54	9.571695	9.967471	9.604223	10.395777	6
55	9.572009	9.967421	9.604588	10.395412	5
56	9.572323	9.967370	9.604953	10.395047	4
57	9.572636	9.967319	9.605317	10.394683	3
58	9.572950	9.967268	9.605682	10.394318	2
59	9.573263	9.967217	9.606046	10.393954	1
60	9.573475	9.967166	9.606410	10.393590	0
	Cosine.	Sine.	Cotang.	Tang.	
	68 Deg.				

	22 Deg.				23 Deg.				
	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	
0	9.573575	9.967166	9.606410	10.393590	9.591878	9.964026	9.627852	10.372148	60
1	9.573888	9.967115	9.606773	10.393227	9.592176	9.963972	9.628203	10.371797	59
2	9.574200	9.967064	9.607137	10.392863	9.592473	9.963919	9.628554	10.371446	58
3	9.574512	9.967013	9.607500	10.392500	9.592770	9.963865	9.628905	10.371095	57
4	9.574824	9.966961	9.607863	10.392137	9.593067	9.963811	9.629255	10.370745	56
5	9.575136	9.966910	9.608225	10.391775	9.593363	9.963757	9.629606	10.370394	55
6	9.575447	9.966859	9.608588	10.391412	9.593659	9.963704	9.629956	10.370044	54
7	9.575758	9.966808	9.608950	10.391050	9.593955	9.963650	9.630306	10.369694	53
8	9.576069	9.966756	9.609312	10.390688	9.594251	9.963596	9.630656	10.369344	52
9	9.576379	9.966705	9.609674	10.390326	9.594547	9.963542	9.631005	10.368 95	51
10	9.576689	9.966653	9.610036	10.389964	9.594842	9.963488	9.631355	10.368645	50
11	9.576999	9.966602	9.610397	10.389603	9.595137	9.963434	9.631704	10.368296	49
12	9.577309	9.966550	9.610759	10.389241	9.595432	9.963379	9.632053	10.367947	48
13	9.577618	9.966499	9.611120	10.388880	9.595727	9.963325	9.632402	10.367598	47
14	9.577927	9.966447	9.611480	10.388520	9.596021	9.963271	9.632750	10.367250	46
15	9.578236	9.966395	9.611841	10.388159	9.596315	9.963217	9.633099	10.366901	45
16	9.578545	9.966344	9.612201	10.387799	9.596609	9.963163	9.633447	10.366553	44
17	9.578853	9.966292	9.612561	10.387439	9.596903	9.963108	9.633795	10.366205	43
18	9.579162	9.966240	9.612921	10.387079	9.597196	9.963054	9.634143	10.365857	42
19	9.579470	9.966188	9.613281	10.386719	9.597490	9.962999	9.634490	10.365510	41
20	9.579777	9.966136	9.613641	10.386359	9.597783	9.962945	9.634838	10.365162	40
21	9.580085	9.966085	9.614000	10.386000	9.598075	9.962890	9.635185	10.364815	39
22	9.580392	9.966033	9.614359	10.385641	9.598368	9.962836	9.63553.	10.364468	38
23	9.580699	9.965981	9.614718	10.385282	9.598660	9.962781	9.635879	10.364121	37
24	9.581005	9.965929	9.615077	10.384923	9.598952	9.962727	9.636226	10.363774	36
25	9.581312	9.965876	9.615435	10.384565	9.599244	9.962672	9.636572	10.363428	35
26	9.581618	9.965824	9.615793	10.384207	9.599536	9.962617	9.636919	10.363081	34
27	9.581924	9.965772	9.616151	10.383849	9.599827	9.962562	9.637265	10.362735	33
28	9.582229	9.965720	9.616509	10.383491	9.600118	9.962508	9.637611	10.362389	32
29	9.582535	9.965668	9.616867	10.383133	9.600409	9.962453	9.637956	10.362044	31
30	9.582840	9.965615	9.617224	10.382776	9.600700	9.962398	9.638302	10.361698	30
31	9.583145	9.965563	9.617582	10.382418	9.600990	9.962343	9.638647	10.361355	29
32	9.583419	9.965511	9.617939	10.382061	9.601280	9.962288	9.638992	10.361008	28
33	9.583754	9.965458	9.618295	10.381705	9.601570	9.962233	9.639537	10.360663	27
34	9.584058	9.965406	9.618652	10.381348	9.601860	9.962178	9.639682	10.360318	26
35	9.584361	9.965353	9.619008	10.380992	9.602150	9.962123	9.640027	10.359973	25
36	9.584665	9.965301	9.619364	10.380636	9.602439	9.962067	9.640871	10.359629	24
37	9.584968	9.965248	9.619720	10.380280	9.602728	9.962012	9.640716	10.359284	23
38	9.585272	9.965195	9.620076	10.379924	9.603017	9.961957	9.641060	10.358940	22
39	9.585574	9.965143	9.620432	10.379568	9.603305	9.961902	9.641404	10.358596	21
40	9.585877	9.965090	9.620787	10.379213	9.603594	9.961846	9.641747	10.358253	20
41	9.586179	9.965037	9.621142	10.378858	9.603882	9.961791	9.642091	10.357909	19
42	9.586482	9.964984	9.621497	10.378503	9.604170	9.961735	9.642434	10.357566	18
43	9.586783	9.964931	9.621852	10.378148	9.604457	9.961680	9.642777	10.357225	17
44	9.587085	9.964879	9.622207	10.377793	9.604745	9.961624	9.643120	10.356880	16
45	9.587386	9.964826	9.622561	10.377439	9.605032	9.961569	9.643463	10.356537	15
46	9.587688	9.964773	9.622915	10.377085	9.605319	9.961513	9.643806	10.356194	14
47	9.587989	9.964720	9.623269	10.376731	9.605606	9.961458	9.644148	10.355852	13
48	9.588289	9.964666	9.623623	10.376377	9.605892	9.961402	9.644490	10.355510	12
49	9.588590	9.964613	9.623976	10.376024	9.606179	9.961346	9.644832	10.355168	11
50	9.588890	9.964560	9.624330	10.375670	9.606465	9.961290	9.645174	10.354826	10
51	9.589190	9.964507	9.624683	10.375317	9.606751	9.961235	9.645516	10.354484	9
52	9.589489	9.964454	9.625036	10.374964	9.607036	9.961179	9.645857	10.354143	8
53	9.589789	9.964400	9.625388	10.374612	9.607322	9.961125	9.646199	10.353801	7
54	9.590088	9.964347	9.625741	10.374259	9.607607	9.961067	9.646540	10.353460	6
55	9.590387	9.964294	9.626093	10.373907	9.607892	9.961011	9.646881	10.353119	5
56	9.590686	9.964240	9.626445	10.373555	9.608177	9.960955	9.647222	10.352778	4
57	9.590984	9.964187	9.626797	10.373203	9.608461	9.960899	9.647562	10.352438	3
58	9.591282	9.964133	9.627149	10.372851	9.608745	9.960843	9.647903	10.352097	2
59	9.591580	9.964080	9.627501	10.372499	9.609029	9.960786	9.948243	10.351757	1
60	9.591878	9.964026	9.627852	10.372148	9.609313	9.960730	9.648583	10.351417	0
	Cosine.	Sine.	Cotan.	Tang.	Cosine.	Sine.	Cotan.	Tang.	
	67 Deg.				66 Deg.				

	24 Deg.				25 Deg.				
	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	
0	9.609313	9.960730	9.648583	10.351417	9.625948	9.957276	9.668673	10.331327	60
1	9.609597	9.960674	9.648923	10.351077	9.626219	9.957217	9.669002	10.330998	59
2	9.603880	9.960618	9.649990	10.350737	9.626490	9.957158	9.669332	10.330668	58
3	9.610164	9.960561	9.649602	10.350398	9.626760	9.957099	9.669661	10.330339	57
4	9.610447	9.960505	9.649942	10.350058	9.627030	9.957040	9.669991	10.330009	56
5	9.610729	9.960448	9.650281	10.349719	9.627300	9.956981	9.670320	10.329681	55
6	9.611012	9.960392	9.650620	10.349380	9.627570	9.956921	9.670649	10.32935	54
7	9.611294	9.960335	9.650959	10.349041	9.627840	9.956862	9.670977	10.329023	53
8	9.611576	9.960279	9.651297	10.348703	9.628109	9.956803	9.671306	10.328694	52
9	9.611858	9.960222	9.651636	10.348364	9.628378	9.956744	9.671635	10.328365	51
10	9.612140	9.960165	9.651974	10.348026	9.628647	9.956684	9.671963	10.328037	50
11	9.612421	9.960109	9.652312	10.347688	9.628916	9.956625	9.672291	10.327709	49
12	9.612702	9.960052	9.652650	10.347350	9.629185	9.956566	9.672619	10.327381	48
13	9.612983	9.959995	9.652988	10.347012	9.629458	9.956506	9.672947	10.327053	47
14	9.613264	9.959938	9.653326	10.346674	9.629721	9.956447	9.673274	10.326726	46
15	9.613545	9.959882	9.653663	10.346337	9.629989	9.956387	9.673602	10.326398	45
16	9.613825	9.959825	9.654000	10.346000	9.630257	9.956327	9.673929	10.326071	44
17	9.614105	9.959768	9.654337	10.345663	9.630524	9.956268	9.674257	10.325743	43
18	9.614385	9.957911	9.654674	10.345326	9.630794	9.956208	9.674584	10.325416	42
19	9.614665	9.959654	9.655011	10.344989	9.631059	9.956148	9.674911	10.325089	41
20	9.614944	9.959596	9.655348	10.344652	9.631326	9.956089	9.675237	10.324763	40
21	9.615223	9.959539	9.655684	10.344316	9.631593	9.956029	9.675564	10.324436	39
22	9.615502	9.959489	9.656020	10.343980	9.631859	9.955969	9.675890	10.324110	38
23	9.615781	9.959425	9.656356	10.343644	9.632125	9.955909	9.676217	10.323783	37
24	9.616060	9.959368	9.656692	10.343308	9.632392	9.955849	9.676543	10.323457	36
25	9.616338	9.959310	9.657028	10.342972	9.632658	9.955789	9.676869	10.323131	35
26	9.616616	9.959253	9.657364	10.342636	9.632923	9.955729	9.677194	10.322806	34
27	9.616894	9.959195	9.657699	10.342301	9.633189	9.955669	9.677520	10.322480	33
28	9.617172	9.959138	9.658034	10.341966	9.633454	9.955609	9.677846	10.322154	32
29	9.617450	9.959080	9.658369	10.341631	9.633719	9.955548	9.678171	10.321829	31
30	9.617727	9.959023	9.658704	10.341296	9.633984	9.955488	9.678496	10.321504	30
31	9.618004	9.958965	9.659039	10.340961	9.634249	9.955428	9.678821	10.321179	29
32	9.618281	9.958908	9.659373	10.340327	9.634514	9.955368	9.679146	10.320854	28
33	9.618558	9.958850	9.659708	10.340292	9.634778	9.955307	9.679471	10.320529	27
34	9.618834	9.958792	9.660042	10.339958	9.635042	9.955247	9.679795	10.320205	26
35	9.619110	9.958734	9.660376	10.339624	9.635306	9.955186	9.680120	10.319880	25
36	9.619386	9.958677	9.660710	10.339290	9.635570	9.955126	9.680444	10.319556	24
37	9.619662	9.958619	9.661043	10.338957	9.635834	9.955065	9.680768	10.319232	23
38	9.619938	9.958561	9.661377	10.338623	9.636097	9.955005	9.681092	10.318908	22
39	9.620213	9.958503	9.661710	10.338290	9.636360	9.954944	9.681416	10.318584	21
40	9.620488	9.958445	9.662043	10.337957	9.636623	9.954883	9.681740	10.318260	20
41	9.620763	9.958387	9.662376	10.337624	9.636886	9.954823	9.682063	10.317937	19
42	9.621038	9.958329	9.662709	10.337291	9.637148	9.954762	9.682387	10.317613	18
43	9.621313	9.958271	9.663042	10.336958	9.637411	9.954701	9.682710	10.317290	17
44	9.621587	9.958213	9.663375	10.336625	9.637673	9.954640	9.683033	10.316967	16
45	9.621861	9.958154	9.663707	10.336293	9.637935	9.954579	9.683356	10.316644	15
46	9.622135	9.958096	9.664039	10.335961	9.638197	9.954518	9.683679	10.316321	14
47	9.622409	9.958038	9.664371	10.335629	9.638458	9.954457	9.684001	10.315999	13
48	9.622682	9.957979	9.664703	10.335297	9.638720	9.954396	9.684324	10.315676	12
49	9.622056	9.957921	9.665035	10.334965	9.638981	9.954334	9.684646	10.315354	11
50	9.623229	9.957863	9.665366	10.334634	9.639242	9.954275	9.684968	10.315032	10
51	9.623502	9.957804	9.665698	10.334302	9.639503	9.954213	9.685290	10.314710	9
52	9.623774	9.957746	9.666029	10.333971	9.639764	9.954152	9.685612	10.314388	8
53	9.624017	9.957687	9.666360	10.333640	9.640024	9.954090	9.685934	10.314066	7
54	9.624319	9.957628	9.666691	10.333309	9.640284	9.954029	9.686255	10.313745	6
55	9.624591	9.957570	9.667021	10.332979	9.640544	9.953968	9.686577	10.313423	5
56	9.624863	9.957511	9.667352	10.332648	9.640804	9.953906	9.686898	10.313102	4
57	9.625135	9.957452	9.667682	10.332318	9.641064	9.953845	9.687219	10.312781	3
58	9.625406	9.957393	9.668013	10.331987	9.641324	9.953783	9.687540	10.312460	2
59	9.625677	9.957335	9.668343	10.331657	9.641583	9.953722	9.687861	10.312139	1
60	9.625948	9.957276	9.668673	10.331327	9.641842	9.953660	9.688182	10.311818	0
	Cosine.	Sine.	Cotan.	Tang.	Cosine.	Sine.	Cotan.	Tang.	
	65 Deg.				64 Deg.				

′	26 Deg. Sine.	Cosine.	Tang.	Cotang.	27 Deg. Sine.	Cosine.	Tang.	Cotang.	
0	9.641842	9.953660	9.688182	10.311818	9.657047	9.949881	9.707166	10.292834	60
1	9.642101	9.953599	9.688502	10.311498	9.657295	9.949816	9.707478	10.292522	59
2	9.642360	9.953537	9.688823	10.311177	9.657542	9.949752	9.707790	10.292210	58
3	9.642618	9.953475	9.689143	10.310857	9.657790	9.949688	9.708102	10.291898	57
4	9.642877	9.953413	9.689463	10.310537	9.658037	9.949623	9.708414	10.291586	56
5	9.643135	9.953352	9.689783	10.310217	9.658284	9.949558	9.708726	10.291274	55
6	9.643393	9.953290	9.690103	10.309897	9.658531	9.949494	9.709037	10.290963	54
7	9.643650	9.953228	9.690423	10.309577	9.658778	9.949429	9.709349	10.290651	53
8	9.643908	9.953166	9.690742	10.309258	9.659025	9.949364	9.709660	10.290340	52
9	9.644165	9.953104	9.691062	10.308938	9.659271	9.949300	9.709971	10.290029	51
10	9.644423	9.953042	9.691381	10.308619	9.659517	9.949235	9.710282	10.289718	50
11	9.644680	9.952980	9.691700	10.308300	9.659763	9.949170	9.710593	10.289407	49
12	9.644936	9.952918	9.692019	10.307981	9.660009	9.949105	9.710904	10.289096	48
13	9.645193	9.952855	9.692338	10.307662	9.660255	9.949040	9.711215	10.288785	47
14	9.645450	9.952793	9.692656	10.307344	9.560501	9.948975	9.711525	10.288475	46
15	9.645706	9.952731	9.692975	10.307025	9.660746	9.948910	9.711836	10.288164	45
16	9.645962	9.952669	9.693293	10.306707	9.660991	9.948845	9.712146	10.287854	44
17	9.646218	9.952606	9.693612	10.306388	9.661236	9.948780	9.712456	10.287544	43
18	9.646474	9.952544	9.693930	10.306070	9.661481	9.948715	9.712766	10.287234	42
19	9.646729	9.952481	9.694248	10.305752	9.661726	9.948650	9.713076	10.286924	41
20	9.646984	9.952419	9.694566	10.305434	9.661970	9.948584	9.713386	10.286614	40
21	9.647240	9.952356	9.694883	10.305117	9.662214	9.948519	9.713696	10.286304	39
22	9.647494	9.952294	9.695201	10.304799	9.662459	9.948454	9.714005	10.285995	38
23	9.647749	9.952231	9.695518	10.304482	9.662703	9.948388	9.714314	10.285686	37
24	9.648004	9.952168	9.695836	10.304164	9.662946	9.948323	9.714624	10.285376	36
25	9.648258	9.952106	9.696153	10.303847	9.663190	9.948257	9.714933	10.285067	35
26	9.648512	9.952043	9.696470	10.303530	9.663433	9.948192	9.715242	10.284758	34
27	9.648766	9.951980	9.696787	10.303213	9.663677	9.948126	9.715551	10.284449	33
28	9.649020	9.951917	9.697103	10.302897	9.663920	9.948060	9.715860	10.284140	32
29	9.649274	9.951854	9.697420	10.302580	9.664163	9.947995	9.716168	10.283832	31
30	9.649527	9.951791	9.697736	10.302264	9.664406	9.947929	9.716477	10.283523	30
31	9.649781	9.951728	9.698053	10.301947	9.664648	9.947863	9.716785	10.283215	29
32	9.650034	9.951665	9.698369	10.301631	9.664891	9.947797	9.717093	10.282907	28
33	9.650287	9.951602	9.698685	10.301315	9.665133	9.947731	9.717401	10.282599	27
34	9.650539	9.951539	9.699001	10.300999	9.665375	9.947665	9.717709	10.282291	26
35	9.650792	9.951476	9.699316	10.300684	9.665617	9.947600	9.718017	10.281983	25
36	9.651044	9.951412	9.699632	10.300368	9.665859	9.947533	9.718325	10.281672	24
37	9.651297	9.951349	9.699947	10.300053	9.666100	9.947467	9.718633	10.281367	23
38	9.651549	9.951286	9.700263	10.299737	9.666342	9.947401	9.718940	10.281060	22
39	9.651800	9.951222	9.700578	10.299422	9.666583	9.947335	9.719248	10.280752	21
40	9.652052	9.951159	9.700893	10.299107	9.666824	9.947269	9.719555	10.280445	20
41	9.652304	9.951096	9.701208	10.298792	9.667065	9.947203	9.719862	10.280138	19
42	9.652555	9.951032	9.701523	10.298477	9.667305	9.947136	9.720169	10.279831	18
43	9.652806	9.950968	9.701837	10.298163	9.667546	9.947070	9.720476	10.279524	17
44	9.653057	9.950905	9.702152	10.297848	9.667786	9.947004	9.720783	10.279217	16
45	9.653308	9.950841	9.702466	10.297534	9.668027	9.946937	9.721089	10.278911	15
46	9.653558	9.950778	9.702781	10.297219	9.668267	9.946871	9.721396	10.278604	14
47	9.053808	9.950714	9.703095	10.296905	9.668506	9.946804	9.721702	10.278298	13
48	9.654059	9.950650	9.703409	10.296591	9.668746	9.946738	9.722009	10.277991	12
49	9.654309	9.950586	9.703722	10.296278	9.668086	9.946671	9.722315	10.277685	11
50	9.654558	9.950522	9.704036	10.295964	9.669225	9.946604	9.722621	10.277379	10
51	9.654808	9.950458	9.704350	10.295650	9.669464	9.946538	9.722927	10.277073	9
52	9.655058	9.950394	9.704663	10.295337	9.669703	9.946471	9.723232	10.276768	8
53	9.655307	9.950330	9.704976	10.295024	9.669942	9.946404	9.723538	10.276462	7
54	9.655556	9.950266	9.705290	10.294710	9.670181	9.946337	9.723844	10.276156	6
55	9.655805	9.950202	9.705603	10.294397	9.670419	9.946270	9.724149	10.275851	5
56	9.656054	9.950138	9.705916	10.294084	9.670658	9.946203	9.724454	10.275546	4
57	9.656303	9.950074	9.706228	10.293772	9.670896	9.946136	9.724760	10.275240	3
58	9.656561	9.950010	9.706541	10.293459	9.671134	9.946069	9.725065	10.274935	2
59	9.656799	9.949945	9.706854	10.293146	9.671372	9.946002	9.725370	10.274630	1
60	9.657047	9.949881	9.707166	10.292834	9.671609	9.945935	9.725674	10.274326	0
′	Cosine. 63 Deg.	Sine.	Cotan.	Tang.	Cosine. 62 Deg.	Sine.	Cotan.	Tang.	

	28 Deg.				29 Deg.				
	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	
0	9.671609	9.945935	9.725674	10.274326	9.685571	9.941819	9.743752	10.256248	60
1	9.671847	9.945868	9.725979	10.274021	9.685799	9.941749	9.744050	10.255950	59
2	9.672084	9.945800	9.726284	10.273716	9.686027	9.941679	9.744348	10.255652	58
3	9.672321	9.945733	9.726588	10.273412	9.686254	9.941609	9.744645	10.255355	57
4	9.672558	9.945666	9.726892	10.273108	9.686482	9.941539	9.744943	10.255057	56
5	9.672795	9.945598	9.727197	10.272803	9.686709	9.941469	9.745240	10.254760	55
6	9.673032	9.945531	9.727501	10.272499	9.686936	9.941398	9.745538	10.254462	54
7	9.673268	9.945464	9.727805	10.272195	9.687163	9.941328	9.745835	10.254165	53
8	9.673505	9.945396	9.728109	10.271891	9.687389	9.941258	9.746132	10.253868	52
9	9.673741	9.945328	9.728412	10.271588	9.687616	9.941187	9.746429	10.253571	51
10	9.673977	9.945261	9.728716	10.271284	9.687843	9.941117	9.746726	10.253274	50
11	9.674213	9.945193	9.729020	10.270980	9.688069	9.941046	9.747023	10.252977	49
12	9.674448	9.945125	9.729323	10.270677	9.688295	9.940975	9.747319	10.252681	48
13	9.674684	9.945058	9.729626	10.270374	9.688521	9.940905	9.747616	10.252384	47
14	9.674919	9.944990	9.729929	10.270071	9.688747	9.940834	9.747913	10.252087	46
15	9.675155	9.944922	9.730233	10.269767	9.688972	9.940763	9.748209	10.251791	45
16	9.675390	9.944854	9.730535	10.269465	9.689198	9.940693	9.748505	10.251495	44
17	9.675624	9.944786	9.730838	10.269162	9.689423	9.940622	9.748801	10.251199	43
18	9.675859	9.944718	9.731141	10.268859	9.689648	9.940551	9.749097	10.250903	42
19	9.676094	9.944650	9.731444	10.268556	9.689873	9.940480	9.749393	10.250607	41
20	9.676328	9.944582	9.731746	10.268254	9.690098	9.940409	9.749689	10.250311	40
21	9.676562	9.944514	9.732048	10.267952	9.690323	9.940338	9.749985	10.250015	39
22	9.676796	9.944446	9.732351	10.267649	9.690548	9.940267	9.750281	10.249719	38
23	9.677030	9.944377	9.732653	10.267347	9.690772	9.940196	9.750576	10.249424	37
24	9.677264	9.944309	9.732955	10.267045	9.690996	9.940125	9.750872	10.249128	36
25	9.677498	9.944241	9.733257	10.266743	9.691220	9.940054	9.751167	10.248833	35
26	9.677731	9.944172	9.733558	10.266442	9.691444	9.939982	9.751462	10.248538	34
27	9.677964	9.944104	9.733860	10.266140	9.691668	9.939911	9.751757	10.248243	33
28	9.678197	9.944036	9.734162	10.265838	9.691892	9.939840	9.752052	10.247948	32
29	9.678430	9.943967	9.734463	10.265537	9.692115	9.939768	9.752347	10.247653	31
30	9.678663	9.943899	9.734764	10.265236	9.692339	9.939697	9.752642	10.247358	30
31	9.678895	9.943830	9.735066	10.264934	9.692562	9.939625	9.752937	10.247063	29
32	9.679128	9.943761	9.735367	10.264633	9.692785	9.939554	9.753231	10.246769	28
33	9.679360	9.943693	9.735668	10.264332	9.693008	9.939482	9.753526	10.246474	27
34	9.679592	9.943624	9.735969	10.264031	9.693231	9.939410	9.753820	10.246180	26
35	9.679824	9.943555	9.736269	10.263731	9.693453	9.939339	9.754115	10.245885	25
36	9.680056	9.943486	9.736570	10.263430	9.693676	9.939267	9.754409	10.245591	24
37	9.680288	9.943417	9.736870	10.263130	9.693898	9.939195	9.754703	10.245297	23
38	9.680519	9.943348	9.737171	10.262829	9.694120	9.939123	9.754997	10.245003	22
39	9.680750	9.943279	9.737471	10.262529	9.694342	9.939052	9.755291	10.244709	21
40	9.680982	9.943210	9.737771	10.262229	9.694564	9.938980	9.755585	10.244415	20
41	9.681213	9.943141	9.738071	10.261929	9.694786	9.938908	9.755878	10.244122	19
42	9.681443	9.943072	9.738371	10.261629	9.695007	9.938836	9.756172	10.243828	18
43	9.681674	9.943003	9.738671	10.261329	9.695229	9.938763	9.756465	10.243535	17
44	9.681905	9.942934	9.738971	10.261029	9.695450	9.938691	9.756759	10.243241	16
45	9.682135	9.942864	9.739271	10.260729	9.696671	9.938619	9.757052	10.242948	15
46	9.682365	9.942795	9.739570	10.260430	9.695892	9.938547	9.757345	10.242655	14
47	9.682595	9.942726	9.739870	10.260130	9.696113	9.938475	9.757638	10.242362	13
48	9.682825	9.942656	9.740169	10.259831	9.696334	9.938402	9.757931	10.242069	12
49	9.683055	9.942587	9.740468	10.259532	9.696554	9.938330	9.758224	10.241776	11
50	9.683284	9.942517	9.740767	10.259233	9.696775	9.938258	9.758517	10.241483	10
51	9.683514	9.942448	9.741066	10.258934	9.696995	9.938185	9.758810	10.241190	9
52	9.683743	9.942378	9.741365	10.258635	9.697215	9.938113	9.759102	10.240898	8
53	9.683972	9.942308	9.741664	10.258336	9.697435	9.938040	9.759395	10.240605	7
54	9.684201	9.942239	9.741962	10.258038	9.697654	9.937967	9.759687	10.240313	6
55	9.684430	9.942169	9.742261	10.257739	9.697874	9.937895	9.759979	10.240021	5
56	9.684658	9.942099	9.742559	10.257441	9.698094	9.937822	9.760272	10.239728	4
57	9.684887	9.942029	9.742858	10.257142	9.698313	9.937749	9.760564	10.239436	3
58	9.685115	9.941959	9.743156	10.256844	9.698532	9.937676	9.760856	10.239144	2
59	9.685343	9.941889	9.743454	10.256546	9.698751	9.937604	9.761148	10.238852	1
60	9.685571	9.941819	9.743752	10.256248	9.698970	9.937531	9.761439	10.238561	0
	Cosine.	Sine.	Cotan.	Tang.	Cosine.	Sine.	Cotan.	Tang.	
	61 Deg.				60 Deg.				

	30 Deg.				31 Deg.				
	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	
0	9.698970	9.937531	9.761439	10.238561	9.711839	9.933066	9.778774	10.221226	6
1	9.699189	9.937458	9.761731	10.238269	9.712050	9.932990	9.779060	10.220940	5
2	9.699407	9.937385	9.762023	10.237977	9.712260	9.932914	9.779346	10.220654	5
3	9.699626	9.937312	9.762314	10.237686	9.712469	9.932838	9.779632	10.220368	5
4	9.699844	9.937238	9.762606	10.237394	9.712679	9.932762	9.779918	10.220082	5
5	9.700062	9.937165	9.762897	10.237103	9.712889	9.932685	9.780203	10.219797	5
6	9.700280	9.937092	9.763188	10.236812	9.713098	9.932609	9.780489	10.219511	5
7	9.700498	9.937019	9.763479	10.236521	9.713308	9.932533	9.780775	10.219225	5
8	9.700716	9.936946	9.763770	10.236230	9.713517	9.932457	9.781060	10.218940	5
9	9.700933	9.936872	9.764061	10.235939	9.713726	9.932380	9.781346	10.218654	5
10	9.701151	9.936799	9.764352	10.235648	9.713935	9.932304	9.781631	10.218369	5
11	9.701368	9.936725	9.764643	10.235357	9.714144	9.932228	9.781916	10.218084	4
12	9.701585	9.936652	9.764933	10.234067	9.714352	9.932151	9.782201	10.217799	4
13	9.701802	9.936578	9.765224	10.234776	9.714561	9.932075	9.782486	10.217514	4
14	9.702019	9.936505	9.765514	10.234486	9.714769	9.931998	9.782771	10.217229	4
15	9.702236	9.936431	9.765805	10.234195	9.714978	9.931921	9.783056	10.216944	4
16	9.702452	9.936357	9.766095	10.233905	9.715186	9.931845	9.783341	10.216659	4
17	9.702669	9.936284	9.766385	10.233615	9.715394	9.931768	9.783626	10,216374	4
18	9.702885	9.936210	9.766675	10.233325	9.715602	9.931691	9.783910	10.216090	4
19	9.703101	9.936136	9.766965	10.233035	9.715809	9.931614	9.784195	10.215805	4
20	9.703317	9.936062	9.767255	10.232745	9.716017	9.931537	9.784479	10.215521	4
21	9.703533	9.935988	9.767545	10.232455	9.716224	9.931460	9.784764	50.215236	3
22	9.703749	9.935914	9.767834	10.232166	9.716432	9.931383	9.785048	10.214952	3
23	9.703964	9.935840	9.768124	10.231876	9.716639	9.931306	9.785332	10.214668	3
24	9.704179	9.935766	9.768414	10.231586	9.716846	9.931229	9.785616	10.214384	3
25	9.704395	9.935692	9.768703	10.231297	9.717053	9.931152	9.785900	10.214100	3
26	9.704610	9.935618	9.768992	10.231008	9.717259	9.931075	9.786184	10.213816	3
27	9.704825	9.935543	9.769281	10.230719	9.717466	9.930998	9.786468	10.213532	3
28	9.705040	9.935469	9.769571	10.230429	9.717673	9.930921	9.786752	10.213248	3
29	9.705254	9.935395	9.769860	10.230140	9.717879	9.930843	9.787036	10.212964	3
30	9.705469	9.935320	9.770148	10.229852	9.718085	9.930766	9.787319	10.212681	3
31	9.705683	9.935246	9.770437	10.229563	9.718291	9.930688	9.787603	10.212397	2
32	9.705898	9.935171	9.770726	10.229274	9.718497	9.930611	9.787886	10.212114	2
33	9.706112	9.935097	9.771015	10.228985	9.718703	9.930533	9.788170	10.211830	2
34	9.706326	9.935022	9.771303	10.228697	9.718909	9.930456	9.788453	10.211547	2
35	9.706539	9.934948	9.771592	10.228408	9.719114	9.930378	9.788736	10.211264	2
36	9.706753	9.934873	9.771880	10.228120	9.719320	9.930300	9.789019	10.210981	2
37	9.706967	9.934798	9.772168	10.227832	9.719525	9.930223	9.789302	10.210698	
38	9.707180	9.934723	9.772457	10.227543	9.719730	9.930145	9.789585	10.210415	
39	9.707393	9.934649	9.772745	10.227255	9.719935	9.930067	9.789868	10.210132	
40	9.707606	9.934574	9.773033	10.226967	9.720140	9.929989	9.790151	10.209849	
41	9.707819	9.934499	9.773321	10.226679	9.720345	9.929911	9.790434	10.209566	
42	9.708032	9.934424	9.773608	10.226392	9.720549	9.929833	9.790716	10.209284	
43	9.708245	9.934349	9.773896	10.226104	9.720754	9.929755	9.790999	10.209001	
44	9.708458	9.934274	9.774184	10.225816	9.720958	9.929677	9.791281	10.208719	
45	9.708670	9.934199	9.774471	10.225529	9.721162	9.929599	9.791563	10.208437	
46	9.708882	9.934123	9.774759	10.225241	9.721366	9.929521	9.791846	10.208154	
47	9.709094	9.934048	9.775046	10.224954	9.721570	9.929442	9.792129	10.20787	
48	9.709306	9.933973	9.775333	10.224667	9.721774	9.929364	9.792410	10.20759	
49	9.709518	9.933898	9.775621	10.224379	9.721978	9.929286	9.792692	10.20730	
50	9.709730	9.933822	9.775908	10.224092	9.722181	9.929207	9.792974	10.20702	
51	9.709941	9.933747	9.776195	10.223805	9.722385	9.929129	9.793256	10.20674	
52	9.710153	9.933671	9.776482	10.223518	9.722588	9.929050	9.793538	10.20646	
53	9.710364	9.933596	9.776769	10.223232	9.722791	9.928972	9.793819	10.20618	
54	9.710575	9.933520	9.777055	10.222945	9.722994	9.928853	9.794101	10.20589	
55	9.710786	9.933445	9.777342	10.222658	9.723197	9.928815	9.794383	10.20561	
56	9.710997	9.933369	9.777628	10.222372	9.723400	9.928736	9.794664	10.20533	
57	9.711208	9.933293	9.777915	10.222085	9.723603	9.928657	9.794946	10.20505	
58	9.711419	9.933217	9.778201	10.221799	9.723805	9.928578	9.795227	10.20477	
59	9.711629	9.933141	9.778488	10.221512	9.724007	9.928499	9.795508	10.20449	
60	9.711839	9.933066	9.778774	10.221226	9.724210	9.928420	9.795789	10.20421	
	Cosine.	Sine.	Cotan.	Tang.	Cosine.	Sine.	Cotan.	Tang.	
	59 Deg.				58 Deg.				

	32 Deg.				33 Deg.				
	Sine.	Cosine.	Tang.	Cotang	Sine.	Cosine.	Tang.	Cotang.	
0	9.724210	9.928420	9.795789	10.204211	9.736109	9.923591	9.812517	10.187483	60
1	9.724412	9.928342	9.796070	10.203930	9.736303	9.923509	9.812794	10.187206	59
2	9.724614	9.928263	9.796351	10.203649	9.736498	9.923427	9.813070	10.186930	58
3	9.724816	9.928183	9.796632	10.203368	9.736692	9.923345	9.813347	10.186653	57
4	9.725017	9.928104	9.796913	10.203087	9.736886	9.923263	9.813623	10.186377	56
5	9.725219	9.928025	9.797194	10.202806	9.737080	9.923181	9.813899	10.186101	55
6	9.725420	9.927946	9.797474	10.202526	9.737274	9.923098	9.814176	10.185824	54
7	9.725622	9.927867	9.797755	10.202245	9.737467	9.923016	9.814452	10.185548	53
8	9.725823	9.927787	9.798036	10.201964	9.737661	9.922933	9.814728	10.185272	52
9	9.726024	9.927708	9.798316	10.201684	9.737855	9.922851	9.815004	10.184996	51
10	9.726225	9.927629	9.798596	10.201404	9.738048	9.922768	9.815280	10.184720	50
11	9.726426	9.927548	9.798877	10.201123	9.738241	9.922686	9.815555	10.184445	49
12	9.726626	9.927470	9.799157	10.200843	9.738434	9.922603	9.815831	10.184169	48
13	9.726827	9.927390	9.799437	10.200563	9.738627	9.922520	9.816107	10.183893	47
14	9.727027	9.927310	9.799717	10.200283	9.738820	9.922438	9.816382	10.183618	46
15	9.727228	9.927231	9.799997	10.200003	9.739013	9.922355	9.816658	10.183342	45
16	9.727428	9.927151	9.800277	10.199723	9.739206	9.922272	9.816933	10.183067	44
17	9.727628	9.927071	9.800557	10.199443	9.739398	9.922189	9.817209	10.182791	43
18	9.727828	9.926991	9.800836	10.199164	9.739590	9.922106	9.817584	10.182516	42
19	9.728027	9.926911	9.801116	10.198884	9.739783	9.922023	9.817759	10.182241	41
20	9.728227	9.926831	9.801396	10.198604	9.739975	9.921940	9.818035	10.181965	40
21	9.728427	9.926751	9.801675	10.198325	9.740167	9.921857	9.818310	10.181690	39
22	9.728626	9.926671	9.801955	10.198045	9.740359	9.921774	9.818585	10.181415	38
23	9.728825	9.926591	9.802234	10.197766	9.740550	9.921691	9.818860	10.181140	37
24	9.729024	9.926511	9.802513	10.197487	9.740742	9.921607	9.819135	10.180865	36
25	9.729223	9.926431	9.802792	10.197208	9.740934	9.921524	9.819410	10.180590	35
26	9.729422	9.926351	9.803072	10.196928	9.741125	9.921441	9.819684	10.180316	34
27	9.729621	9.926270	9.803351	10.196649	9.741316	9.921357	9.819959	10.180041	33
28	9.729820	9.926190	9.803630	10.196370	9.741508	9.921274	9.820234	10.179766	32
29	9.730018	9.926110	9.803909	10.196091	9.741699	9.921190	9.820508	10.179492	31
30	9.730217	9.926029	9.804187	10.195813	9.741889	9.921107	9.820783	10.179217	30
31	9.730415	9.925949	9.804466	10.195534	9.742080	9.921023	9.821057	10.178943	29
32	9.730613	9.925868	9.804745	10.195255	9.742271	9.920939	9.821332	10.178668	28
33	9.730811	9.925788	9.805023	10.194977	9.742462	9.920856	9.821606	10.178394	27
34	9.731009	9.925707	9.805302	10.194698	9.742652	9.920772	9.821880	10.178120	26
35	9.731206	9.925626	9.805580	10.194420	9.742842	9.920688	9.822154	10.177846	25
36	9.731404	9.925545	9.805859	10.194141	9.743033	9.920604	9.822429	10.177571	24
37	9.731602	9.925465	9.806137	10.193863	9.743223	9.920520	9.822703	10.177297	23
38	9.731799	9.925384	9.806415	10.193585	9.743413	9.920436	9.822977	10.177023	22
39	9.731996	9.925303	9.806693	10.193307	9.743602	9.920352	9.823251	10.176749	21
40	9.732193	9.925222	9.806971	10.193029	9.743792	9.920268	9.823524	10.176476	20
41	9.732390	9.925141	9.807249	10.192751	9.743982	9.920184	9.823798	10.176202	19
42	9.732587	9.925060	9.807527	10.192473	9.744171	9.920099	9.824072	10.175928	18
43	9.732784	9.924979	9.807805	10.192195	9.744361	9.920015	9.824345	10.175655	17
44	9.732980	9.924897	9.808083	10.191917	9.744550	9.919931	9.824619	10.175381	16
45	9.733177	9.924816	9.808361	10.191639	9.744739	9.919846	9.824893	10.175107	15
46	9.733373	9.924735	9.808638	10.191362	9.744928	9.919762	9.825160	10.174834	14
47	9.733569	9.924654	9.808916	10.191084	9.745117	9.919677	9.825439	10.174561	13
48	9.733765	9.924572	9.809193	10.190807	9.745306	9.919593	9.825713	10.174287	12
49	9.733961	9.924491	9.809471	10.190529	9.745494	9.919508	9.825986	10.174014	11
50	9.734157	9.924409	9.809748	10.190252	9.745683	9.919425	9.826259	10.173741	10
51	9.734353	9.924328	9.810025	10.189975	9.745871	9.919339	9.826532	10.173468	9
52	9.734549	9.924246	9.810302	10.189698	9.746060	9.919254	9.826805	10.173195	8
53	9.734744	9.924164	9.810580	10.189420	9.746248	9.919169	9.827078	10.172922	7
54	9.734939	9.924083	9.810857	10.189143	9.746436	9.919085	9.827351	10.172649	6
55	9.735135	9.924001	9.811134	10.188866	9.746624	9.919000	9.827624	10.172376	5
56	9.735330	9.923919	9.811410	10.188590	9.746812	9.918915	9.827897	10.172103	4
57	9.735525	9.923837	9.811687	10.188313	9.746999	9.918830	9.828170	10.171830	3
58	9.735719	9.923755	9.811964	10.188036	9.747187	9.918745	9.828442	10.171558	2
59	9.735914	9.923673	9.812241	10.187759	9.747374	9.918659	9.828715	10.171285	1
60	9.736109	9.923591	9.812517	10.187483	9.747562	9.918574	9.828987	10.171013	0
	Cosine	Sine.	Cotan.	Tang.	Cosine.	Sine.	Cotan.	Tang.	
	57 Deg.				56 Deg.				

	34 Deg.				35 Deg.				
	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine	Tang.	Cotang.	
0	9.747562	9.918574	9.828987	10.171013	9.758591	9.913365	9.845227	10.154773	60
1	9.747749	9.918489	9.829260	10.170740	9.7587.6	9.913276	9.845496	10.154504	59
2	9.747936	9.918404	9.829532	10.170468	9.758952	9.913187	9.845764	10.154236	58
3	9.748123	9.918318	9.829805	10.170195	9.759132	9.913099	9.846033	10.153967	57
4	9.748310	9.918233	9.830077	10.169923	9.759312	9.913010	9.846302	10.153698	56
5	9.748497	9.918147	9.830349	10.169651	9.759492	9.912922	9.846570	10.153430	55
6	9.748683	9.918062	9.830621	10.169379	9.759672	9.912833	9.846839	10.153161	54
7	9.748870	9.917976	9.830893	10.169107	9.759852	9.912744	9.847108	10.152892	53
8	9.749056	9.917891	9.831165	10.168835	9.760031	9.912655	9.847376	10.152624	52
9	9.749243	9.917805	9.831437	10.168563	9.760211	9.912566	9.847644	10.152356	51
10	9.749429	9.917719	9.831709	10.168291	9.760390	9.912477	9.847913	10.152087	50
11	9.749615	9.917634	9.831981	10.168019	9.760569	9.912388	9.848181	10.151819	49
12	9.749801	9.917548	9.832253	10.167747	9.760748	9.912299	9.848449	10.151551	48
13	9.749987	9.917466	9.832525	10.167475	9.760927	9.912210	9.848717	10.151283	47
14	9.750172	9.917376	9.832796	10.167204	9.761106	9.912121	9.848986	10.151014	46
15	9.750358	9.917290	9.833068	10.166932	9.761285	9.912031	9.849254	10.150746	45
16	9.750543	9.917204	9.833339	10.166661	9.761464	9.911942	9.849522	10.150478	44
17	9.750729	9.917118	9.833611	10.166389	9.761642	9.911853	9.849790	10.150210	43
18	9.750914	9.917032	9.833882	10.166118	9.761821	9.911763	9.850057	10.149943	42
19	9.751099	9.916946	9.834154	10.165846	9.761999	9.911674	9.850325	10.149675	41
20	9.751284	9.916859	9.834425	10.165575	9.762177	9.911584	9.850593	10.149407	40
21	9.751469	9.916773	9.834696	10.165304	9.762356	9.911495	9.850861	10.149139	39
22	9.751654	9.916687	9.834967	10.165033	9.762534	9.911405	9.851129	10.148871	38
23	9.751839	9.916600	9.835238	10.164762	9.762712	9.911315	9.851396	10.148604	37
24	9.752023	9.916514	9.835509	10.164491	9.762889	9.911226	9.851664	10.148336	36
25	9.752208	9.916427	9.835780	10.164220	9.763067	9.911136	9.851931	10.148069	35
26	9.752392	9.916341	9.836051	10.163949	9.763245	9.911046	9.852199	10.147801	34
27	9.752576	9.916254	9.836322	10.163678	9.763422	9.910956	9.852466	10.147534	33
28	9.752760	9.916167	9.836593	10.163407	9.763600	9.910866	9.852733	10.147267	32
29	9.752944	9.916081	9.836864	10.163136	9.763777	9.910776	9.853001	10.146999	31
30	9.753128	9.915994	9.837134	10.162866	9.763954	9.910686	9.853268	10.146732	30
31	9.753312	9.915907	9.837405	10.162595	9.764131	9.910596	9.853535	10.146465	29
32	9.753495	9.915820	9.837675	10.162325	9.764308	9.910506	9.853802	10.146198	28
33	9.753679	9.915733	9.837946	10.162054	9.764485	9.910415	9.854069	10.145931	27
34	9.753862	9.915646	9.838216	10.161784	9.764662	9.910325	9.854336	10.145664	26
35	9.754046	9.915559	9.838487	10.161513	9.764838	9.910235	9.854603	10.145397	25
36	9.754229	9.915472	9.838757	10.161243	9.765015	9.910144	9.854870	10.145130	24
37	9.754412	9.915385	9.839027	10.160973	9.765191	9.910054	9.855137	10.144863	23
38	9.754595	9.915297	9.839297	10.160703	9.765367	9.909963	9.855404	10.144596	22
39	9.754778	9.915210	9.839568	10.160432	9.765544	9.909873	9.855671	10.144329	21
40	9.754960	9.915123	9.839838	10.160162	9.765720	9.909782	9.855938	10.144062	20
41	9.755143	9.915035	9.840108	10.159892	9.765896	9.909691	9.856204	10.143796	19
42	9.755326	9.914948	9.840378	10.159622	9.766072	9.909601	9.856471	10.143529	18
43	9.755508	9.914860	9.840648	10.159352	9.766247	9.909510	9.856737	10.143263	17
44	9.755690	9.914773	9.840917	10.159083	9.766423	9.909419	9.857004	10.142996	16
45	9.755872	9.914685	9.841187	10.158813	9.766598	9.909328	9.857270	10.142730	15
46	9.756054	9.914598	9.841457	10.158543	9.766774	9.909237	9.857537	10.142463	14
47	9.756236	9.914510	9.841727	10.158273	9.766949	9.909146	9.857803	10.142197	13
48	9.756418	9.914422	9.841996	10.158004	9.767124	9.909055	9.858069	10.141931	12
49	9.756600	9.914334	9.842266	10.157734	9.767300	9.908964	9.858336	10.141664	11
50	9.756782	9.914246	9.842535	10.157465	9.767475	9.908873	9.858602	10.141398	10
51	9.756963	9.914158	9.842805	10.157195	9.767649	9.908781	9.858868	10.141132	9
52	9.757144	9.914070	9.843074	10.156926	9.767824	9.908690	9.859134	10.140866	8
53	9.757326	9.913982	9.843343	10.156657	9.767999	9.908599	9.859400	10.140600	7
54	9.757507	9.913894	9.843612	10.156388	9.768173	9.908507	9.859666	10.140334	6
55	9.757688	9.913806	9.843882	10.156118	9.768348	9.908416	9.859932	10.140068	5
56	9.757869	9.913718	9.844151	10.155849	9.768522	9.908324	9.860198	10.139802	4
57	9.758050	9.913630	9.844420	10.155580	9.768697	9.908233	9.860464	10.139536	3
58	9.758230	9.913541	9.844689	10.155311	9.768871	9.908141	9.860730	10.139270	2
59	9.758411	9.913453	9.844958	10.155042	9.769045	9.908049	9.860995	10.139005	1
60	9.758591	9.913365	9.845227	10.154773	9.769219	9.907958	9.861261	10.138739	0
	Cosine.	Sine.	Cotan.	Tang.	Cosine.	Sine.	Cotan.	Tang.	
	55 Deg.				54 Deg.				

	36 Deg.				37 Deg.				
′	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	
0	9.769219	9.907958	9.861261	10.138739	9.779463	9.902349	9.877114	10.122886	60
1	9.769393	9.907866	9.861527	10.138473	9.779631	9.902253	9.877377	10.122623	59
2	9.769566	9.907774	9.861792	10.138208	9.779798	9.902158	9.877640	10.122360	58
3	9.769740	9.907682	9.862058	10.137942	9.779966	9.902063	9.877903	10.122097	57
4	9.769913	9.907590	9.862323	10.137677	9.780133	9.901967	9.878165	10.121835	56
5	9.770087	9.907498	9.862589	10.137411	9.780300	9.901872	9.878428	10.121572	55
6	9.770260	9.907406	9.862854	10.137146	9.780467	9.901776	9.878691	10.121309	54
7	9.770433	9.907314	9.863119	10.136881	9.780634	9.901681	9.878953	10.121047	53
8	9.770606	9.907222	9.863385	10.136615	9.780801	9.901585	9.879216	10.120784	52
9	9.770779	9.907129	9.863650	10.136350	9.780968	9.901490	9.879478	10.120522	51
10	9.770952	9.907037	9.863915	10.136085	9.781134	9.901394	9.879741	10.120259	50
11	9.771125	9.906945	9.864180	10.135820	9.781301	9.901298	9.880003	10.119997	49
12	9.771298	9.906852	9.864445	10.135555	9.781468	9.901202	9.880265	10.119735	48
13	9.771470	9.906760	9.864710	10.135290	9.781634	9.901106	9.880528	10.119472	47
14	9.771643	9.906667	9.864975	10.135025	9.781800	9.901010	9.880790	10.119210	46
15	9.771815	9.906575	9.865240	10.134760	9.781966	9.900914	9.881052	10.118948	45
16	9.771987	9.906482	9.865505	10.134495	9.782132	9.900818	9.881314	10.118686	44
17	9.772159	9.906389	9.865770	10.134230	9.782298	9.900722	9.881577	10.118423	43
18	9.772331	9.906296	9.866035	10.133965	9.782464	9.900626	9.881839	10.118161	42
19	9.772503	9.906204	9.866300	10.133700	9.782630	9.900529	9.882101	10.117899	41
20	9.772675	9.906111	9.866564	10.133436	9.782796	9.900433	9.882363	10.117637	40
21	9.772847	9.906018	9.866829	10.133171	9.782961	9.900337	9.882625	10.117375	39
22	9.773018	9.905925	9.867094	10.132906	9.783127	9.900240	9.882887	10.117113	38
23	9.773190	9.905832	9.867358	10.132642	9.783292	9.900144	9.883148	10.116852	37
24	9.773361	9.905739	9.867623	10.132377	9.783458	9.900047	9.883410	10.116590	36
25	9.773533	9.905645	9.867887	10.132113	9.783623	9.899951	9.883672	10.116328	35
26	9.773704	9.905552	9.868152	10.131848	9.783788	9.899854	9.883934	10.116066	34
27	9.773875	9.905459	9.868416	10.131584	9.783953	9.899757	9.884196	10.115804	33
28	9.774046	9.905366	9.868680	10.131320	9.784118	9.899660	9.884457	10.115543	32
29	9.774217	9.905272	9.868945	10.131055	9.784282	9.899564	9.884719	10.115281	31
30	9.774388	9.905179	9.869209	10.130791	9.784447	9.899467	9.884980	10.115020	30
31	9.774558	9.905085	9.869473	10.130527	9.784612	9.899370	9.885242	10.114758	29
32	9.774729	9.904992	9.869737	10.130263	9.784776	9.899273	9.885504	10.114496	28
33	9.774899	9.904898	9.870001	10.129999	9.784941	9.899176	9.885765	10.114235	27
34	9.775070	9.904804	9.870265	10.129735	9.785105	9.899078	9.886026	10.113974	26
35	9.775240	9.904711	9.870529	10.129471	9.785269	9.898981	9.886288	10.113712	25
36	9.775410	9.904617	9.870793	10.129207	9.785433	9.898884	9.886549	10.113451	24
37	9.775580	9.904523	9.871057	10.128943	9.785597	9.898787	9.886811	10.113189	23
38	9.775759	9.904429	9.871321	10.128679	9.785761	9.898689	9.887072	10.112928	22
39	9.775920	9.904335	9.871585	10.128415	9.785925	9.898592	9.887333	10.112667	21
40	9.776090	9.904241	9.871849	10.128151	9.786089	9.898494	9.887594	10.112406	20
41	9.776259	9.904147	9.872112	10.127888	9.786252	9.898397	9.887855	10.112145	19
42	9.776429	9.904053	9.872376	10.127624	9.786416	9.898299	9.888116	10.111884	18
43	9.776598	9.903959	9.872640	10.127360	9.786579	9.898202	9.888378	10.111622	17
44	9.776768	9.903864	9.872903	10.127097	9.786742	9.898104	9.888639	10.111361	16
45	9.776937	9.903770	9.873167	10.126833	9.786906	9.898006	9.888900	10.111100	15
46	9.777106	9.903676	9.873430	10.126570	9.787069	9.897908	9.889161	10.110839	14
47	9.777275	9.903581	9.873694	10.126306	9.787232	9.897810	9.889421	10.110579	13
48	9.777444	9.903487	9.873957	10.126043	9.787395	9.897712	9.889682	10.110318	12
49	9.777613	9.903392	9.874220	10.125780	9.787557	9.897614	9.889943	10.110057	11
50	9.777781	9.903298	9.874484	10.125516	9.787720	9.897516	9.890204	10.109796	10
51	9.777950	9.903203	9.874747	10.125253	9.787883	9.897418	9.890465	10.109535	9
52	9.778119	9.903108	9.875010	10.124990	9.788045	9.897320	9.890725	10.109275	8
53	9.778287	9.903014	9.875273	10.124727	9.788208	9.897222	9.890986	10.109014	7
54	9.778455	9.902919	9.875537	10.124463	9.788370	9.897123	9.891247	10.108753	6
55	9.778624	9.902824	9.875800	10.124200	9.788532	9.897025	9.891507	10.108493	5
56	9.778792	9.902729	9.876063	10.123937	9.788694	9.896926	9.891768	10.108232	4
57	9.778960	9.902634	9.876326	10.123674	9.788856	9.896828	9.892028	10.107972	3
58	9.779128	9.902539	9.876589	10.123411	9.789018	9.896729	9.892289	10.107711	2
59	9.779295	9.902444	9.876852	10.123148	9.789180	9.896631	9.892549	10.107451	1
60	9.779463	9.902349	9.877114	10.122886	9.789342	9.896532	9.892810	10.107190	0
	Cosine.	Sine.	Cotan.	Tang.	Cosine.	Sine.	Cotan.	Tang.	′
	53 Deg.				52 Deg.				

	38 Deg.				39 Deg.				
	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	
0	9.789342	9.896532	9.892810	10.107190	9.798872	9.890503	9.908369	10.091631	60
1	9.789504	9.896433	9.893070	10.106930	9.799028	9.890400	9.908628	10.091372	59
2	9.789665	9.896335	9.893331	10.106669	9.799184	9.890298	9.908886	10.091114	58
3	9.789827	9.896236	9.893591	10.106409	9.799339	9.890195	9.909144	10.090856	57
4	9.789988	9.896137	9.893851	10.106149	9.799495	9.890093	9.909402	10.090598	56
5	9.790149	9.896038	9.894111	10.105889	9.799651	9.889990	9.909660	10.090340	55
6	9.790310	9.895939	9.894372	10.105628	9.799806	9.889888	9.909918	10.090082	54
7	9.790471	9.895840	9.894632	10.105368	9.799962	9.889785	9.910177	10.089823	53
8	9.790632	9.895741	9.894892	10.105108	9.800117	9.889682	9.910435	10.089565	52
9	9.790793	9.895641	9.895152	10.104848	9.800272	9.889579	9.910693	10.089307	51
10	9.790954	9.895542	9.895412	10.104588	9.800427	9.889477	9.910951	10.089049	50
11	9.791115	9.895443	9.895672	10.104328	9.800582	9.889374	9.911209	10.088791	49
12	9.791276	9.895343	9.895932	10.104068	9.800737	9.889271	9.911467	10.088533	48
13	9.791436	9.895244	9.896192	10.103808	9.800892	9.889168	9.911725	10.088275	47
14	9.791596	9.895145	9.896452	10.103548	9.801047	9.889064	9.911982	10.088018	46
15	9.791757	9.895045	9.896712	10.103288	9.801201	9.888961	9.912240	10.087760	45
16	9.791917	9.894945	9.896971	10.103029	9.801356	9.888858	9.912498	10.087502	44
17	9.792077	9.894846	9.897231	10.102769	9.801511	9.888755	9.912756	10.087244	43
18	9.792237	9.894746	9.897491	10.102509	9.801665	9.888651	9.913014	10.086986	42
19	9.792397	9.894646	9.897751	10.102249	9.801819	9.888548	9.913271	10.086729	41
20	9.792557	9.894546	9.898010	10.101990	9.801973	9.888444	9.913529	10.086471	40
21	9.792716	9.894446	9.898270	10.101730	9.802128	9.888341	9.913787	10.086213	39
22	9.792876	9.894346	9.898530	10.101470	9.802282	9.888237	9.914044	10.085956	38
23	9.793035	9.894246	9.898789	10.101211	9.802436	9.888134	9.914302	10.085698	37
24	9.793195	9.894146	9.899049	10.100951	9.802589	9.888030	9.914560	10.085440	36
25	9.793354	9.894046	9.899308	10.100692	9.802743	9.887926	9.914817	10.085183	35
26	9.793514	9.893946	9.899568	10.100432	9.802897	9.887822	9.915075	10.084925	34
27	9.793673	9.893846	9.899827	10.100173	9.803050	9.887718	9.915332	10.084668	33
28	9.793832	9.893745	9.900087	10.099913	9.803204	9.887614	9.915590	10.084410	32
29	9.793991	9.893645	9.900346	10.099654	9.803357	9.887510	9.915847	10.084153	31
30	9.794150	9.893544	9.900605	10.099395	9.803511	9.887406	9.916104	10.083896	30
31	9.794308	9.893444	9.900864	10.099136	9.803664	9.887302	9.916362	10.083638	29
32	9.794467	9.893343	9.901124	10.098876	9.803817	9.887198	9.916619	10.083381	28
33	9.794626	9.893243	9.901383	10.098617	9.803970	9.887093	9.916877	10.083123	27
34	9.794784	9.893142	9.901642	10.098358	9.804123	9.886989	9.917134	10.082866	26
35	9.794942	9.893041	9.901901	10.098099	9.804276	9.886885	9.917391	10.082609	25
36	9.795101	9.892940	9.902160	10.097840	9.804428	9.886780	9.917648	10.082352	24
37	9.795259	9.892839	9.902420	10.097580	9.804581	9.886676	9.917906	10.082094	23
38	9.795417	9.892739	9.902679	10.097321	9.804734	9.886571	9.918163	10.081837	22
39	9.795575	9.892638	9.902938	10.097062	9.804886	9.886466	9.918420	10.081580	21
40	9.795733	9.892536	9.903197	10.096803	9.805039	9.886362	9.918677	10.081323	20
41	9.795891	9.892435	9.903456	10.096544	9.805191	9.886257	9.918934	10.081066	19
42	9.796049	9.892334	9.903714	10.096286	9.805343	9.886152	9.919191	10.080809	18
43	9.796206	9.892233	9.903973	10.096027	9.805495	9.886047	9.919448	10.080552	17
44	9.796364	9.892132	9.904232	10.095768	9.805647	9.885942	9.919705	10.080295	16
45	9.796521	9.892030	9.904491	10.095509	9.805799	9.885837	9.919962	10.080038	15
46	9.796679	9.891929	9.904750	10.095250	9.805951	9.885732	9.920219	10.079781	14
47	9.796836	9.891827	9.905008	10.094992	9.806103	9.885627	9.920476	10.079524	13
48	9.796993	9.891726	9.905267	10.094733	9.806253	9.885522	9.920733	10.079267	12
49	9.797150	9.891624	9.905526	10.094474	9.806406	9.885416	9.920990	10.079010	11
50	9.797307	9.891523	9.905785	10.094215	9.806557	9.885311	9.921247	10.078753	10
51	9.797464	9.891421	9.906043	10.093957	9.806709	9.885205	9.921503	10.078497	9
52	9.797621	9.891319	9.906302	10.093698	9.806860	9.885100	9.921760	10.078240	8
53	9.797777	9.891217	9.906560	10.093440	9.807111	9.884994	9.922017	10.077983	7
54	9.797934	9.891115	9.906819	10.093181	9.807163	9.884889	9.922274	10.077726	6
55	9.798091	9.891013	9.907077	10.092923	9.807314	9.884783	9.922530	10.077470	5
56	9.798247	9.890911	9.907336	10.092664	9.807465	9.884677	9.922784	10.077213	4
57	9.798403	9.890809	9.907594	10.092406	9.807615	9.884572	9.923044	10.076956	3
58	9.798560	9.890707	9.907853	10.092147	9.807766	9.884466	9.923300	10.076700	2
59	9.798716	9.890605	9.908111	10.091889	9.807917	9.884360	9.923557	10.076443	1
60	9.798872	9.890503	9.908369	10.091631	9.808067	9.884254	9.923814	10.076186	0
	Cosine.	Sine.	Cotan.	Tang.	Cosine.	Sine.	Cotan.	Tang.	
	51 Deg.				50 Deg.				

	40 Deg.				41 Deg.				
	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	
0	9.808067	9.884254	9.923814	10.076186	9.816943	9.877780	9.939163	10.060837	60
1	9.808218	9.884148	9.924040	10.075930	9.817088	9.877670	9.939418	10.060582	59
2	9.808368	9.884042	9.924327	10.075673	9.817233	9.877560	9.939763	10.060327	58
3	9.808519	9.883936	9.924583	10.075417	9.817379	9.877450	9.939928	10.060072	57
4	9.808669	9.883829	9.924840	10.075160	9.817524	9.877340	9.940183	10.059817	56
5	9.808819	9.883723	9.925096	10.074904	9.817668	9.877230	9.940439	10.059561	55
6	9.808969	9.883617	9.925352	10.074648	9.817813	9.877120	9.940694	10.059306	54
7	9.809119	9.883510	9.925609	10.074391	9.817958	9.877010	9.940949	10.059051	53
8	9.809269	9.883404	9.925865	10.074135	9.818103	9.876899	9.941204	10.058796	52
9	9.809419	9.883297	9.926122	10.073878	9.818247	9.876789	9.941459	10.058541	51
10	9.809569	9.883191	9.926378	10.073622	9.818392	9.876678	9.941713	10.058287	50
11	9.809718	9.883084	9.926634	10.073366	9.818536	9.876568	9.941968	10.058032	49
12	9.809868	9.882977	9.926890	10.073110	9.818681	9.876457	9.942223	10.057777	48
13	9.810017	9.882871	9.927147	10.072853	9.818825	9.876347	9.942478	10.057522	47
14	9.810167	9.882764	9.927403	10.072597	9.818969	9.876236	9.942733	10.057267	46
15	9.810316	9.882657	9.927659	10.072341	9.819113	9.876125	9.942988	10.057012	45
16	9.810465	9.882550	9.927915	10.072085	9.819257	9.876014	9.943243	10.056757	44
17	9.810614	9.882443	9.928171	10.071829	9.819401	9.875904	9.943498	10.056502	43
18	9.810763	9.882336	9.928427	10.071573	9.819545	9.875793	9.943752	10.056248	42
19	9.810912	9.882229	9.928684	10.071316	9.819689	9.875682	9.944007	10.055993	41
20	9.811061	9.882121	9.928940	10.071060	9.819832	9.875571	9.944262	10.055738	40
21	9.811210	9.882014	9.929196	10.070804	9.819976	9.875459	9.944517	10.055483	39
22	9.811358	9.881907	9.929452	10.070548	9.820120	9.875348	9.944771	10.055229	38
23	9.811507	9.881799	9.929708	10.070292	9.820263	9.875237	9.945026	10.054974	37
24	9.811655	9.881692	9.929964	10.070036	9.820406	9.875126	9.945281	10.054719	36
25	9.811804	9.881584	9.930220	10.069780	9.820550	9.875014	9.945535	10.054465	35
26	9.811952	9.881477	9.930475	10.069525	9.820693	9.874903	9.945790	10.054210	34
27	9.812100	9.881369	9.930731	10.069269	9.820836	9.874791	9.946045	10.053955	33
28	9.812248	9.881261	9.930987	10.069013	9.820979	9.874680	9.946299	10.053701	32
29	9.812396	9.881153	9.931243	10.068757	9.821122	9.874568	9.946554	10.053446	31
30	9.812544	9.881046	9.931499	10.068501	9.821265	9.874456	9.946808	10.053192	30
31	9.812692	9.880938	9.931755	10.068245	9.821407	9.874344	9.947063	10.052937	29
32	9.812840	9.880830	9.932010	10.067990	9.821550	9.874232	9.947318	10.052682	28
33	9.812988	9.880722	9.932266	10.067734	9.821693	9.874121	9.947572	10.052428	27
34	9.813135	9.880613	9.932522	10.067478	9.821835	9.874009	9.947827	10.052173	26
35	9.813283	9.880505	9.932778	10.067222	9.821977	9.873896	9.948081	10.051919	25
36	9.813430	9.880397	9.933033	10.066967	9.822120	9.873784	9.948335	10.051665	24
37	9.813578	9.880289	9.933289	10.066711	9.822262	9.873672	9.948590	10.051410	23
38	9.813725	9.880180	9.933545	10.066455	9.822404	9.873560	9.948844	10.051156	22
39	9.813872	9.880072	9.933800	10.066200	9.822546	9.873448	9.949099	10.050901	21
40	9.814019	9.879963	9.934056	10.065944	9.822688	9.873335	9.949353	10.050647	20
41	9.814166	9.879855	9.934311	10.065689	9.822830	9.873223	9.949608	10.050392	19
42	9.814313	9.879746	9.934567	10.065433	9.822972	9.873110	9.949862	10.050138	18
43	9.814460	9.879637	9.934822	10.065178	9.823114	9.872998	9.950116	10.049884	17
44	9.814607	9.879529	9.935078	10.064922	9.823255	9.872885	9.950371	10.049629	16
45	9.814753	9.879420	9.935333	10.064667	9.823397	9.872772	9.950625	10.049375	15
46	9.814900	9.879311	9.935589	10.064411	9.823539	9.872659	9.950879	10.049121	14
47	9.815046	9.879202	9.935844	10.064156	9.823680	9.872547	9.951133	10.048867	13
48	9.815193	9.879093	9.936100	10.063900	9.823821	9.872434	9.951388	10.048612	12
49	9.815339	9.878984	9.936355	10.063645	9.823963	9.872321	9.951642	10.048358	11
50	9.815485	9.878875	9.936611	10.063389	9.824104	9.872208	9.951896	10.048104	10
51	9.815632	9.878766	9.936866	10.063134	9.824245	9.872095	9.952150	10.047850	9
52	9.815778	9.878656	9.937121	10.062879	9.824386	9.871981	9.952405	10.047595	8
53	9.815924	9.878547	9.937377	10.062623	9.824527	9.871868	9.952659	10.047341	7
54	9.816069	9.878438	9.937632	10.062368	9.824668	9.871755	9.952913	10.047087	6
55	9.816215	9.878328	9.937887	10.062113	9.824808	9.871641	9.953167	10.046833	5
56	9.816361	9.878219	9.938142	10.061858	9.824949	9.871528	9.953421	10.046579	4
57	9.816507	9.878109	9.938398	10.061602	9.825090	9.871414	9.953675	10.046325	3
58	9.816652	9.877999	9.938653	10.061347	9.825230	9.871301	9.953929	10.046071	2
59	9.816798	9.877890	9.938908	10.061092	9.825371	9.871187	9.954183	10.045817	1
60	9.816943	9.877780	9.939163	10.060837	9.825511	9.871073	9.954437	10.045563	0
	Cosine.	Sine.	Cotan.	Tang.	Cosine.	Sine.	Cotan.	Tang.	
	49 Deg.				48 Deg.				

	42 Deg.				43 Deg.				
	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	
0	9.825511	9.871073	9.954437	10.045563	9.833783	9.864127	9.969656	10.030344	60
1	9.825651	9.870960	9.954691	10.045309	9.833919	9.864010	9.969903	10.030091	59
2	9.825791	9.870846	9.954946	10.045054	9.834054	9.863892	9.970162	10.029838	58
3	9.825931	9.870732	9.955200	10.044800	9.834189	9.863774	9.970416	10.029584	57
4	9.826071	9.870618	9.955454	10.044546	9.834325	9.863656	9.970669	10.029331	56
5	9.826211	9.870504	9.955708	10.044292	9.834460	9.863538	9.970922	10.029078	55
6	9.826351	9.870390	9.955961	10.044039	9.834595	9.863419	9.971175	10.028825	54
7	9.826491	9.870276	9.956215	10.043785	9.834730	9.863301	9.971429	10.028571	53
8	9.826631	9.870161	9.956469	10.043531	9.834865	9.863183	9.971682	10.028318	52
9	9.826770	9.870047	9.956723	10.043277	9.834999	9.863064	9.971935	10.028065	51
10	9.826910	9.869933	9.956977	10.043023	9.835134	9.862946	9.972188	10.027812	50
11	9.827049	9.869818	9.957231	10.042769	9.835269	9.862827	9.972441	10.027559	49
12	9.827189	9.869704	9.957485	10.042515	9.835403	9.862709	9.972695	10.027305	48
13	9.827328	9.869589	9.957739	10.042261	9.835538	9.862590	9.972948	10.027052	47
14	9.827467	9.869474	9.957993	10.042007	9.835672	9.862471	9.973201	10.026799	46
15	9.827606	9.869360	9.958247	10.041753	9.835807	9.862353	9.973454	10.026546	45
16	9.827745	9.869245	9.958500	10.041500	9.835941	9.862234	9.973707	10.026293	44
17	9.827884	9.869130	9.958754	10.041246	9.836075	9.862115	9.973960	10.026040	43
18	9.828023	9.869015	9.959008	10.040992	9.836209	9.861996	9.974213	10.025787	42
19	9.828162	9.868900	9.959262	10.040738	9.836343	9.861877	9.974466	10.025534	41
20	9.828301	9.868785	9.959516	10.040484	9.836477	9.861758	9.974720	10.025280	40
21	9.828439	9.868670	9.959769	10.040231	9.836611	9.861638	9.974973	10.025027	39
22	9.828578	9.868555	9.960023	10.039977	9.836745	9.861519	9.975226	10.024774	38
23	9.828716	9.868440	9.960277	10.039723	9.836878	9.861400	9.975479	10.024521	37
24	9.828855	9.868324	9.960530	10.039470	9.837012	9.861280	9.975732	10.024268	36
25	9.828993	9.868209	9.960784	10.039216	9.837146	9.861161	9.975985	10.024015	35
26	9.829131	9.868093	9.961038	10.038962	9.837279	9.861041	9.976238	10.023762	34
27	9.829269	9.867978	9.961292	10.038708	9.837412	9.860922	9.976491	10.023509	33
28	9.829407	9.867862	9.961545	10.038455	9.837546	9.860802	9.976744	10.023256	32
29	9.829545	9.867747	9.961799	10.038201	9.837679	9.860682	9.976997	10.023003	31
30	9.829683	9.867631	9.962052	10.037948	9.837812	9.860562	9.977250	10.022750	30
31	9.829821	9.867515	9.962306	10.037694	9.837945	9.860442	9.977503	10.022497	29
32	9.829959	9.867399	9.962560	10.037440	9.838078	9.860322	9.977756	10.022244	28
33	9.830097	9.867283	9.962813	10.037187	9.838211	9.860202	9.978009	10.021991	27
34	9.830234	9.867167	9.963067	10.036933	9.838344	9.860082	9.978262	10.021738	26
35	9.830372	9.867051	9.963320	10.036680	9.838477	9.859962	9.978515	10.021485	25
36	9.830509	9.866935	9.963574	10.036426	9.838610	9.859842	9.978768	10.021232	24
37	9.830646	9.866819	9.963828	10.036172	9.838742	9.859721	9.979021	10.020979	23
38	9.830784	9.866703	9.964081	10.035919	9.838875	9.859601	9.979274	10.020726	22
39	9.830921	9.866586	9.964335	10.035665	9.839007	9.859480	9.979527	10.020473	21
40	9.831058	9.866470	9.964588	10.035412	9.839140	9.859360	9.979780	10.020220	20
41	9.831195	9.866353	9.964842	10.035158	9.839272	9.859239	9.980033	10.019967	19
42	9.831332	9.866237	9.965095	10.034905	9.839404	9.859119	9.980286	10.019714	18
43	9.831469	9.866120	9.965349	10.034651	9.839536	9.858998	9.980538	10.019462	17
44	9.831606	9.866004	9.965602	10.034398	9.839668	9.858877	9.980791	10.019209	16
45	9.831742	9.865887	9.965855	10.034145	9.839800	9.858756	9.981044	10.018956	15
46	9.831879	9.865770	9.966109	10.033891	9.839932	9.858635	9.981297	10.018703	14
47	9.832015	9.865653	9.966362	10.033638	9.840064	9.858514	9.981550	10.018450	13
48	9.832152	9.865536	9.966616	10.033384	9.840196	9.858393	9.981803	10.018197	12
49	9.832288	9.865419	9.966869	10.033131	9.840328	9.858272	9.982056	10.017944	11
50	9.832425	9.865302	9.967123	10.032877	9.840459	9.858151	9.982309	10.017691	10
51	9.832561	9.865185	9.967376	10.032624	9.840591	9.858029	9.982562	10.017438	9
52	9.832697	9.865068	9.967629	10.032371	9.840722	9.857908	9.982814	10.017186	8
53	9.832833	9.864950	9.967883	10.032117	9.840854	9.857786	9.983067	10.016933	7
54	9.832969	9.864833	9.968136	10.031864	9.840985	9.857665	9.983320	10.016680	6
55	9.833105	9.864716	9.968389	10.031611	9.841116	9.857543	9.983573	10.016427	5
56	9.833241	9.864598	9.968643	10.031357	9.841247	9.857422	9.983826	10.016174	4
57	9.833377	9.864481	9.968896	10.031104	9.841378	9.857300	9.984079	10.015921	3
58	9.833512	9.864363	9.969149	10.030851	9.841509	9.857178	9.984332	10.015668	2
59	9.833648	9.864245	9.969403	10.030597	9.841640	9.857056	9.984584	10.015416	1
60	9.833783	9.864127	9.969656	10.030344	9.841771	9.856934	9.984837	10.015163	0
	Cosine.	Sine.	Cotan.	Tang.	Cosine.	Sine.	Cotan.	Tang.	
	47 Deg.				46 Deg.				

44 Deg.

′	Sine.	Cosine.	Tang.	Cotang.	
0	9.841711	9.856934	9.984837	10.015163	60
1	9.841902	9.856812	9.985090	10.014910	59
2	9.842033	9.856690	9.985343	10.014657	58
3	9.842163	9.856568	9.985596	10.014404	57
4	9.842294	9.856446	9.985848	10.014152	56
5	9.842424	9.856323	9.986101	10.013899	55
6	9.842555	9.856201	9.986354	10.013646	54
7	9.842685	9.856078	9.986607	10.013393	53
8	9.842815	9.855956	9.986860	10.013140	52
9	9.842946	9.855833	9.987112	10.012888	51
10	9.843076	9.855711	9.987365	10.012635	50
11	9.843206	9.855588	9.987618	10.012382	49
12	9.843336	9.855465	9.987871	10.012129	48
13	9.843466	9.855342	9.988123	10.011877	47
14	9.843595	9.855219	9.988376	10.011624	46
15	9.843725	9.855096	9.988629	10.011371	45
16	9.843855	9.854973	9.988882	10.011118	44
17	9.843984	9.854850	9.988134	10.010866	43
18	9.844114	9.854727	9.989387	10.010613	42
19	9.844243	9.854603	9.989640	10.010360	41
20	9.844372	9.854480	9.989893	10 010107	40
21	9.844502	9.854356	9.990145	10.009855	39
22	9.844631	9.854233	9.990398	10.009602	38
23	9.844760	9.854109	9.990651	10.009349	37
24	9.844889	9.853986	9.990903	10.009097	36
25	9 845018	9.853862	9.991156	10.008844	35
26	9.845147	9.853738	9.991409	10.008591	34
27	9.845276	9.853614	9.991662	10.008338	33
28	9.845405	9.853490	9.991914	10.008086	32
29	9.845533	9.853366	9.992167	10.007833	31
30	9.845662	9.853242	9.992420	10.007580	30
31	9.845790	9.853118	9.992672	10.007328	29
32	9.845919	9.852994	9.992925	10.007075	28
33	9.846047	9.852869	9.993178	10.006822	27
34	9.846175	9.852745	9.993431	10.006569	26
35	9.846304	9.852620	9.993683	10.006317	25
36	9.846432	9.852496	9.993936	10.006064	24
37	9.846560	9.852371	9.994189	10.005811	23
38	9.846688	9.852247	9.994441	10.005559	22
39	9.846816	9.852122	9.994694	10.005306	21
40	9.846944	9.851997	9.994947	10.005053	20
41	9.847071	9.851872	9.995119	10.004801	19
42	9.847199	9.851747	9.995452	10.004548	18
43	9.847327	9.851622	9.995705	10.004295	17
44	9.847454	9.851497	9.995957	10.004043	16
45	9 847582	9.851372	9.996210	10.003790	15
46	9.847709	9.851246	9.996463	10.003537	14
47	9.847836	9.851121	9.996715	10.003285	13
48	9.847964	9.850996	9.996968	10.003032	12
49	9.848091	9.850870	9.997221	10.002779	11
50	9.848218	9.850745	9.997473	10 002527	10
51	9 848345	9.850619	9.997726	10.002274	9
52	9.848472	9.850493	9.997979	10.002021	8
53	9 848599	9.850368	9.998231	10 001769	7
54	9.848726	9.850242	9.998484	10.001516	6
55	9.848852	9.850116	9.998737	10.001263	5
56	9.848979	9.849990	9.998989	10.001011	4
57	9.849106	9.849864	9.999242	10.000758	3
58	9.849232	9.849738	9.999495	10.000505	2
59	9.849359	9.849611	9.999747	10.000253	1
60	9.849485	9.849485	10.00000	10.000000	0
	Cosine.	Sine.	Cotan.	Tang.	′

45 Deg.

NOTE.—To find the *log. secant* of an arc or angle, subtract the *log. cosine* from 20, in the index.

To find the *log. cosecant*, subtract the *log. sine* from 20.

To find the *log. versed-sine*, add ·301030, to twice the *log. sine* of *half* the arc or angle, and take 10 from the index of the sum.

END OF VOL. I.

1.079191

9.920809

8.920819

1.5973

8917

1.4098